CHAPTER 7

Quantified Statements

We have seen that the symbols $\wedge$, $\vee$, $\sim$, $\Rightarrow$ and $\Leftrightarrow$ can guide the logical flow of algorithms. We have learned how to use them to deconstruct many English sentences into a symbolic form. We have studied how this symbolic form can help us understand the logical structure of sentences and how different sentences may actually have the same meaning (as in logical equivalence). This will be particularly significant as we begin proving theorems in the next chapter.

But these logical symbols alone are not powerful enough to capture the full meaning of every statement. To see why, imagine that we are dealing with some set $S = \{x_1, x_2, x_3, \ldots\}$ of integers. (For emphasis, say S is an infinite set.) Suppose we want to express the statement "*Every element of S is odd.*" We would have to write

$$P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge P(x_4) \wedge \cdots,$$

where $P(x)$ is the open sentence "*x is odd*." And if we wanted the express "*There is at least one element of S that is odd,*" we'd have to write

$$P(x_1) \vee P(x_2) \vee P(x_3) \vee P(x_4) \vee \cdots.$$

The problem is that these expressions might never end.

To overcome this defect, we will introduce two new symbols $\forall$ and $\exists$. The symbol $\forall$ stands for the phrase "*for all*" and $\exists$ stands for "*there exists*." Thus the statement "*Every element of S is odd.*" is written symbolically as

$$\forall x \in S, P(x),$$

and "*There is at least one element of S that is odd,*" is written succinctly as

$$\exists x \in S, P(x),$$

These new symbols are called *quantifiers*. They are the subject of this chapter.

7.1 Quantifiers

To repeat, here are the main ideas of this chapter.

> **Definition 7.1** The symbols $\forall$ and $\exists$ are called **quantifiers**.
>
> $\forall$ stands for the phrase *"For all"* or *"For every,"* or *"For each,"*
>
> $\exists$ stands for the phrase *"There exists a"* or *"There is a."*

Thus the statement

For every $n \in \mathbb{Z}$, $2n$ is even,

can be expressed in either of the following ways:

$\forall\, n \in \mathbb{Z}, 2n$ is even,

$\forall\, n \in \mathbb{Z}, E(2n)$.

Likewise, a statement such as

There exists a subset X of $\mathbb{N}$ for which $|X| = 5$.

can be translated as

$\exists X, (X \subseteq \mathbb{N}) \wedge (|X| = 5)$ or $\exists X \subseteq \mathbb{N}, |X| = 5$ or $\exists X \in \mathscr{P}(\mathbb{N}), |X| = 5$.

The symbols $\forall$ and $\exists$ are called quantifiers because they refer in some sense to the quantity (i.e., all or some) of the variable that follows them. Symbol $\forall$ is called the **universal quantifier** and $\exists$ is called the **existential quantifier**. Statements which contain them are called **quantified** statements. A statement beginning with $\forall$ is called a **universally quantified** statement, and one beginning with $\exists$ is called an **existentially quantified** statement.

Example 7.1 The following English statements are paired with their translations into symbolic form.

Every integer that is not odd is even.
$\forall\, n \in \mathbb{Z}, \sim (n \text{ is odd }) \Rightarrow (n \text{ is even})$, or $\forall\, n \in \mathbb{Z}, \sim O(n) \Rightarrow E(n)$.

There is an integer that is not even.
$\exists n \in \mathbb{Z}, \sim E(n)$.

For every real number x, there is a real number y for which $y^3 = x$.
$\forall\, x \in \mathbb{R}, \exists\, y \in \mathbb{R}, y^3 = x$.

Given any two rational numbers a and b, it follows that ab is rational.
$\forall\, a, b \in \mathbb{Q}, ab \in \mathbb{Q}$. ✍

Given a set S (such as, but not limited to, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ etc.), a quantified statement of form $\forall x \in S, P(x)$ is understood to be true if $P(x)$ is true for every $x \in S$. If there is at least one $x \in S$ for which $P(x)$ is false, then $\forall x \in S, P(x)$ is a false statement. Similarly, $\exists x \in S, P(x)$ is true provided that $P(x)$ is true for at least one element $x \in S$; otherwise it is false. Thus each statement in Example 7.1 is true. Here are some that are false:

Example 7.2 The following false quantified statements are paired with their translations.

