CHAPTER 12

Functions

You know from calculus that functions play a fundamental role in mathematics. You likely view a function as a kind of formula that describes a relationship between two (or more) quantities. You certainly understand and appreciate the fact that relationships between quantities are important in all scientific disciplines, so you do not need to be convinced that functions are important. Still, you may not be aware of the full significance of functions. Functions are more than merely descriptions of numeric relationships. In a more general sense, functions can compare and relate different kinds of mathematical structures. You will see this as your understanding of mathematics deepens. In preparation of this deepening, we will now explore a more general and versatile view of functions.

The concept of a relation between sets (Definition 11.7) plays a big role here, so you may want to quickly review it.

12.1 Functions

Let's start on familiar ground. Consider the function $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$. Its graph is the set of points $R = \{(x, x^2) : x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$.

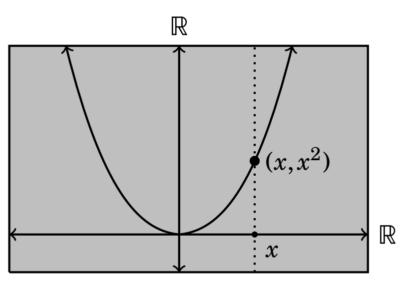


Figure 12.1. A familiar function

Having read Chapter 11, you may see f in a new light. Its graph $R \subseteq \mathbb{R} \times \mathbb{R}$ is a relation on the set $\mathbb{R}$. In fact, as we shall see, functions are just special kinds of relations. Before stating the exact definition, we look at another example. Consider the function $f(n) = |n| + 2$ that converts integers n into natural numbers $|n|+2$. Its graph is $R = \{(n, |n|+2) : n \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{N}$.

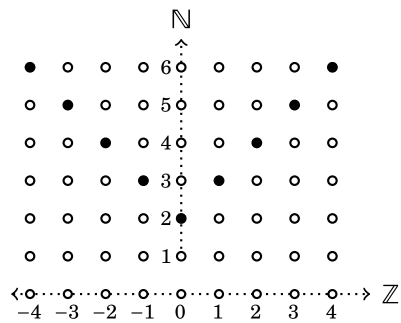


Figure 12.2. The function $f : \mathbb{Z} \to \mathbb{N}$, where $f(n) = |n| + 2$

Figure 12.2 shows the graph R as darkened dots in the grid of points $\mathbb{Z} \times \mathbb{N}$. Notice that in this example R is not a relation on a single set. The set of input values $\mathbb{Z}$ is different from the set $\mathbb{N}$ of output values, so the graph $R \subseteq \mathbb{Z} \times \mathbb{N}$ is *a relation from* $\mathbb{Z}$ *to* $\mathbb{N}$.

This example illustrates three things. First, a function can be viewed as sending elements from one set A to another set B. (In the case of f, $A = \mathbb{Z}$ and $B = \mathbb{N}$.) Second, such a function can be regarded as a relation from A to B. Third, for every input value n, there is *exactly one* output value $f(n)$. In your high school algebra course, this was expressed by the *vertical line test*: Any vertical line intersects a function's graph at most once. It means that for any input value x, the graph contains exactly one point of form $(x, f(x))$. Our main definition, given below, incorporates all of these ideas.

Definition 12.1 Suppose A and B are sets. A **function** f from A to B (denoted as $f : A \to B$) is a relation $f \subseteq A \times B$ from A to B, satisfying the property that for each $a \in A$ the relation f contains exactly one ordered pair of form (a, b). The statement $(a, b) \in f$ is abbreviated $f(a) = b$.

Example 12.1 Consider the function f graphed in Figure 12.2. According to Definition 12.1, we regard f as the set of points in its graph, that is, $f = \{(n, |n|+2) : n \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{N}$. This is a relation from $\mathbb{Z}$ to $\mathbb{N}$, and indeed given any $a \in \mathbb{Z}$ the set f contains exactly one ordered pair $(a, |a|+2)$ whose first coordinate is a. Since $(1,3) \in f$, we write $f(1) = 3$; and since $(-3,5) \in f$ we write $f(-3) = 5$, etc. In general, $(a, b) \in f$ means that f sends the input value a to the output value b, and we express this as $f(a) = b$. This function can be expressed by a formula: For each input value n, the output value is $|n|+2$, so we may write $f(n) = |n|+2$. All this agrees with the way we thought of functions in algebra and calculus; the only difference is that now we also think of a function as a relation.

Definition 12.2 For a function $f : A \to B$, the set A is called the **domain** of f. (Think of the domain as the set of possible "input values" for f.) The set B is called the **codomain** of f. The **range** of f is the set $\{f(a) : a \in A\} = \{b : (a,b) \in f\}$. (Think of the range as the set of all possible "output values" for f. Think of the codomain as a sort of "target" for the outputs.)

Continuing Example 12.1, the domain of f is $\mathbb{Z}$ and its codomain is $\mathbb{N}$. Its range is $\{f(a) : a \in \mathbb{Z}\} = \{|a|+2 : a \in \mathbb{Z}\} = \{2,3,4,5,\ldots\}$. Notice that the range is a subset of the codomain, but it does not (in this case) equal the codomain.

In our examples so far, the domains and codomains are sets of numbers, but this needn't be the case in general, as the next example indicates.

Example 12.2 Let $A = \{p,q,r,s\}$ and $B = \{0,1,2\}$, and

$$f = \{(p,0),(q,1),(r,2),(s,2)\} \subseteq A \times B.$$

This is a function $f : A \to B$ because each element of A occurs exactly once as a first coordinate of an ordered pair in f. We have $f(p) = 0$, $f(q) = 1$, $f(r) = 2$ and $f(s) = 2$. The domain of f is $\{p,q,r,s\}$, and the codomain and range are both $\{0,1,2\}$.

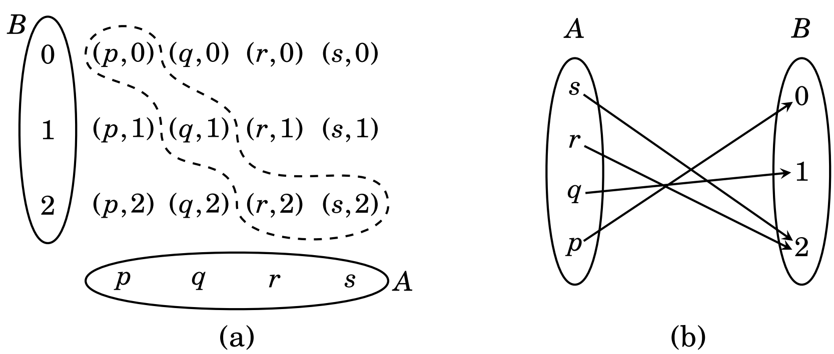


Figure 12.3. Two ways of drawing the function $f = \{(p,0),(q,1),(r,2),(s,2)\}$

If A and B are not both sets of numbers it can be difficult to draw a graph of $f : A \to B$ in the traditional sense. Figure 12.3(a) shows an attempt at a graph of f from Example 12.2. The sets A and B are aligned roughly as x- and y-axes, and the Cartesian product $A \times B$ is filled in accordingly. The subset $f \subseteq A \times B$ is indicated with dashed lines, and this can be regarded as a "graph" of f. A more natural visual description of f is shown in 12.3(b). The sets A and B are drawn side-by-side, and arrows point from a to b whenever $f(a) = b$.

