5.6 Solutions for Chapter 5

Section 5.1

1. A card is randomly selected from a deck of 52 cards. What is the chance that the card is red or a king?

Solution: The sample space S is the set of 52 cards. The experiment is drawing one card. The event $E \subseteq S$ is the set of the cards that are red or kings. This is the set of 26 red cards, plus the king of spades and the king of clubs. Therefore $|E| = 28$, and $p(E) = \frac{|E|}{|S|} = \frac{28}{52} \approx 53.8\%$.

3. Toss a dice 5 times in a row. What is the probability that you don't get any ⚅'s?

Solution: The sample space S is the set of all length-6 lists (repetition allowed) whose entries are the numbers 1, 2, 3, 4, 5, 6. There are $6^5 = 46{,}656$ such lists, so $|S| = 46{,}656$. The event E consists of those lists in S that do not contain a 6. There are $5^6 = 15{,}625$ of them, so $p(E) = \frac{|E|}{|S|} = \frac{15{,}625}{46{,}656} \approx 33.5\%$.

5. Toss a dice 5 times in a row. What is the probability that you will get the same number on each roll? (i.e. ⚀⚀⚀⚀⚀ or ⚁⚁⚁⚁⚁, etc.)

Solution: The sample space S is the set of all length-5 lists (repetition allowed) whose entries are the numbers 1, 2, 3, 4, 5, 6. There are $6^5 = 7776$ such lists, so $|S| = 7776$. Note that $E = \{11111, 22222, 33333, 44444, 55555, 66666\}$, so $|E| = 6$, and $p(E) = \frac{|E|}{|S|} = \frac{6}{7776} \approx 0.077\%$.

7. You have a pair of dice, a white one and a black one. Toss them both. What is the probability that they show the same number?

Solution: The sample space S is shown in Example 5.1 on page 132. You can see that $|S| = 36$ and the event E of both dice showing the same number has cardinality 6, so $p(E) = \frac{|E|}{|S|} = \frac{6}{36} = 16.\overline{6}\%$.

9. You have a pair of dice, a white one and a black one. Toss them both. What is the probability that both show even numbers?

Solution: The sample space S is shown in Example 5.1 on page 132. Note that $E = \{$⚁⚁, ⚁⚃, ⚁⚅, ⚃⚁, ⚃⚃, ⚃⚅, ⚅⚁, ⚅⚃, ⚅⚅$\}$. Thus $p(E) = \frac{|E|}{|S|} = \frac{9}{36} = 25\%$.

11. Toss a coin 8 times. Find the probability that the first and last tosses are heads.

Solution: The sample space S is the set of all length-8 lists made from the two symbols H and T. Thus $|S| = 2^8$. The event E of the first and last tosses being heads consists of all those outcomes in S that have the form $(\text{H}, \square, \square, \square, \square, \square, \square, \text{H})$ where there are two choices for each box. Thus $|E| = 2^6$, and so $p(E) = \frac{|E|}{|S|} = \frac{2^6}{2^8} = \frac{1}{2^2} = 25\%$.

13. Five cards are dealt from a shuffled 52-card deck. What is the probability of getting three red cards and two clubs?

Solution: The sample space S is the set of all possible 5-card hands that can be made from the 52 cards in the deck, so $|S| = \binom{52}{5} = 2{,}598{,}960$. There are $\binom{26}{3}$ ways to get 3 red cards and $\binom{13}{2}$ ways to get 2 clubs, so by the multiplication principle there are $\binom{26}{3}\binom{13}{2} = 202{,}800$ different 5-card hands that have 3 red cards and 2 clubs. Therefore $p(E) = \frac{|E|}{|S|} = \frac{202{,}800}{2{,}598{,}960} \approx 12.81\%$.

15. Alice and Bob each randomly pick an integer from 0 to 9. What is the probability that they pick the same number? What is the probability that they pick different numbers?