Every integer is even.
$\forall n \in \mathbb{Z}, E(n)$.

There is an integer n for which $n^2 = 2$.
$\exists n \in \mathbb{Z}, n^2 = 2$.

For every real number x, there is a real number y for which $y^2 = x$.
$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^2 = x$.

Given any two rational numbers a and b, it follows that $\sqrt{ab}$ is rational.
$\forall a, b \in \mathbb{Q}, \sqrt{ab} \in \mathbb{Q}$. ✍

Example 7.3 When a statement contains two quantifiers you must be very alert to their order, for reversing the order can change the meaning. Consider the following statement from Example 7.1.

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^3 = x$.

This statement is true, for no matter what number x is there exists a number $y = \sqrt[3]{x}$ for which $y^3 = x$. Now reverse the order of the quantifiers to get the new statement

$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, y^3 = x$.

This new statement says that there exists a particular number y with the property that $y^3 = x$ for *every* real number x. Since no number y can have this property, the statement is false. The two statements above have entirely different meanings. ✍

Quantified statements are often misused in casual conversation. Maybe you've heard someone say *"All students do not pay full tuition."* when they mean *"Not all students pay full tuition."* While the mistake is perhaps marginally forgivable in casual conversation, it must never be made in a mathematical context. Do not say *"All integers are not even."* because that means there are no even integers. Instead, say *"Not all integers are even."*

Exercises for Section 7.1

Write the following as English sentences. Say whether they are true or false.

1. $\forall x \in \mathbb{R}, x^2 > 0$
2. $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}, x^n \geq 0$
3. $\exists a \in \mathbb{R}, \forall x \in \mathbb{R}, ax = x$
4. $\forall X \in \mathscr{P}(\mathbb{N}), X \subseteq \mathbb{R}$
5. $\forall n \in \mathbb{N}, \exists X \in \mathscr{P}(\mathbb{N}), |X| < n$
6. $\exists n \in \mathbb{N}, \forall X \in \mathscr{P}(\mathbb{N}), |X| < n$
7. $\forall X \subseteq \mathbb{N}, \exists n \in \mathbb{Z}, |X| = n$
8. $\forall n \in \mathbb{Z},\ \exists X \subseteq \mathbb{N}, |X| = n$
9. $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m = n+5$
10. $\exists m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m = n+5$

7.2 More on Conditional Statements

It is time to address a very important point about conditional statements that contain variables. To motivate this, let's return to the following example concerning integers x:

$$(x \text{ is a multiple of } 6) \Rightarrow (x \text{ is even}).$$

As noted earlier, since every multiple of 6 is even, this is a true statement no matter what integer x is. We could even underscore this fact by writing this true statement as

$$\forall x \in \mathbb{Z}, (x \text{ is a multiple of } 6) \Rightarrow (x \text{ is even}).$$

But now switch things around to get the different statement

$$(x \text{ is even}) \Rightarrow (x \text{ is a multiple of } 6).$$

This is true for some values of x such as -6, 12, 18, etc., but false for others (such as 2, 4, etc.). Thus we do not have a statement, but rather an open sentence. (Recall from Section 3.1 that an *open sentence* is a sentence whose truth value depends on the value of a certain variable or variables.) However, by putting a universal quantifier in front we get

$$\forall x \in \mathbb{Z}, (x \text{ is even}) \Rightarrow (x \text{ is a multiple of } 6),$$

which is definitely false, so this new expression is a statement, *not an open sentence*. In general, given any two open sentences $P(x)$ and $Q(x)$ about integers x, the expression $\forall x \in \mathbb{Z}, P(x) \Rightarrow Q(x)$ is either true or false, so it is a statement, not an open sentence.