In general, if $f : A \to B$ is the kind of function you may have encountered in algebra or calculus, then conventional graphing techniques offer the best visual description of it. On the other hand, if A and B are finite or if we are thinking of them as generic sets, then describing f with arrows is often a more appropriate way of visualizing it.

We emphasize that, according to Definition 12.1, a function is really just a special kind of set. Any function $f : A \to B$ is a subset of $A \times B$. By contrast, your calculus text probably defined a function as a certain kind of "rule." While that intuitive outlook is adequate for the first few semesters of calculus, it does not hold up well to the rigorous mathematical standards necessary for further progress. The problem is that words like "rule" are too vague. Defining a function as a set removes the ambiguity. It makes a function into a concrete mathematical object.

Still, in practice we tend to think of functions as rules. Given $f : \mathbb{Z} \to \mathbb{N}$ where $f(x) = |x| + 2$, we think of this as a rule that associates any number $n \in \mathbb{Z}$ to the number $|n| + 2$ in $\mathbb{N}$, rather than a set containing ordered pairs $(n, |n| + 2)$. It is only when we have to understand or interpret the theoretical nature of functions (as we do in this text) that Definition 12.1 comes to bear. The definition is a foundation that gives us license to think about functions in a more informal way.

The next example brings up a point about notation. Consider a function such as $f : \mathbb{Z}^2 \to \mathbb{Z}$, whose domain is a Cartesian product. This function takes as input an ordered pair $(m, n) \in \mathbb{Z}^2$ and sends it to a number $f((m, n)) \in \mathbb{Z}$. To simplify the notation, it is common to write $f(m, n)$ instead of $f((m, n))$, even though this is like writing fx instead of $f(x)$. We also remark that although we've been using the letters f, g and h to denote functions, any other reasonable symbol could be used. Greek letters such as φ and θ are common.

Example 12.3 Say a function $\varphi : \mathbb{Z}^2 \to \mathbb{Z}$ is defined as $\varphi(m, n) = 6m - 9n$. Note that as a set, this function is $\varphi = \big\{\big((m, n), 6m - 9n\big) : (m, n) \in \mathbb{Z}^2\big\} \subseteq \mathbb{Z}^2 \times \mathbb{Z}$. What is the range of φ?

To answer this, first observe that for any $(m,n) \in \mathbb{Z}^2$, the value $f(m,n) = 6m-9n = 3(2m-3n)$ is a multiple of 3. Thus every number in the range is a multiple of 3, so the range is a *subset* of the set of all multiples of 3. On the other hand if $b = 3k$ is a multiple of 3 we have $\varphi(-k,-k) = 6(-k)-9(-k) = 3k = b$, which means any multiple of 3 is in the range of φ. Therefore the range of φ is the set $\{3k : k \in \mathbb{Z}\}$ of all multiples of 3.

To conclude this section, let's use Definition 12.1 to help us understand what it means for two functions $f : A \to B$ and $g : C \to D$ to be equal. According to our definition, functions f and g are subsets $f \subseteq A \times B$ and $g \subseteq C \times D$. It makes sense to say that f and g are equal if $f = g$, that is, if they are equal as sets.

Thus the two functions $f = \{(1,a),(2,a),(3,b)\}$ and $g = \{(3,b),(2,a),(1,a)\}$ are equal because the sets f and g are equal. Notice that the domain of both functions is $A = \{1,2,3\}$, the set of first elements x in the ordered pairs $(x,y) \in f = g$. In general, equal functions must have equal domains.

Observe also that the equality $f = g$ means $f(x) = g(x)$ for every $x \in A$. We repackage these ideas in the following definition.

Definition 12.3 Two functions $f : A \to B$ and $g : A \to D$ are **equal** if $f(x) = g(x)$ for every $x \in A$.

Observe that f and g can have different codomains and still be equal. Consider the functions $f : \mathbb{Z} \to \mathbb{N}$ and $g : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = |x| + 2$ and $g(x) = |x| + 2$. Even though their codomains are different, the functions are equal because $f(x) = g(x)$ for every x in the domain.

Exercises for Section 12.1

1. Suppose $A = \{0,1,2,3,4\}$, $B = \{2,3,4,5\}$ and $f = \{(0,3),(1,3),(2,4),(3,2),(4,2)\}$. State the domain and range of f. Find $f(2)$ and $f(1)$.
2. Suppose $A = \{a,b,c,d\}$, $B = \{2,3,4,5,6\}$ and $f = \{(a,2),(b,3),(c,4),(d,5)\}$. State the domain and range of f. Find $f(b)$ and $f(d)$.
3. There are four different functions $f : \{a,b\} \to \{0,1\}$. List them all. Diagrams will suffice.
4. There are eight different functions $f : \{a,b,c\} \to \{0,1\}$. List them all. Diagrams will suffice.
5. Give an example of a relation from $\{a,b,c,d\}$ to $\{d,e\}$ that is not a function.
6. Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is defined as $f = \{(x,4x+5) : x \in \mathbb{Z}\}$. State the domain, codomain and range of f. Find $f(10)$.

7. Consider the set $f = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : 3x + y = 4\}$. Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$? Explain.

8. Consider the set $f = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x + 3y = 4\}$. Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$? Explain.

9. Consider the set $f = \{(x^2, x) : x \in \mathbb{R}\}$. Is this a function from $\mathbb{R}$ to $\mathbb{R}$? Explain.

10. Consider the set $f = \{(x^3, x) : x \in \mathbb{R}\}$. Is this a function from $\mathbb{R}$ to $\mathbb{R}$? Explain.

11. Is the set $\theta = \{(X, |X|) : X \subseteq \mathbb{Z}_5\}$ a function? If so, what is its domain and range?

12. Is the set $\theta = \{\big((x,y),(3y,2x,x+y)\big) : x,y \in \mathbb{R}\}$ a function? If so, what is its domain, codomain and range?