Solution: Lets put $S = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : 0 \le a,b \le 9\} = \{(0,0),(0,1),(0,2),\ldots,(9,9)\}$ where a ordered pair (a,b) means Alice picked a and Bob picked b. Then $S = 10 \cdot 10 = 100$. The event of their both picking the same number is $E = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9)\}$, so $|E| = 10$. Then the probability of their picking the same number is $p(E) = \frac{|E|}{|S|} = \frac{10}{100} = 10\%$. The event of their picking different numbers is $\overline{E} = S - E$, so the probability of their picking different numbers is $p(E) = \frac{|\overline{E}|}{|S|} = \frac{90}{100} = 90\%$.

17. What is the probability that a 5-card hand dealt off a shuffled 52-card deck does not contain an ace?

Solution: The sample space S is the set of all possible 5-card hands that can be made from the 52 cards in the deck, so $|S| = \binom{52}{5} = 2{,}598{,}960$. To make a 5-card hand that contains no ace, we have to choose 5 cards from the 48 non-ace cards, so there are $\binom{48}{5} = 1{,}712{,}304$ hands that contain no aces. Thus the probability of the event E of no aces in the hand is $p(E) = \frac{|E|}{|S|} = \frac{1{,}712{,}304}{2{,}598{,}960} \approx 65.88\%$.

Section 5.2

1. A card is taken off the top of a shuffled 52-card deck. What is the probability that it is black or an ace?

Solution: Let A be the event of the card being an ace, and let B be the event of its being black. Then $p(A) = \frac{4}{52}$, and $p(B) = \frac{26}{52}$. The event $A \cap B$ is the event of the card being either the ace of spades or the ace of clubs. Thus $|A \cap B| = 2$, and $p(A \cap B) = \frac{2}{52}$. The answer we seek is $p(A \cup B) = p(A) + p(B) - p(A \cap B) = \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52} \approx 53.8\%$.

3. What is the probability that a 5-card hand dealt off a shuffled 52-card deck contains at least one red card?

Solution: The sample space S is the set of all 5-card hands, so $|S| = \binom{52}{5} = 2{,}598{,}960$. Let E be the event of a 5-card hand without any red cards. Then $|E| = \binom{26}{5} = 65{,}780$ (choose 5 cards from the 26 black cards). Note that the complement $\overline{E}$ is the event of at least one red card, so the answer we seek is $p(\overline{E}) = 1 - p(E) = 1 - \frac{|E|}{|S|} = 1 - \frac{65{,}780}{2{,}598{,}960} \approx 97.46\%$.

5. You toss a fair coin 8 times. What is the probability that you do not get 4 heads?

Solution: The sample space S is the set of all length-8 lists made from the symbols H and T. Thus $|S| = 2^8$. Now let E be the event of getting exactly 4 heads, so $|E| = \binom{8}{4} = 70$. (Choose 4 of 8 positions for H and fill the rest with T.) Then $\overline{E}$ is the event of not getting four heads. Our answer is then $p(\overline{E}) = 1 - p(E) = 1 - \frac{|E|}{|S|} = 1 - \frac{70}{2^8} \approx 72.65\%$.

7. Two cards are dealt off a shuffled 52-card deck.What is the probability that the cards are both red or both aces?

Solution: The sample space S consists of all possible 2-card hands, so $|S| = \binom{52}{2} = 1326$. Let A be the event of both cards being aces, so $|A| = \binom{4}{2} = 6$. Let B be the event that both cards are red, so $|B| = \binom{26}{2} = 325$. Then the event $A \cap B$ consists on only one hand, namely the 2-card hand consisting of the ace of hearts and the ace of diamonds. The answer to our question is $p(A \cup B) = p(A) + p(B) - p(A \cap B) = \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|} = \frac{6}{1326} + \frac{325}{1326} - \frac{1}{1326} \approx 24.88\%$.

9. A dice is tossed six times. You win \$1 if the first toss is a five or the last toss is even. What are your chances of winning?