Now we come to the very important point. In mathematics, whenever $P(x)$ and $Q(x)$ are open sentences concerning elements x in some set S (depending on context), an expression of form $P(x) \Rightarrow Q(x)$ is understood to be the *statement* $\forall x \in S, P(x) \Rightarrow Q(x)$. In other words, if a conditional statement is not explicitly quantified then there is an implied universal quantifier in front of it. This is done because statements of the form $\forall x \in S, P(x) \Rightarrow Q(x)$ are so common in mathematics that we would get tired of putting the $\forall x \in S$ in front of them.

Thus the following sentence is a true statement (as it is true for all x).

If x is a multiple of 6, then x is even.

Likewise, the next sentence is a false statement (as it is not true for all x).

If x is even, then x is a multiple of 6.

This leads to the following significant interpretation of a conditional statement, which is more general than (but consistent with) the interpretation from Section 3.3.

Definition 7.2 If P and Q are statements or open sentences, then

"If P, then Q,"

is a statement. This statement is true if it's impossible for P to be true while Q is false. It is false if there is at least one instance in which P is true but Q is false.

Thus the following are **true** statements:

If $x \in \mathbb{R}$, then $x^2 + 1 > 0$.

If a function f is differentiable on $\mathbb{R}$, then f is continuous on $\mathbb{R}$.

If a list has n entries, then it has $n!$ permutations.

Likewise, the following are **false** statements:

If p is a prime number, then p is odd. (2 is prime.)

If f is a rational function, then f has an asymptote. (x^2 is rational.)

If a set X has n elements, then $|\mathscr{P}(X)| = n^2$. (true only if $|X| = 2$.)

7.3 Translating English to Symbolic Logic

In writing (and reading) proofs of theorems, we must always be alert to the logical structure and meanings of the sentences. Sometimes it is necessary or helpful to parse them into expressions involving logic symbols. This may be done mentally or on scratch paper, or occasionally even explicitly within the body of a proof. The purpose of this section is to give you sufficient practice in translating English sentences into symbolic form so that you can better understand their logical structure. Here are some examples:

Example 7.4 Consider the Mean Value Theorem from Calculus:

> If f is continuous on the interval $[a,b]$ and differentiable on (a,b), then there is a number $c \in (a,b)$ for which $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Here is a translation to symbolic form:

$$\Big(\big(f \text{ cont. on } [a,b]\big) \wedge \big(f \text{ is diff. on } (a,b)\big)\Big) \Rightarrow \Big(\exists\, c \in (a,b), f'(c) = \frac{f(b)-f(a)}{b-a}\Big).$$ ✍

Example 7.5 Consider Goldbach's conjecture, from Section 3.1:

> Every even integer greater than 2 is the sum of two primes.

This can be translated in the following ways, where P is the set of prime numbers and $S = \{4,6,8,10,\ldots\}$ is the set of even integers greater than 2.

$$\big(n \in S\big) \Rightarrow \big(\exists\, p,q \in P, n = p+q\big)$$

$$\forall\, n \in S, \exists\, p,q \in P, n = p+q$$ ✍

These translations of Goldbach's conjecture illustrate an important point. The first has the basic structure $(n \in S) \Rightarrow Q(n)$ and the second has structure $\forall\, n \in S, Q(n)$, yet they have exactly the same meaning. This is significant. Every universally quantified statement can be expressed as a conditional statement.

> **Fact 7.1** Suppose S is a set and $Q(x)$ is a statement about x for any $x \in S$. The following statements mean the same thing:
>
> $\forall\, x \in S, Q(x)$
>
> $(x \in S) \Rightarrow Q(x)$.

This fact is significant because so many theorems have the form of a conditional statement. (The Mean Value Theorem is an example!) In proving a theorem we have to think carefully about what it says. Sometimes a theorem will be expressed as a universally quantified statement but it will be more convenient to think of it as a conditional statement. Understanding the above fact allows us to switch between the two forms.

We close this section with some final points. In translating a statement, be attentive to its intended meaning. Don't jump into, for example, automatically replacing every "and" with $\wedge$ and "or" with $\vee$. An example:

At least one of the integers x and y is even.