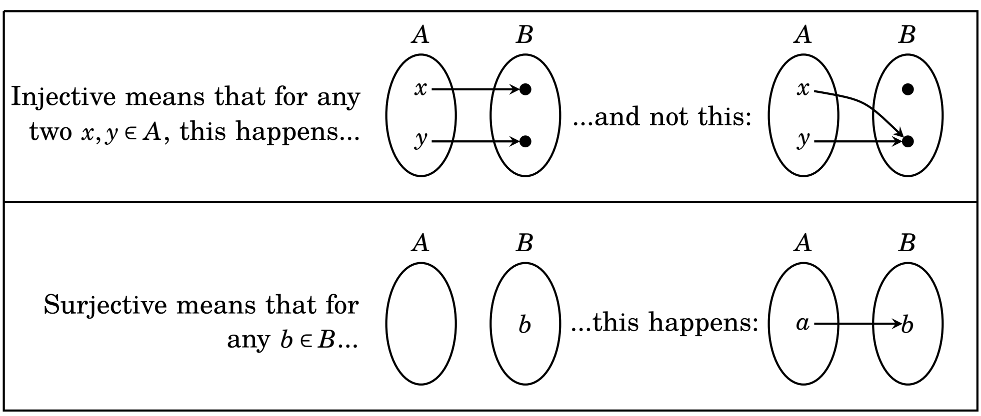
12.2 Injective and Surjective Functions

You may recall from algebra and calculus that a function may be *one-to-one* and *onto*, and these properties are related to whether or not the function is invertible. We now review these important ideas. In advanced mathematics, the word *injective* is often used instead of *one-to-one*, and *surjective* is used instead of *onto*. Here are the exact definitions:

Definition 12.4 A function $f : A \to B$ is:

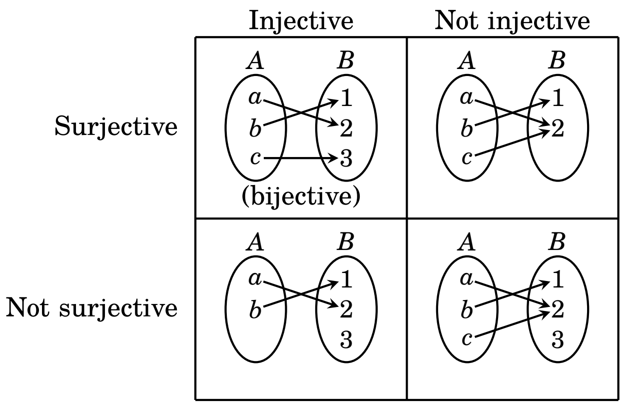
1. **injective** (or one-to-one) if for every $x, y \in A$, $x \neq y$ implies $f(x) \neq f(y)$;
2. **surjective** (or onto) if for every $b \in B$ there is an $a \in A$ with $f(a) = b$;
3. **bijective** if f is both injective and surjective.

Below is a visual description of Definition 12.4. In essence, injective means that unequal elements in A always get sent to unequal elements in B. Surjective means that every element of B has an arrow pointing to it, that is, it equals $f(a)$ for some a in the domain of f.



For more concrete examples, consider the following functions from $\mathbb{R}$ to $\mathbb{R}$. The function $f(x) = x^2$ is not injective because $-2 \neq 2$, but $f(-2) = f(2)$. Nor is it surjective, for if $b = -1$ (or if b is any negative number), then there is no $a \in \mathbb{R}$ with $f(a) = b$. On the other hand, $g(x) = x^3$ is both injective and surjective, so it is also bijective.

There are four possible injective/surjective combinations that a function may possess. This is illustrated in the following figure showing four functions from A to B. Functions in the first column are injective, those in the second column are not injective. Functions in the first row are surjective, those in the second row are not.



We note in passing that, according to the definitions, a function is surjective if and only if its codomain equals its range.

Often it is necessary to prove that a particular function $f : A \to B$ is injective. For this we must prove that for any two elements $x, y \in A$, the conditional statement $(x \neq y) \Rightarrow \big(f(x) \neq f(y)\big)$ is true. The two main approaches for this are summarized below.

How to show a function $f : A \to B$ is injective:

Direct approach:	**Contrapositive approach:**
Suppose $x, y \in A$ and $x \neq y$.	Suppose $x, y \in A$ and $f(x) = f(y)$.
$\vdots$	$\vdots$
Therefore $f(x) \neq f(y)$.	Therefore $x = y$.

Of these two approaches, the contrapositive is often the easiest to use, especially if f is defined by an algebraic formula. This is because the contrapositive approach starts with the *equation* $f(x) = f(y)$ and proceeds to the *equation* $x = y$. In algebra, as you know, it is usually easier to work with equations than inequalities.

To prove that a function is *not* injective, you must *disprove* the statement $(x \neq y) \Rightarrow \big(f(x) \neq f(y)\big)$. For this it suffices to find example of two elements $x, y \in A$ for which $x \neq y$ and $f(x) = f(y)$.

Next we examine how to prove that $f : A \to B$ is *surjective*. According to Definition 12.4, we must prove the statement $\forall\, b \in B, \exists\, a \in A, f(a) = b$. In words, we must show that for any $b \in B$, there is at least one $a \in A$ (which may depend on b) having the property that $f(a) = b$. Here is an outline.

How to show a function $f : A \to B$ is surjective:

Suppose $b \in B$. [Prove there exists $a \in A$ for which $f(a) = b$.]

In the second step, we have to prove the existence of an a for which $f(a) = b$. For this, just finding an example of such an a would suffice. (How to find such an example depends on how f is defined. If f is given as a formula, we may be able to find a by solving the equation $f(a) = b$ for a. Sometimes you can find a by just plain common sense.) To show f is *not* surjective, we must prove the negation of $\forall\, b \in B, \exists\, a \in A, f(a) = b$, that is, we must prove $\exists\, b \in B, \forall\, a \in A, f(a) \neq b$.

The following examples illustrate these ideas. (For the first example, note that the set $\mathbb{R} - \{0\}$ is $\mathbb{R}$ with the number 0 removed.)

Example 12.4 Show that the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined as $f(x) = \frac{1}{x} + 1$ is injective but not surjective.

We will use the contrapositive approach to show that f is injective. Suppose $x, y \in \mathbb{R} - \{0\}$ and $f(x) = f(y)$. This means $\frac{1}{x} + 1 = \frac{1}{y} + 1$. Subtracting 1 from both sides and inverting produces $x = y$. Therefore f is injective.

Function f is not surjective because there exists an element $b = 1 \in \mathbb{R}$ for which $f(x) = \frac{1}{x} + 1 \neq 1$ for every $x \in \mathbb{R} - \{0\}$.

Example 12.5 Show that the function $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by the formula $g(m,n) = (m+n, m+2n)$, is both injective and surjective.