Solution: The sample space S is the set of all length-6 lists made from the symbols 1, 2, 3, 4, 5 and 6. Thus $|S| = 6^6$. Let A be the event of the first toss being a five. By the multiplication principle, $|A| = 6^5$. Let B be the event of the last toss being even, that is, 2, 4 or 6. Then $|B| = 6^5 \cdot 3$. Note that $A \cap B$ is the set of all lists in S whose first entry is 5 and whose last entry is even. By the multiplication principle, $|A \cap B| = 6^4 \cdot 3$. The probability that the first toss is a five and the last is even is $p(A \cup B) = p(A) + p(B) - p(A \cap B) = \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|} = \frac{6^5}{6^6} + \frac{6^5 \cdot 3}{6^6} - \frac{6^4 \cdot 3}{6^6} = \frac{1}{6} + \frac{3}{6} - \frac{3}{36} = \frac{21}{36} \approx 58.33\%$. So you have a reasonably good chance of winning.

11. A dice is rolled 5 times. Find the probability that not all of the tosses are even.

Solution: Think if the sample space as being the set of all length-5 lists made from the numbers 1, 2, 3, 4, 5 and 6, where the first entry is the result of the first roll, the second entry is the result of the second roll, etc. Thus $|S| = 6^5$. Now let E be the event of all rolls being even. Then E is the set of all length-5 lists made from the numbers 2, 4 and 6, so $|E| = 3^5$. We are interested in the probability of the event of not all rolls being even, that is, the probability of $\overline{E}$. Thus our answer is $p(\overline{E}) = 1 - p(E) = 1 - \frac{|E|}{|S|} = 1 - \frac{3^5}{6^5} \approx 96.875\%$.

13. Two cards are dealt off a well-shuffled deck. You win \$1 if the two cards are of different suits. Find the probability of your winning.

Solution: The sample space S is the set of all possible 2-card hands, so $|S| = \binom{52}{2} = 1326$. There are $\binom{13}{2} = 78$ 2-card hands with both cards hearts, and similarly 78 hands with both cards diamonds, 78 hands with both cards clubs, and 78 hands with both cards spades. By the addition principle there are $78 + 78 + 78 + 78 = 312$ hands in S for which both cards are of the same suit. So there are $|S| - 312 = 1014$ hands in S for which the carss are of different suits. Thus the probability of the two cards being the same suit is $\frac{1014}{|S|} = \frac{1014}{1326} \approx 76.47\%$.

15. A coin is tossed 5 times. What is the probability that the first toss is a head or exactly 2 out of the five tosses are heads?

Solution: The sample space S is the set of length-5 lists made from the symbols H and T, so $|S| = 2^5 = 32$. The event $A \subseteq S$ of the first toss being a head is the set of all lists in S of form H□□□□, so $|A| = 2^4 = 16$. The event B of exactly two heads has cardinality $|B| = \binom{5}{2} = 10$. (Choose two of 5 positions for H, and fill the rest with T's.) Finally, $A \cap B$ is the set of lists in S whose first entry is H and exactly one of the four remaining entries is an H, so $|A \cap B| = 4$. So the probability of the first toss being a head or exactly two tosses being heads is $p(A \cap B) = p(A) + p(B) - p(A \cap B) = \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|} = \frac{16}{32} + \frac{10}{32} - \frac{4}{32} = \frac{22}{32} = 68.75\%$.

17. In a shuffled 52-card deck, what is the probability that neither the top nor bottom card is a heart?

Solution: Regard the sample space as the set of 2-element lists (no repetition) whose entries are the cards in the deck. The first entry represents the top card and the second entry represents the bottom card. Then $|S| = 52 \cdot 51 = 2652$. Now let E be the event that neither the top nor bottom card is a heart. So E consists of those lists in S for which neither entry is a heart. As there are 39 non-heart cards, $|E| = 39 \cdot 38 = 1482$. The answer is thus $p(E) = \frac{|E|}{|S|} = \frac{1482}{2652} \approx 55.88\%$.

19. A bag contains 20 red marbles, 20 green marbles and 20 blue marbles. You reach in and grab 15 marbles. What is the probability that they are all the same color?