Don't be led astray by the presence of the word "and." The meaning of the statement is that one or both of the numbers is even, so it should be translated with "or," not "and":

(x is even) $\vee$ (y is even).

Finally, the logical meaning of "but" can be captured by "and." The sentence "*The integer x is even, but the integer y is odd*," is translated as

(x is even) $\wedge$ (y is odd).

Exercises for Section 7.3

Translate each of the following sentences into symbolic logic.

1. If f is a polynomial and its degree is greater than 2, then f' is not constant.

2. The number x is positive, but the number y is not positive.

3. If x is prime then $\sqrt{x}$ is not a rational number.

4. For every prime number p there is another prime number q with $q > p$.

5. For every positive number ε, there is a positive number δ for which $|x-a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.

6. For every positive number ε there is a positive number M for which $|f(x) - b| < \varepsilon$, whenever $x > M$.

7. There exists a real number a for which $a + x = x$ for every real number x.

8. I don't eat anything that has a face.

9. If x is a rational number and $x \neq 0$, then $\tan(x)$ is not a rational number.

10. If $\sin(x) < 0$, then it is not the case that $0 \leq x \leq \pi$.

11. There is a Providence that protects idiots, drunkards, children and the United States of America. (Otto von Bismarck)

12. You can fool some of the people all of the time, and you can fool all of the people some of the time, but you can't fool all of the people all of the time. (Abraham Lincoln)

13. Everything is funny as long as it is happening to somebody else. (Will Rogers)

7.4 Negating Statements

Given a statement R, the statement $\sim R$ is called the **negation** of R. If R is a complex statement, then it is often the case that its negation $\sim R$ can be written in a simpler or more useful form. The process of finding this form is called **negating** R. In proving theorems it is often necessary to negate certain statements. We now investigate how to do this.

We have already examined part of this topic. **DeMorgan's laws**

$$\sim(P \wedge Q) \quad = \quad (\sim P) \vee (\sim Q) \tag{7.1}$$

$$\sim(P \vee Q) \quad = \quad (\sim P) \wedge (\sim Q) \tag{7.2}$$

(from Section 3.6) can be viewed as rules that tell us how to negate the statements $P \wedge Q$ and $P \vee Q$. Here are some examples that illustrate how DeMorgan's laws are used to negate statements involving "and" or "or."

Example 7.6 Consider negating the following statement.

R : You can solve it by factoring or with the quadratic formula.

Now, R means (You can solve it by factoring) $\vee$ (You can solve it with Q.F.), which we will denote as $P \vee Q$. The negation of this is

$$\sim(P \vee Q) \quad = \quad (\sim P) \wedge (\sim Q).$$

Therefore, in words, the negation of R is

$\sim R$: You can't solve it by factoring and you can't solve it with the quadratic formula.

Maybe you can find $\sim R$ without invoking DeMorgan's laws. That is good; you have internalized the laws and are using them unconsciously. ✍

Example 7.7 We will negate the following sentence.

R : The numbers x and y are both odd.

This statement means (x is odd) $\wedge$ (y is odd), so its negation is

$$\begin{aligned} \sim\big((x \text{ is odd}) \wedge (y \text{ is odd})\big) \quad &= \quad \sim(x \text{ is odd}) \vee \sim(y \text{ is odd}) \\ &= \quad (x \text{ is even}) \vee (y \text{ is even}). \end{aligned}$$

Therefore the negation of R can be expressed in the following ways:

$\sim R$: The number x is even or the number y is even.
$\sim R$: At least one of x and y is even. ✍

Now let's move on to a slightly different kind of problem. It's often necessary to find the negations of quantified statements. For example, consider $\sim(\forall x \in \mathbb{N}, P(x))$. Reading this in words, we have the following:

It is not the case that $P(x)$ is true for all natural numbers x.