We will use the contrapositive approach to show that g is injective. Thus we need to show that $g(m,n) = g(k,\ell)$ implies $(m,n) = (k,\ell)$. Suppose $(m,n), (k,\ell) \in \mathbb{Z} \times \mathbb{Z}$ and $g(m,n) = g(k,\ell)$. Then $(m+n, m+2n) = (k+\ell, k+2\ell)$. It follows that $m+n = k+\ell$ and $m+2n = k+2\ell$. Subtracting the first equation from the second gives $n = \ell$. Next, subtract $n = \ell$ from $m+n = k+\ell$ to get $m = k$. Since $m = k$ and $n = \ell$, it follows that $(m,n) = (k,\ell)$. Therefore g is injective.

To see that g is surjective, consider an arbitrary element $(b,c) \in \mathbb{Z}\times\mathbb{Z}$. We need to show that there is some $(x,y) \in \mathbb{Z}\times\mathbb{Z}$ for which $g(x,y) = (b,c)$. To find (x,y), note that $g(x,y) = (b,c)$ means $(x+y, x+2y) = (b,c)$. This leads to the following system of equations:

$$\begin{array}{ccccc} x & + & y & = & b \\ x & + & 2y & = & c. \end{array}$$

Solving gives $x = 2b - c$ and $y = c - b$. Then $(x,y) = (2b-c, c-b)$. We now have $g(2b-c, c-b) = (b,c)$, and it follows that g is surjective.

Example 12.6 Consider function $h : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Q}$ defined as $h(m,n) = \dfrac{m}{|n|+1}$. Determine whether this is injective and whether it is surjective.

This function is *not* injective because of the unequal elements $(1,2)$ and $(1,-2)$ in $\mathbb{Z}\times\mathbb{Z}$ for which $h(1,2) = h(1,-2) = \frac{1}{3}$. However, h is surjective: Take any element $b \in \mathbb{Q}$. Then $b = \frac{c}{d}$ for some $c,d \in \mathbb{Z}$. Notice we may assume d is positive by making c negative, if necessary. Then $h(c,d-1) = \frac{c}{|d-1|+1} = \frac{c}{d} = b$.

Exercises for Section 12.2

1. Let $A = \{1,2,3,4\}$ and $B = \{a,b,c\}$. Give an example of a function $f : A \to B$ that is neither injective nor surjective.
2. Consider the logarithm function $\ln : (0,\infty) \to \mathbb{R}$. Decide whether this function is injective and whether it is surjective.
3. Consider the cosine function $\cos : \mathbb{R} \to \mathbb{R}$. Decide whether this function is injective and whether it is surjective. What if it had been defined as $\cos : \mathbb{R} \to [-1,1]$?
4. A function $f : \mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ is defined as $f(n) = (2n, n+3)$. Verify whether this function is injective and whether it is surjective.
5. A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined as $f(n) = 2n+1$. Verify whether this function is injective and whether it is surjective.
6. A function $f : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ is defined as $f(m,n) = 3n - 4m$. Verify whether this function is injective and whether it is surjective.
7. A function $f : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ is defined as $f(m,n) = 2n - 4m$. Verify whether this function is injective and whether it is surjective.
8. A function $f : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ is defined as $f(m,n) = (m+n, 2m+n)$. Verify whether this function is injective and whether it is surjective.
9. Prove that the function $f : \mathbb{R}-\{2\} \to \mathbb{R}-\{5\}$ defined by $f(x) = \dfrac{5x+1}{x-2}$ is bijective.
10. Prove the function $f : \mathbb{R}-\{1\} \to \mathbb{R}-\{1\}$ defined by $f(x) = \left(\dfrac{x+1}{x-1}\right)^3$ is bijective.
11. Consider the function $\theta : \{0,1\}\times\mathbb{N} \to \mathbb{Z}$ defined as $\theta(a,b) = (-1)^a b$. Is θ injective? Is it surjective? Bijective? Explain.

12. Consider the function $\theta:\{0,1\}\times\mathbb{N}\to\mathbb{Z}$ defined as $\theta(a,b)=a-2ab+b$. Is θ injective? Is it surjective? Bijective? Explain.

13. Consider the function $f:\mathbb{R}^2\to\mathbb{R}^2$ defined by the formula $f(x,y)=(xy,x^3)$. Is f injective? Is it surjective? Bijective? Explain.

14. Consider the function $\theta:\mathscr{P}(\mathbb{Z})\to\mathscr{P}(\mathbb{Z})$ defined as $\theta(X)=\overline{X}$. Is θ injective? Is it surjective? Bijective? Explain.

15. This question concerns functions $f:\{A,B,C,D,E,F,G\}\to\{1,2,3,4,5,6,7\}$. How many such functions are there? How many of these functions are injective? How many are surjective? How many are bijective?

16. This question concerns functions $f:\{A,B,C,D,E\}\to\{1,2,3,4,5,6,7\}$. How many such functions are there? How many of these functions are injective? How many are surjective? How many are bijective?

17. This question concerns functions $f:\{A,B,C,D,E,F,G\}\to\{1,2\}$. How many such functions are there? How many of these functions are injective? How many are surjective? How many are bijective?

18. Prove that the function $f:\mathbb{N}\to\mathbb{Z}$ defined as $f(n)=\frac{(-1)^n(2n-1)+1}{4}$ is bijective.

12.3 The Pigeonhole Principle

Here is a simple but useful idea. Imagine there is a set A of pigeons and a set B of pigeon-holes, and all the pigeons fly into the pigeon-holes. You can think of this as describing a function $f:A\to B$, where pigeon X flies into pigeon-hole $f(X)$. Figure 12.4 illustrates this.

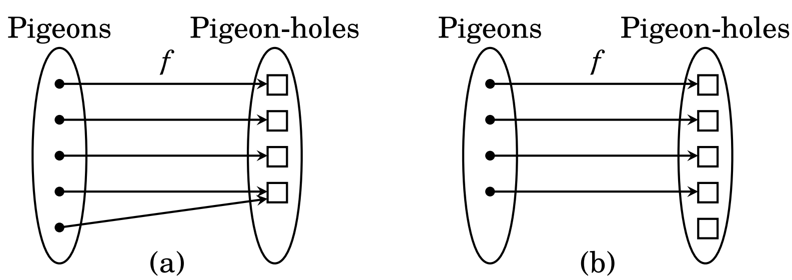


Figure 12.4. The pigeonhole principle

In Figure 12.4(a) there are more pigeons than pigeon-holes, and it is obvious that in such a case at least two pigeons have to fly into the same pigeon-hole, meaning that f is not injective. In Figure 12.4(b) there are fewer pigeons than pigeon-holes, so clearly at least one pigeon-hole remains empty, meaning that f is not surjective.