Solution: An outcome for this experiment is a 15-element multiset made from the symbols $\{\text{R}, \text{G}, \text{B}\}$. Thus the sample space S is the set of all such multisets. Encode the elements of S as stars and bars, so a typical element of S is a list of length $15+2 = 17$, made from 15 stars and 2 bars.

$$\underbrace{* * \cdots * *}_{* \text{ for each R}} \,|\, \underbrace{* * \cdots * *}_{* \text{ for each G}} \,|\, \underbrace{* * \cdots * *}_{* \text{ for each B}}$$

Then $|S| = \binom{17}{2} = 136$. Also the event E of all balls being the same color is $E = \{***************||,\ |***************|,\ ||***************\}$, so $|E| = 3$. Finally, $p(E) = \frac{|E|}{|S|} = \frac{3}{136} \approx 2.2\%$.

Section 5.3

1. A box contains six tickets: $\boxed{\boxed{\text{A}}\boxed{\text{A}}\boxed{\text{B}}\boxed{\text{B}}\boxed{\text{B}}\boxed{\text{E}}}$. You remove two tickets, one after the other. What is the probability that the first ticket is an A and the second is a B?

Solution: Let A be the event of the first draw being an A and let B be the event of the second draw being a B. With the help of Fact 5.3, the answer to this question is $p(A \cap B) = p(A) \cdot P(B|A) = \frac{2}{6} \cdot \frac{2}{5} = \frac{2}{15} = 13.\overline{3}\%$.

3. In a shuffled 52-card deck, what is the probability that the top card is red and the bottom card is a heart?

Solution: Let A be the event of the top card being red, and let B be the event of the bottom card being a heart. With the help of Fact 5.3, the answer to this question is $p(A \cap B) = p(B) \cdot P(A|B) = \frac{13}{52} \cdot \frac{25}{26} = \frac{25}{124} \approx 20.16\%$. (Notice that if you used the formula $p(A \cap B) = p(A) \cdot p(B|A)$, then the problem is somewhat harder to think about.)

5. Suppose A and B are events, and $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(A \cap B) = \frac{1}{6}$. Are A and B independent, dependent, or is there not enough information to say for sure?

Solution: Using the information given, and Fact 5.3, we get $\frac{1}{6} = p(A \cap B) = p(A) \cdot p(B|A) = \frac{1}{2}p(B|A)$, which yields $p(B|A) = \frac{1}{3}$, so $p(B|A) = p(B)$. Also, $\frac{1}{6} = p(A \cap B) = p(B) \cdot p(A|B) = \frac{1}{3}p(A|B)$, which yields $p(A|B) = \frac{1}{2}$, so $p(A) = p(A|B)$. This means A and B are independent.

7. Say A and B are events with $P(A) = \frac{2}{3}$, $P(A|B) = \frac{3}{4}$, and $P(B|A) = \frac{1}{2}$. Find $p(B)$.

Solution: Fact 5.3 says $p(A) \cdot p(B|A) = p(A \cap B) = p(B) \cdot p(A|B)$. Plugging in the given information, this becomes $\frac{2}{3} \cdot \frac{1}{2} = p(B) \cdot \frac{3}{4}$. Solving, $p(B) = \frac{4}{9}$.

9. A box contains 2 red balls, 3 black balls, and 4 white balls. One is removed, and then another is removed. What is the probability that no black balls were drawn?

Let A be the event of no black ball on the first draw. Let B be the event of no black ball drawn on the second draw. Then $p(A) = \frac{6}{9}$, because 6 of the 9 balls are not black. If A has occurred, then 5 of the remaining 8 balls are not black, so $p(B|A) = \frac{5}{8}$. The probability that no black ball was drawn is then $p(A \cap B) = p(A) \cdot p(B|A) = \frac{6}{9}\frac{5}{8} = \frac{5}{12} = 41.\overline{6}\%$.

11. A coin is flipped 5 times, and there are more tails than heads. What is the probability that the first flip was a tail?