This means $P(x)$ is false for at least one x. In symbols, this is $\exists x \in \mathbb{N}, \sim P(x)$. Thus $\sim(\forall x \in \mathbb{N}, P(x)) = \exists x \in \mathbb{N}, \sim P(x)$. Similarly, you can reason out that $\sim(\exists x \in \mathbb{N}, P(x)) = \forall x \in \mathbb{N}, \sim P(x)$. In general:

$$\sim(\forall x \in S, P(x)) \;=\; \exists x \in S, \sim P(x), \tag{7.3}$$
$$\sim(\exists x \in S, P(x)) \;=\; \forall x \in S, \sim P(x). \tag{7.4}$$

Example 7.8 Consider negating the following statement.

R : The square of every real number is non-negative.

Symbolically, R can be expressed as $\forall x \in \mathbb{R}, x^2 \geq 0$, and thus its negation is $\sim(\forall x \in \mathbb{R}, x^2 \geq 0) = \exists x \in \mathbb{R}, \sim(x^2 \geq 0) = \exists x \in \mathbb{R}, x^2 < 0$. In words, this is

$\sim R$: There exists a real number whose square is negative.

Observe that R is true and $\sim R$ is false. You may be able to get $\sim R$ immediately, without using Equation (7.3) as we did above. If so, that is good; if not, you will probably be there soon. ✍

If a statement has multiple quantifiers, negating it will involve several iterations of Equations (7.3) and (7.4). Consider the following:

S : For every real number x there is a real number y for which $y^3 = x$.

This statement asserts any real number x has a cube root y, so it's true. Symbolically S can be expressed as

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^3 = x.$$

Let's work out the negation of this statement.

$$\begin{aligned} \sim(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^3 = x) &= \exists x \in \mathbb{R}, \sim(\exists y \in \mathbb{R}, y^3 = x) \\ &= \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \sim(y^3 = x) \\ &= \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y^3 \neq x. \end{aligned}$$

Therefore the negation is the following (false) statement.

$\sim S$: There is a real number x for which $y^3 \neq x$ for all real numbers y.

In writing proofs you will sometimes have to negate a conditional statement $P \Rightarrow Q$. The remainder of this section describes how to do this. To begin, look at the expression $\sim(P \Rightarrow Q)$, which literally says "*$P \Rightarrow Q$ is false.*" You know from the truth table for $\Rightarrow$ that the only way that $P \Rightarrow Q$ can be false is if P is true and Q is false. Therefore $\sim(P \Rightarrow Q) = P \wedge \sim Q$.

$$\sim(P \Rightarrow Q) \;=\; P \wedge \sim Q \tag{7.5}$$

(In fact, in Exercise 12 of Section 3.6, you used a truth table to verify that these two statements are indeed logically equivalent.)

Example 7.9 Negate the following statement about a particular (i.e., constant) number a.

R : If a is odd then a^2 is odd.

Using Equation (7.5), we get the following negation.

$\sim R$: a is odd and a^2 is not odd.

Notice that R is true. Also $\sim R$ is false, no matter the value of a. ✍

Example 7.10 This example is like the previous one, but the constant a is replaced by a variable x. We will negate the following statement.

R : If x is odd then x^2 is odd.

As discussed in Section 7.2, we interpret this as the universally quantified statement

$R : \forall x \in \mathbb{Z}, (x \text{ odd}) \Rightarrow (x^2 \text{ odd})$.

By Equations (7.3) and (7.5), we get the following negation for R.

$$\begin{aligned} \sim\big(\forall x \in \mathbb{Z}, (x \text{ odd}) \Rightarrow (x^2 \text{ odd})\big) \;&=\; \exists x \in \mathbb{Z}, \sim\big((x \text{ odd}) \Rightarrow (x^2 \text{ odd})\big) \\ &=\; \exists x \in \mathbb{Z}, (x \text{ odd}) \wedge \sim (x^2 \text{ odd}). \end{aligned}$$

Translating back into words, we have

$\sim R$: There is an odd integer x whose square is not odd.