Although the underlying idea expressed by these figures has little to do with pigeons, it is nonetheless called the *pigeonhole principle:*

The Pigeonhole Principle
Suppose A and B are finite sets and $f : A \to B$ is any function. Then:

1. If $|A| > |B|$, then f is not injective.
2. If $|A| < |B|$, then f is not surjective.

Though the pigeonhole principle is obvious, it can be used to prove some things that are not so obvious.

Example 12.7 Prove the following proposition.

Proposition If A is any set of 10 integers between 1 and 100, then there exist two different subsets $X \subseteq A$ and $Y \subseteq A$ for which the sum of elements in X equals the sum of elements in Y.

To illustrate what this proposition is saying, consider the random set

$$A = \{5,7,12,11,17,50,51,80,90,100\}$$

of 10 integers between 1 and 100. Notice that A has subsets $X = \{5,80\}$ and $Y = \{7,11,17,50\}$ for which the sum of the elements in X equals the sum of those in Y. If we tried to "mess up" A by changing the 5 to a 6, we get

$$A = \{6,7,12,11,17,50,51,80,90,100\}$$

which has subsets $X = \{7,12,17,50\}$ and $Y = \{6,80\}$ both of whose elements add up to the same number (86). The proposition asserts that this is always possible, no matter what A is. Here is a proof:

Proof. Suppose $A \subseteq \{1,2,3,4,\ldots,99,100\}$ and $|A| = 10$, as stated. Notice that if $X \subseteq A$, then X has no more than 10 elements, each between 1 and 100, and therefore the sum of all the elements of X is less than $100 \cdot 10 = 1000$. Consider the function

$$f : \mathscr{P}(A) \to \{0,1,2,3,4,\ldots,1000\}$$

where $f(X)$ is the sum of the elements in X. (Examples: $f(\{3,7,50\}) = 60$; $f(\{1,70,80,95\}) = 246$.) As $|\mathscr{P}(A)| = 2^{10} = 1024 > 1001 = \left|\{0,1,2,3,\ldots,1000\}\right|$, it follows from the pigeonhole principle that f is not injective. Therefore there are two unequal sets $X,Y \in \mathscr{P}(A)$ for which $f(X) = f(Y)$. In other words, there are subsets $X \subseteq A$ and $Y \subseteq A$ for which the sum of elements in X equals the sum of elements in Y. ■

Example 12.8 Prove the following proposition.

Proposition There are at least two Texans with the same number of hairs on their heads.

Proof. We will use two facts. First, the population of Texas is more than twenty million. Second, it is a biological fact that every human head has fewer than one million hairs. Let A be the set of all Texans, and let $B = \{0,1,2,3,4,\ldots,1000000\}$. Let $f : A \to B$ be the function for which $f(x)$ equals the number of hairs on the head of x. Since $|A| > |B|$, the pigeonhole principle asserts that f is not injective. Thus there are two Texans x and y for whom $f(x) = f(y)$, meaning that they have the same number of hairs on their heads. ■

Proofs that use the pigeonhole principle tend to be inherently non-constructive, in the sense discussed in Section 7.4. For example, the above proof does not explicitly give us of two Texans with the same number of hairs on their heads; it only shows that two such people exist. If we were to make a constructive proof, we could find examples of two bald Texans. Then they have the same number of head hairs, namely zero.

Exercises for Section 12.3

1. Prove that if six numbers are chosen at random, then at least two of them will have the same remainder when divided by 5.
2. Prove that if a is a natural number, then there exist two unequal natural numbers k and ℓ for which $a^k - a^\ell$ is divisible by 10.
3. Prove that if six natural numbers are chosen at random, then the sum or difference of two of them is divisible by 9.
4. Consider a square whose side-length is one unit. Select any five points from inside this square. Prove that at least two of these points are within $\frac{\sqrt{2}}{2}$ units of each other.
5. Prove that any set of seven distinct natural numbers contains a pair of numbers whose sum or difference is divisible by 10.
6. Given a sphere S, a *great circle* of S is the intersection of S with a plane through its center. Every great circle divides S into two parts. A *hemisphere* is the union of the great circle and one of these two parts. Prove that if five points are placed arbitrarily on S, then there is a hemisphere that contains four of them.
7. Prove or disprove: Any subset $X \subseteq \{1,2,3,\ldots,2n\}$ with $|X| > n$ contains two (unequal) elements $a, b \in X$ for which $a \mid b$ or $b \mid a$.

12.4 Composition

You should be familiar with the notion of function composition from algebra and calculus. Still, it is worthwhile to revisit it now with our more sophisticated ideas about functions.

Definition 12.5 Suppose $f : A \to B$ and $g : B \to C$ are functions with the property that the codomain of f equals the domain of g. The **composition** of f with g is another function, denoted as $g \circ f$ and defined as follows: If $x \in A$, then $g \circ f(x) = g(f(x))$. Therefore $g \circ f$ sends elements of A to elements of C, so $g \circ f : A \to C$.

The following figure illustrates the definition. Here $f : A \to B$, $g : B \to C$, and $g \circ f : A \to C$. We have, for example, $g \circ f(0) = g(f(0)) = g(2) = 4$. Be very careful with the order of the symbols. Even though g comes first in the symbol $g \circ f$, we work out $g \circ f(x)$ as $g(f(x))$, with f acting on x first, followed by g acting on $f(x)$.

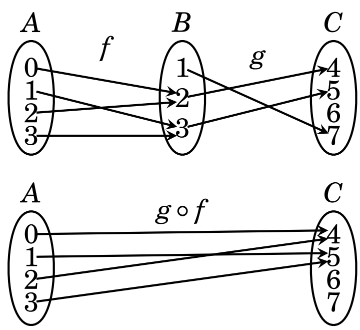


Figure 12.5. Composition of two functions

Notice that the composition $g \circ f$ also makes sense if the range of f is a *subset* of the domain of g. You should take note of this fact, but to keep matters simple we will continue to emphasize situations where the codomain of f equals the domain of g.

Example 12.9 Suppose $A = \{a,b,c\}$, $B = \{0,1\}$, $C = \{1,2,3\}$. Let $f : A \to B$ be the function $f = \{(a,0),(b,1),(c,0)\}$, and let $g : B \to C$ be the function $g = \{(0,3),(1,1)\}$. Then $g \circ f = \{(a,3),(b,1),(c,3)\}$.

Example 12.10 Suppose $A = \{a,b,c\}$, $B = \{0,1\}$, $C = \{1,2,3\}$. Let $f : A \to B$ be the function $f = \{(a,0),(b,1),(c,0)\}$, and let $g : C \to B$ be the function $g = \{(1,0),(2,1),(3,1)\}$. In this situation the composition $g \circ f$ is not defined because the codomain B of f is not the same set as the domain C of g.