Solution: The sample space S is the set of length-5 lists made from symbols H and T, so $|S| = 2^5 = 32$. Let A be the event of there being more tails than heads, and let B be the event of the first flip being a tail. Thus the answer to the question will be $p(B|A)$. Fact 5.3 says $p(B|A) = \frac{p(A \cap B)}{p(A)}$, so we need to calculate $p(A \cap B)$ and $p(A)$. Note that $A \cap B$ is the event of more tails than heads **and** the first flip is a tail. If the first flip is a tail, and there are to be more tails than heads, then 2, 3 or 4 of the remaining 4 flips must be tails. The number of ways for this to happen is $\binom{4}{2} + \binom{4}{3} + \binom{4}{1} = 6 + 4 + 1 = 11$, so $|A \cap B| = 11$. Considering $|A|$, to have more tails than heads, 3, 4 or 5 of the flips must be tails, and it follows that $|A| = \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 10 + 5 + 1 = 16$. To get our final answer, we have

$$p(B|A) = \frac{p(A \cap B)}{p(A)} = \frac{\frac{|A \cap B|}{|S|}}{\frac{|A|}{|S|}} = \frac{|A \cap B|}{|A|} = \frac{11}{16} = 68.75\%.$$

13. A 5-card hand is dealt from a shuffled 52-card deck. Exactly 2 of the cards in the hand are hearts. Find the probability that all the cards in the hand are red.

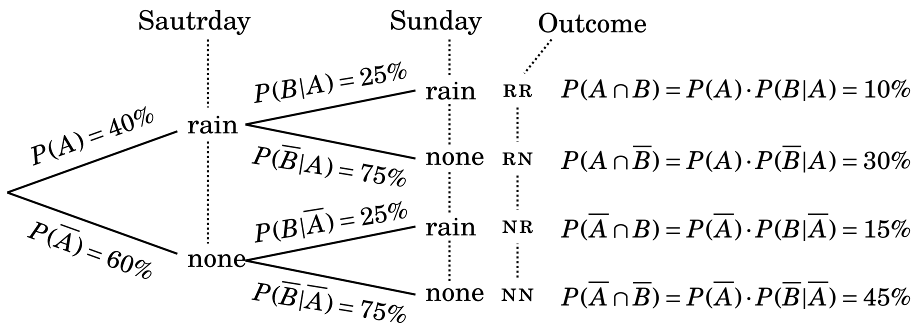
Solution: Let A be the event of getting exactly 2 hearts in the hand. Let B be the event of all cards in the hand being red. Thus the answer to the question will be $p(B|A)$. Fact 5.3 says $p(B|A) = \frac{p(A \cap B)}{p(A)}$, so we need to calculate $p(A \cap B)$ and $p(A)$. Note that $A \cap B$ is the event of getting a 5-card hand that has 2 hearts and 3 diamonds. Thus $|A \cap B| = \binom{13}{2}\binom{13}{3}$. Also $|A| = \binom{13}{2}\binom{39}{3}$ (choose 2 hearts and 3 non-hearts). To get our final answer, we have

$$p(B|A) = \frac{p(A \cap B)}{p(A)} = \frac{\frac{|A \cap B|}{|S|}}{\frac{|A|}{|S|}} = \frac{|A \cap B|}{|A|} = \frac{\binom{13}{2}\binom{13}{3}}{\binom{13}{2}\binom{39}{3}} = \frac{\binom{13}{3}}{\binom{39}{3}} = \frac{13 \cdot 12 \cdot 11}{39 \cdot 38 \cdot 37} \approx 3.13\%.$$

Section 5.4

1. There is a 40% chance of rain on Saturday and a 25% chance of rain on Sunday. What is the probability that it will rain on at least one day of the weekend? (You may assume that the events "Rain on Saturday" and "Rain on Sunday" are independent events.)

Solution: Say A is the event of rain on Saturday and B is the event of rain on Sunday. Then our sample space is $S = \{\text{RR, RN, NR, NN}\}$, and $A = \{\text{RR, RN}\}$ and $B = \{\text{RR, NR}\}$. Here is a probability tree for this.



From this, the probability of rain over the weekend is $10\% + 30\% + 15\% = 55\%$.

If you got the answer without drawing a probability tree, then that is good. Another solution would be to calculate the probability of no rain over the weekend, which is $p(\overline{A}) \cdot p(\overline{B}|\overline{A}) = p(\overline{A}) \cdot p(\overline{B}) = 0.6 \cdot 0.75 = 45\%$. Then the probability of rain over the weekend is $1 - 0.45 = 55\%$.