Notice that R is true and $\sim R$ is false. ✍

The above Example 7.10 showed how to negate a conditional statement $P(x) \Rightarrow Q(x)$. This type of problem can sometimes be embedded in more complex negation. See Exercise 5 below (and its solution).

Exercises for Section 7.4

Negate the following sentences.

1. The number x is positive, but the number y is not positive.
2. If x is prime, then $\sqrt{x}$ is not a rational number.
3. For every prime number p, there is another prime number q with $q > p$.
4. For every positive number ε, there is a positive number δ such that $|x-a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.
5. For every positive number ε, there is a positive number M for which $|f(x) - b| < \varepsilon$ whenever $x > M$.
6. There exists a real number a for which $a + x = x$ for every real number x.
7. I don't eat anything that has a face.
8. If x is a rational number and $x \neq 0$, then $\tan(x)$ is not a rational number.
9. If $\sin(x) < 0$, then it is not the case that $0 \leq x \leq \pi$.
10. If f is a polynomial and its degree is greater than 2, then f' is not constant.
11. You can fool all of the people all of the time.
12. Whenever I have to choose between two evils, I choose the one I haven't tried yet. (Mae West)

7.5 Logical Inference

There are four very significant reasons that we study logic. First, truth tables tell us the exact meanings of the words such as "and," "or," "not" and so on. So, whenever we encounter the *"If..., then"* construction in a mathematical context, logic tells us exactly what is meant. Second, logical rules such as DeMorgan's laws help us correctly change certain statements into (potentially more useful) statements with the same meaning. Third, logic is an essential ingredient in the design and flow of algorithms.

This section covers the fourth reason that logic is important It provides a means of combining facts and information to produce new facts.

To begin, suppose we know that a statement of form $P \Rightarrow Q$ is true. This tells us that whenever P is true, Q will also be true. By itself, $P \Rightarrow Q$ being true does not tell us that either P or Q is true (they could both be false, or P could be false and Q true). However if in addition we happen to know that P is true then it must be that Q is true. This is called a **logical inference**: Given two true statements we can infer that a third statement is true. In this instance true statements $P \Rightarrow Q$ and P are "added together" to get Q. This is described below with $P \Rightarrow Q$ and P stacked one atop the other with a line separating them from Q. The intended meaning is that $P \Rightarrow Q$ combined with P produces Q.

$$\begin{array}{l} P \Rightarrow Q \\ P \\ \hline Q \end{array}$$

This is a very frequently-used pattern of thought. (In fact, it is exactly the pattern we used in the example on page 52.) This rule even has a name. It is called the **modus ponens** rule.

Two other logical inferences, called **modus tollens** and **elimination** are listed below. In each case you should convince yourself (based on your knowledge of the relevant truth tables) that the truth of the statements above the line forces the statement below the line to be true.

Modus Ponens	Modus Tollens	Elimination
$\begin{array}{l} P \Rightarrow Q \\ P \\ \hline Q \end{array}$	$\begin{array}{l} P \Rightarrow Q \\ \sim Q \\ \hline \sim P \end{array}$	$\begin{array}{l} P \vee Q \\ \sim P \\ \hline Q \end{array}$

It is important to internalize these rules. (You surely already use at least modus ponens and elimination in daily life anyway.) But don't bother remembering their names; very few working mathematicians and computer scientists can recall the names of these rules, though they use the rules constantly. The names are not important, but the rules are.

Three additional logical inferences are listed below. The first states the obvious fact that if P and Q are both true, then so is the statement $P \wedge Q$. On the other hand, $P \wedge Q$ being true forces P (also Q) to be true. Finally, if P is true, then $P \vee Q$ must be true, no matter what statement Q is.

$$\begin{array}{l} P \\ Q \\ \hline P \wedge Q \end{array} \qquad \begin{array}{l} P \wedge Q \\ \hline P \end{array} \qquad \begin{array}{l} P \\ \hline P \vee Q \end{array}$$

These inferences are so intuitively obvious that they scarcely need to be mentioned. However, they represent certain patterns of reasoning that we will frequently apply to sentences in proofs, so we should be cognizant of the fact that we are using them.