Remember: In order for $g \circ f$ to make sense, the codomain of f must equal the domain of g. (Or at least be a subset of it.)

Example 12.11 Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2 + x$, and $g : \mathbb{R} \to \mathbb{R}$ be defined as $g(x) = x + 1$. Then $g \circ f : \mathbb{R} \to \mathbb{R}$ is the function defined by the formula $g \circ f(x) = g(f(x)) = g(x^2 + x) = x^2 + x + 1$.

Since the domains and codomains of g and f are the same, we can in this case do a composition in the other order. Note that $f \circ g : \mathbb{R} \to \mathbb{R}$ is the function defined as $f \circ g(x) = f(g(x)) = f(x+1) = (x+1)^2 + (x+1) = x^2 + 3x + 2$.

This example illustrates that even when $g \circ f$ and $f \circ g$ are both defined, they are not necessarily equal. We can express this fact by saying *function composition is not commutative.*

We close this section by proving several facts about function composition that you are likely to encounter in your future study of mathematics. First, we note that, although it is not commutative, function composition *is* associative.

Theorem 12.1 Composition of functions is associative. That is if $f : A \to B$, $g : B \to C$ and $h : C \to D$, then $(h \circ g) \circ f = h \circ (g \circ f)$.

Proof. Suppose f, g, h are as stated. It follows from Definition 12.5 that both $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are functions from A to D. To show that they are equal, we just need to show

$$\Big((h \circ g) \circ f\Big)(x) = \Big(h \circ (g \circ f)\Big)(x)$$

for every $x \in A$. Note that Definition 12.5 yields

$$\Big((h \circ g) \circ f\Big)(x) = (h \circ g)(f(x)) = h(g(f(x))).$$

Also

$$\Big(h \circ (g \circ f)\Big)(x) = h(g \circ f(x)) = h(g(f(x))).$$

Thus

$$\Big((h \circ g) \circ f\Big)(x) = \Big(h \circ (g \circ f)\Big)(x),$$

as both sides equal $h(g(f(x)))$. ■

Theorem 12.2 Suppose $f : A \to B$ and $g : B \to C$. If both f and g are injective, then $g \circ f$ is injective. If both f and g are surjective, then $g \circ f$ is surjective.

Proof. First suppose both f and g are injective. To see that $g \circ f$ is injective, we must show that $g \circ f(x) = g \circ f(y)$ implies $x = y$. Suppose $g \circ f(x) = g \circ f(y)$. This means $g(f(x)) = g(f(y))$. It follows that $f(x) = f(y)$. (For otherwise g wouldn't be injective.) But since $f(x) = f(y)$ and f is injective, it must be that $x = y$. Therefore $g \circ f$ is injective.

Next suppose both f and g are surjective. To see that $g \circ f$ is surjective, we must show that for any element $c \in C$, there is a corresponding element $a \in A$ for which $g \circ f(a) = c$. Thus consider an arbitrary $c \in C$. Because g is surjective, there is an element $b \in B$ for which $g(b) = c$. And because f is surjective, there is an element $a \in A$ for which $f(a) = b$. Therefore $g(f(a)) = g(b) = c$, which means $g \circ f(a) = c$. Thus $g \circ f$ is surjective. ■

Exercises for Section 12.4

1. Suppose $A = \{5,6,8\}$, $B = \{0,1\}$, $C = \{1,2,3\}$. Let $f : A \to B$ be the function $f = \{(5,1),(6,0),(8,1)\}$, and $g : B \to C$ be $g = \{(0,1),(1,1)\}$. Find $g \circ f$.
2. Suppose $A = \{1,2,3,4\}$, $B = \{0,1,2\}$, $C = \{1,2,3\}$. Let $f : A \to B$ be
$$f = \{(1,0),(2,1),(3,2),(4,0)\},$$
and $g : B \to C$ be $g = \{(0,1),(1,1),(2,3)\}$. Find $g \circ f$.
3. Suppose $A = \{1,2,3\}$. Let $f : A \to A$ be the function $f = \{(1,2),(2,2),(3,1)\}$, and let $g : A \to A$ be the function $g = \{(1,3),(2,1),(3,2)\}$. Find $g \circ f$ and $f \circ g$.
4. Suppose $A = \{a,b,c\}$. Let $f : A \to A$ be the function $f = \{(a,c),(b,c),(c,c)\}$, and let $g : A \to A$ be the function $g = \{(a,a),(b,b),(c,a)\}$. Find $g \circ f$ and $f \circ g$.
5. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sqrt[3]{x+1}$ and $g(x) = x^3$. Find the formulas for $g \circ f$ and $f \circ g$.
6. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \frac{1}{x^2+1}$ and $g(x) = 3x+2$. Find the formulas for $g \circ f$ and $f \circ g$.
7. Consider the functions $f, g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $f(m,n) = (mn, m^2)$ and $g(m,n) = (m+1, m+n)$. Find the formulas for $g \circ f$ and $f \circ g$.
8. Consider the functions $f, g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $f(m,n) = (3m-4n, 2m+n)$ and $g(m,n) = (5m+n, m)$. Find the formulas for $g \circ f$ and $f \circ g$.
9. Consider the functions $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined as $f(m,n) = m+n$ and $g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $g(m) = (m,m)$. Find the formulas for $g \circ f$ and $f \circ g$.
10. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x,y) = (xy, x^3)$. Find a formula for $f \circ f$.

12.5 Inverse Functions

You may recall from calculus that if a function f is injective and surjective, then it has an inverse function f^{-1} that "undoes" the effect of f in the sense that $f^{-1}(f(x)) = x$ for every x in the domain. (For example, if $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$.) We now review these ideas. Our approach uses two ingredients, outlined in the following definitions.

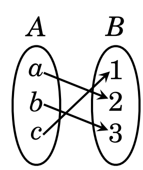
Definition 12.6 Given a set A, the **identity function** on A is the function $i_A : A \to A$ defined as $i_A(x) = x$ for every $x \in A$.

Example: If $A = \{1,2,3\}$, then $i_A = \{(1,1),(2,2),(3,3)\}$. Also $i_{\mathbb{Z}} = \{(n,n) : n \in \mathbb{Z}\}$. The identity function on a set is the function that sends any element of the set to itself.

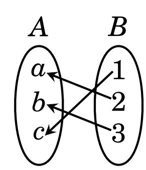
Notice that for any set A, the identity function i_A is bijective: It is injective because $i_A(x) = i_A(y)$ immediately reduces to $x = y$. It is surjective because if we take any element b in the codomain A, then b is also in the domain A, and $i_A(b) = b$.

Definition 12.7 Given a relation R from A to B, the **inverse relation of** R is the relation from B to A defined as $R^{-1} = \{(y,x) : (x,y) \in R\}$. In other words, the inverse of R is the relation R^{-1} obtained by interchanging the elements in every ordered pair in R.