3. A club consists of 60 men and 40 women. To fairly choose a president and a secretary, names of all members are put into a hat and two names are drawn. The first name drawn is the president, and the second name drawn is the secretary. What is the probability that the president and the secretary have the same gender?

Solution: Say the sample space is $S = \{\text{MM, MW, WM, WW}\}$ where the first letter indicates the gender of the first draw, and the second letter indicates the gender of the second draw. Then $p(\text{MM}) = \frac{60}{100} \cdot \frac{59}{99} = \frac{3540}{9900}$ and $p(\text{WW}) = \frac{40}{100} \cdot \frac{39}{99} = \frac{1560}{9900}$. Thus the probability of both offices being the same gender is $p(\{\text{MM, WW}\}) = \frac{3540}{9900} + \frac{1560}{9900} = \frac{5100}{9900} = 51.\overline{51}\%$.

5. At a certain college, 30% of the students are freshmen. Also, 80% of the freshmen live on campus, while only 60% of the non-freshman students live on campus. A student is chosen at random. What is the probability that the student is a freshman who lives off campus?

Solution: Let A be the event of choosing a freshman. Let B be the event of choosing someone who lives on campus. The given information states that $p(A) = 30\%$ and $p(\overline{B}|A) = 20\%$. (If you chose a freshman, there is an 80% chance he or she lives on campus, so there is a 20% chance he or she lives off campus.)

The problem is asking for $p(A \cap \overline{B})$. Now, $p(A \cap \overline{B}) = p(A) \cdot p(\overline{B}|A) = 0.30 \cdot 0.20 = 6\%$. Thus there is a 6% chance of choosing a freshman who lives off campus.

Section 5.5

1. At a certain college, 40% of the students are male, and 60% are female. Also, 20% of the males are smokers, and 10% of the females are smokers. A student is chosen at random. If the student is a smoker, what is the probability that the student is female?

Solution: Let S be the set of all students, S_1 be the set of female students, and S_2 be the set of male students. Then $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. Let $E \subseteq S$ be the set of smokers. The problem asks for $p(S_1|E)$. Bayes' theorem applies and we get

$$p(S_1|E) = \frac{p(S_1) \cdot p(E|S_1)}{p(S_1) \cdot p(E|S_1) + p(S_2) \cdot p(E|S_2)} = \frac{0.60 \cdot 0.10}{0.60 \cdot 0.10 + 0.40 \cdot 0.20} = \frac{0.6}{0.8} = 75\%.$$

3. A jar contains 4 red balls and 5 white balls. A random ball is removed, and then another is removed. If the second ball was red, what is the probability that the first ball was red?

Solution: The sample space is $S = \{\textsc{rr}, \ \textsc{rw}, \ \textsc{wr}, \ \textsc{ww}\}$, where the first letter is the color of the first ball and the second letter is the color of the second ball. Let $S_1 = \{\textsc{rr}, \textsc{rw}\}$ be the event of the first ball being red. Let $S_2 = \{\textsc{wr}, \textsc{ww}\}$ be the event of the first ball being white. Let $E = \{\textsc{rr}, \textsc{wr}\}$ be the event of the second ball being red. The answer to the question is thus $p(S_1 \mid E)$. As $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, Bayes' formula applies, and it gives

$$\begin{aligned} p(S_1 \mid E) &= \frac{p(S_1) \cdot p(E \mid S_1)}{p(S_1) \cdot p(E \mid S_1) + p(S_2) \cdot p(E \mid S_2)} \\ &= \frac{\frac{4}{9} \cdot \frac{3}{8}}{\frac{4}{9} \cdot \frac{3}{8} + \frac{5}{9} \cdot \frac{4}{8}} = \frac{\frac{1}{6}}{\frac{3}{18} + \frac{5}{18}} = \frac{\frac{1}{6}}{\frac{4}{9}} = \frac{3}{8} = 37.5\%. \end{aligned}$$

So there's a 37.5% chance that the first ball was red if the second is red.