For example, let $A = \{a,b,c\}$ and $B = \{1,2,3\}$, and suppose f is the relation $f = \{(a,2),(b,3),(c,1)\}$ from A to B. Then $f^{-1} = \{(2,a),(3,b),(1,c)\}$ and this is a relation from B to A. Notice that f is actually a function from A to B, and f^{-1} is a function from B to A. These two relations are drawn below. Notice the drawing for relation f^{-1} is just the drawing for f with arrows reversed.

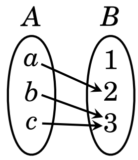


$f = \{(a,2),(b,3),(c,1)\}$

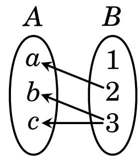


$f^{-1} = \{(2,a),(3,b),(1,c)\}$

For another example, let A and B be the same sets as above, but consider the relation $g = \{(a,2),(b,3),(c,3)\}$ from A to B. Then $g^{-1} = \{(2,a),(3,b),(3,c)\}$ is a relation from B to A. These two relations are sketched below.



$g = \{(a,2),(b,3),(c,3)\}$



$g^{-1} = \{(2,a),(3,b),(3,c)\}$

This time, even though the relation g is a function, its inverse g^{-1} is not a function because the element 3 occurs twice as a first coordinate of an ordered pair in g^{-1}.

In the above examples, relations f and g are both functions, and f^{-1} is a function and g^{-1} is not. This raises a question: What properties does f have and g lack that makes f^{-1} a function and g^{-1} not a function? The answer is not hard to see. Function g is not injective because $g(b) = g(c) = 3$, and thus $(b,3)$ and $(c,3)$ are both in g. This causes a problem with g^{-1} because it means $(3,b)$ and $(3,c)$ are both in g^{-1}, so g^{-1} can't be a function. Thus, in order for g^{-1} to be a function, it would be necessary that g be injective.

But that is not enough. Function g also fails to be surjective because no element of A is sent to the element $1 \in B$. This means g^{-1} contains no ordered pair whose first coordinate is 1, so it can't be a function from B to A. If g^{-1} were to be a function it would be necessary that g be surjective.

The previous two paragraphs suggest that if g is a function, then it must be bijective in order for its inverse relation g^{-1} to be a function. Indeed, this is easy to verify. Conversely, if a function is bijective, then its inverse relation is easily seen to be a function. We summarize this in the following theorem.

Theorem 12.3 Let $f : A \to B$ be a function. Then f is bijective if and only if the inverse relation f^{-1} is a function from B to A.

Suppose $f : A \to B$ is bijective, so according to the theorem f^{-1} is a function. Observe that the relation f contains all the pairs $(x, f(x))$ for $x \in A$, so f^{-1} contains all the pairs $(f(x), x)$. But $(f(x), x) \in f^{-1}$ means $f^{-1}(f(x)) = x$. Therefore $f^{-1} \circ f(x) = x$ for every $x \in A$. From this we get $f^{-1} \circ f = i_A$. Similar reasoning produces $f \circ f^{-1} = i_B$. This leads to the following definitions.

Definition 12.8 If $f : A \to B$ is bijective then its **inverse** is the function $f^{-1} : B \to A$. Functions f and f^{-1} obey the equations $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

You probably recall from algebra and calculus at least one technique for computing the inverse of a bijective function f: to find f^{-1}, start with the equation $y = f(x)$. Then interchange variables to get $x = f(y)$. Solving this equation for y (if possible) produces $y = f^{-1}(x)$. The next two examples illustrate this.

Example 12.12 The function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^3 + 1$ is bijective. Find its inverse.

We begin by writing $y = x^3 + 1$. Now interchange variables to obtain $x = y^3 + 1$. Solving for y produces $y = \sqrt[3]{x-1}$. Thus

$$f^{-1}(x) = \sqrt[3]{x-1}.$$

(You can check your answer by computing

$$f^{-1}(f(x)) = \sqrt[3]{f(x)-1} = \sqrt[3]{x^3+1-1} = x.$$

Therefore $f^{-1}(f(x)) = x$. Any answer other than x indicates a mistake.)

We close with one final example. Example 12.5 showed that the function $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by the formula $g(m,n) = (m+n, m+2n)$ is bijective. Let's find its inverse. The approach outlined above should work, but we need to be careful to keep track of coordinates in $\mathbb{Z} \times \mathbb{Z}$. We begin by writing $(x, y) = g(m,n)$, then interchanging the variables (x, y) and (m,n) to get $(m,n) = g(x, y)$. This gives

$$(m,n) = (x+y, x+2y),$$

from which we get the following system of equations:

$$\begin{array}{ccccc} x & + & y & = & m \\ x & + & 2y & = & n. \end{array}$$

Solving this system using techniques from algebra with which you are familiar, we get

$$\begin{array}{ccc} x & = & 2m-n \\ y & = & n-m. \end{array}$$

Then $(x, y) = (2m-n, n-m)$, so $\boxed{g^{-1}(m,n) = (2m-n, n-m).}$

We can check our work by confirming that $g^{-1}(g(m,n)) = (m,n)$. Doing the math,

$$\begin{aligned} g^{-1}(g(m,n)) &= g^{-1}(m+n, m+2n) \\ &= \big(2(m+n)-(m+2n), (m+2n)-(m+n)\big) \\ &= (m,n). \end{aligned}$$

Exercises for Section 12.5

1. Check that the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 6 - n$ is bijective. Then compute f^{-1}.
2. In Exercise 9 of Section 12.2 you proved that $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective. Now find its inverse.
3. Let $B = \{2^n : n \in \mathbb{Z}\} = \{\ldots, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \ldots\}$. Show that the function $f : \mathbb{Z} \to B$ defined as $f(n) = 2^n$ is bijective. Then find f^{-1}.
4. The function $f : \mathbb{R} \to (0,\infty)$ defined as $f(x) = e^{x^3+1}$ is bijective. Find its inverse.
5. The function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \pi x - e$ is bijective. Find its inverse.
6. The function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by the formula $f(m,n) = (5m+4n, 4m+3n)$ is bijective. Find its inverse.
7. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x,y) = ((x^2+1)y, x^3)$ is bijective. Then find its inverse.
8. Is the function $\theta : \mathscr{P}(\mathbb{Z}) \to \mathscr{P}(\mathbb{Z})$ defined as $\theta(X) = \overline{X}$ bijective? If so, what is its inverse?
9. Consider the function $f : \mathbb{R} \times \mathbb{N} \to \mathbb{N} \times \mathbb{R}$ defined as $f(x,y) = (y, 3xy)$. Check that this is bijective; find its inverse.
10. Consider $f : \mathbb{N} \to \mathbb{Z}$ defined as $f(n) = \frac{(-1)^n(2n-1)+1}{4}$. This function is bijective by Exercise 18 in Section 12.2. Find its inverse.

12.6 Image and Preimage

It is time to take up a matter of notation that you will encounter in future mathematics classes. Suppose we have a function $f : A \to B$. If $X \subseteq A$, the expression $f(X)$ has a special meaning. It stands for the set $\{f(x) : x \in X\}$. Similarly, if $Y \subseteq B$ then $f^{-1}(Y)$ has a meaning *even if f is not invertible.* The expression $f^{-1}(Y)$ stands for the set $\{x \in A : f(x) \in Y\}$. Here are the precise definitions.

Definition 12.9 Suppose $f : A \to B$ is a function.

1. If $X \subseteq A$, the **image** of X is the set $f(X) = \{f(x) : x \in X\} \subseteq B$.
2. If $Y \subseteq B$, the **preimage** of Y is the set $f^{-1}(Y) = \{x \in A : f(x) \in Y\} \subseteq A$.

In words, the image $f(X)$ of X is the set of all things in B that f sends elements of X to. (Roughly speaking, you might think of $f(X)$ as a kind of distorted "copy" or "image" of X in B.) The preimage $f^{-1}(Y)$ of Y is the set of all things in A that f sends into Y.

Maybe you have already encountered these ideas in linear algebra, in a setting involving a linear transformation $T : V \to W$ between two vector spaces. If $X \subseteq V$ is a subspace of V, then its image $T(X)$ is a subspace of W. If $Y \subseteq W$ is a subspace of W, then its preimage $T^{-1}(Y)$ is a subspace of V. (If this does not sound familiar, then ignore it.)

Example 12.13 Let $f : \{s,t,u,v,w,x,y,z\} \to \{0,1,2,3,4,5,6,7,8,9\}$, where

$$f = \{(s,4),(t,8),(u,8),(v,1),(w,2),(x,4),(y,6),(z,4)\}.$$

Notice that f is neither injective nor surjective, so it certainly is not invertible. Be sure you understand the following statements.

1. $f(\{s,t,u,z\}) = \{8,4\}$
2. $f(\{s,x,z\}) = \{4\}$
3. $f(\{s,v,w,y\}) = \{1,2,4,6\}$
4. $f^{-1}(\{4\}) = \{s,x,z\}$
5. $f^{-1}(\{4,9\}) = \{s,x,z\}$
6. $f^{-1}(\{9\}) = \emptyset$
7. $f^{-1}(\{1,4,8\}) = \{s,t,u,v,x,z\}$

It is important to realize that the X and Y in Definition 12.9 are subsets (not elements!) of A and B. Note that in the above example we had $f^{-1}(\{4\}) = \{s,x,z\}$, while $f^{-1}(4)$ has absolutely no meaning because the inverse function f^{-1} does not exist. Likewise, there is a subtle difference between $f(\{s\}) = \{4\}$ and $f(s) = 4$. Be careful.

Example 12.14 Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$. Note that $f(\{0,1,2\}) = \{0,1,4\}$ and $f^{-1}(\{0,1,4\}) = \{-2,-1,0,1,2\}$. This shows $f^{-1}(f(X)) \neq X$ in general.

Using the same f, now check your understanding of the following statements involving images and preimages of intervals: $f([-2,3]) = [0,9]$, and $f^{-1}([0,9]) = [-3,3]$. Also $f(\mathbb{R}) = [0,\infty)$ and $f^{-1}([-2,-1]) = \emptyset$.

If you continue with mathematics you are likely to encounter the following results. For now, you are asked to prove them in the exercises.

Theorem 12.4 Suppose $f : A \to B$ is a function. Let $W, X \subseteq A$, and $Y, Z \subseteq B$. Then:

1. $f(W \cap X) \subseteq f(W) \cap f(X)$
2. $f(W \cup X) = f(W) \cup f(X)$
3. $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$
4. $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$
5. $X \subseteq f^{-1}(f(X))$
6. $f(f^{-1}(Y)) \subseteq Y$.

Exercises for Section 12.6

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2 + 3$. Find $f([-3,5])$ and $f^{-1}([12,19])$.

2. Consider the function $f : \{1,2,3,4,5,6,7\} \to \{0,1,2,3,4,5,6,7,8,9\}$ given as

$$f = \{(1,3),(2,8),(3,3),(4,1),(5,2),(6,4),(7,6)\}.$$

Find: $f(\{1,2,3\})$, $f(\{4,5,6,7\})$, $f(\emptyset)$, $f^{-1}(\{0,5,9\})$ and $f^{-1}(\{0,3,5,9\})$.

3. This problem concerns functions $f : \{1,2,3,4,5,6,7\} \to \{0,1,2,3,4\}$. How many such functions have the property that $|f^{-1}(\{3\})| = 3$?

4. This problem concerns functions $f : \{1,2,3,4,5,6,7,8\} \to \{0,1,2,3,4,5,6\}$. How many such functions have the property that $|f^{-1}(\{2\})| = 4$?

5. Consider a function $f : A \to B$ and a subset $X \subseteq A$. We observed in Section 12.6 that $f^{-1}(f(X)) \neq X$ in general. However $X \subseteq f^{-1}(f(X))$ is always true. Prove this.

6. Given a function $f : A \to B$ and a subset $Y \subseteq B$, is $f(f^{-1}(Y)) = Y$ always true? Prove or give a counterexample.

7. Given a function $f : A \to B$ and subsets $W, X \subseteq A$, prove $f(W \cap X) \subseteq f(W) \cap f(X)$.

8. Given a function $f : A \to B$ and subsets $W, X \subseteq A$, then $f(W \cap X) = f(W) \cap f(X)$ is *false* in general. Produce a counterexample.

9. Given a function $f : A \to B$ and subsets $W, X \subseteq A$, prove $f(W \cup X) = f(W) \cup f(X)$.

10. Given $f : A \to B$ and subsets $Y, Z \subseteq B$, prove $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$.

11. Given $f : A \to B$ and subsets $Y, Z \subseteq B$, prove $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$.

12. Consider $f : A \to B$. Prove that f is injective if and only if $X = f^{-1}(f(X))$ for all $X \subseteq A$. Prove that f is surjective if and only if $f(f^{-1}(Y)) = Y$ for all $Y \subseteq B$.

13. Let $f : A \to B$ be a function, and $X \subseteq A$. Prove or disprove: $f\big(f^{-1}(f(X))\big) = f(X)$.

14. Let$f : A \to B$ be a function, and $Y \subseteq B$. Prove or disprove: $f^{-1}\big(f(f^{-1}(Y))\big) = f^{-1}(Y)$.
