CHAPTER 3

Counting

It may seem peculiar that a college-level text has a chapter on counting. At its most basic level, counting is a process of pointing to each object in a collection and calling off *"one, two, three,..."* until the quantity of objects is determined. How complex could that be? Actually, counting can become quite subtle, and in this chapter we explore some of its more sophisticated aspects. Our goal is still to answer the question *"How many?"* but we introduce mathematical techniques that bypass the actual process of counting individual objects.

Almost every branch of mathematics uses some form of this "sophisticated counting." Many such counting problems can be modeled with the idea of a *list*, so we start there.

3.1 Counting Lists

A **list** is an ordered sequence of objects. A list is denoted by an opening parenthesis, followed by the objects, separated by commas, followed by a closing parenthesis. For example (a,b,c,d,e) is a list consisting of the first five letters of the English alphabet, in order. The objects a,b,c,d,e are called the **entries** of the list; the first entry is a, the second is b, and so on. If the entries are rearranged we get a different list, so, for instance,

$$(a,b,c,d,e) \neq (b,a,c,d,e).$$

A list is somewhat like a set, but instead of being a mere collection of objects, the entries of a list have a definite *order*. Note that for sets we have

$$\{a,b,c,d,e\} = \{b,a,c,d,e\},$$

but—as noted above—the analogous equality for lists does not hold.

Unlike sets, lists are allowed to have repeated entries. For example $(5,3,5,4,3,3)$ is a perfectly acceptable list, as is (S,O,S). The number of entries in a list is called its **length**. Thus $(5,3,5,4,3,3)$ has length six, and (S,O,S) has length three.

Occasionally we may get sloppy and write lists without parentheses and commas; for instance, we may express (S,O,S) as SOS if there is no danger of confusion. But be alert that doing this can lead to ambiguity. Is it reasonable that (9,10,11) should be the same as 91011? If so, then $(9,10,11) = 91011 = (9,1,0,1,1)$, which makes no sense. We will thus almost always adhere to the parenthesis/comma notation for lists.

Lists are important because many real-world phenomena can be described and understood in terms of them. For example, your phone number (with area code) can be identified as a list of ten digits. Order is essential, for rearranging the digits can produce a different phone number. A *byte* is another important example of a list. A byte is simply a length-eight list of 0's and 1's. The world of information technology revolves around bytes.

To continue our examples of lists, $(a,15)$ is a list of length two. Likewise $(0,(0,1,1))$ is a list of length two whose second entry is a list of length three. The list $(\mathbb{N},\mathbb{Z},\mathbb{R})$ has length three, and each of its entries is a set. We emphasize that for two lists to be equal, they must have exactly the same entries in exactly the same order. Consequently if two lists are equal, then they must have the same length. Said differently, if two lists have different lengths, then they are not equal. For example, $(0,0,0,0,0,0) \neq (0,0,0,0,0)$. For another example note that

$$(g,r,o,c,e,r,y,\ l,i,s,t) \quad \neq \quad \left(\boxed{\begin{array}{l}\text{bread}\\ \text{milk}\\ \text{eggs}\\ \text{mustard}\\ \text{coffee}\end{array}} \right)$$

because the list on the left has length eleven but the list on the right has just one entry (a piece of paper with some words on it).

There is one very special list which has no entries at all. It is called the **empty list**, and is denoted (). It is the only list whose length is zero.

One often needs to count up the number of possible lists that satisfy some condition or property. For example, suppose we need to make a list of length three having the property that the first entry must be an element of the set $\{a,b,c\}$, the second entry must be in $\{5,7\}$ and the third entry must be in $\{a,x\}$. Thus $(a,5,a)$ and $(b,5,a)$ are two such lists. How many such lists are there all together? To answer this question, imagine making the list by selecting the first element, then the second and finally the third. This is described in Figure 3.1. The choices for the first list entry are a,b or c, and the left of the diagram branches out in three directions, one for each choice. Once this choice is made there are two choices (5 or 7) for the second entry, and this is described graphically by two branches from each of the three choices for the first entry. This pattern continues for the choice for the third entry, which is either a or x. Thus, in the diagram there are $3 \cdot 2 \cdot 2 = 12$ paths from left to right, each corresponding to a particular choice for each entry in the list. The corresponding lists are tallied at the far-right end of each path. So, to answer our original question, there are 12 possible lists with the stated properties.

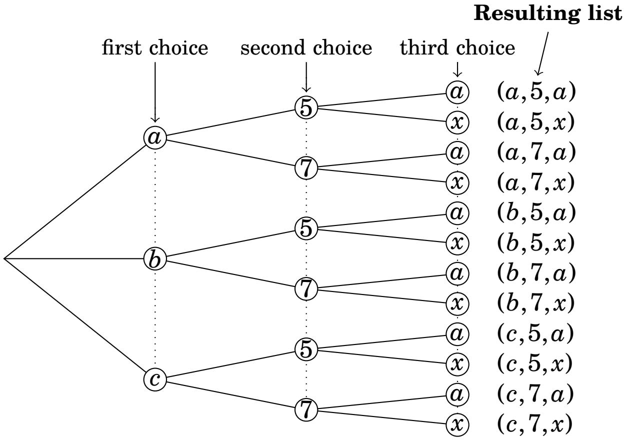


Figure 3.1. Constructing lists of length 3

We summarize the type of reasoning used above in an important fact called the **multiplication principle**.

Fact 3.1 (Multiplication Principle) Suppose in making a list of length n there are a_1 possible choices for the first entry, a_2 possible choices for the second entry, a_3 possible choices for the third entry and so on. Then the total number of different lists that can be made this way is the product $a_1 \cdot a_2 \cdot a_3 \cdot \cdots \cdot a_n$.

So, for instance, in the above example we had $a_1 = 3, a_2 = 2$ and $a_3 = 2$, so the total number of lists was $a_1 \cdot a_2 \cdot a_3 = 3 \cdot 2 \cdot 2 = 12$. Now let's look at some additional examples of how the multiplication principle can be used.

Example 3.1 A standard license plate consists of three letters followed by four numbers. For example, *JRB-4412* and *MMX-8901* are two standard license plates. (Vanity plates such as *LV2COUNT* are not included among the standard plates.) How many different standard license plates are possible?

To answer this question, note that any standard license plate such as *JRB-4412* corresponds to a length-7 list *(J,R,B,4,4,1,2)*, so the question can be answered by counting how many such lists are possible. We use the multiplication principle. There are $a_1 = 26$ possibilities (one for each letter of the alphabet) for the first entry of the list. Similarly, there are $a_2 = 26$ possibilities for the second entry and $a_3 = 26$ possibilities for the third entry. There are $a_4 = 10$ possibilities for the fourth entry, and likewise $a_5 = a_6 = a_7 = 10$. Therefore there are a total of $a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 \cdot a_6 \cdot a_7 = 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 =$ **175,760,000 possible standard license plates.**

There are two types of list-counting problems. On one hand, there are situations in which the same symbol or symbols may appear multiple times in different entries of the list. For example, license plates or telephone numbers can have repeated symbols. The sequence *CCX-4144* is a perfectly valid license plate in which the symbols *C* and *4* appear more than once. On the other hand, for some lists repeated symbols do not make sense or are not allowed. For instance, imagine drawing 5 cards from a standard 52-card deck and laying them in a row. Since no 2 cards in the deck are identical, this list has no repeated entries. We say that *repetition is allowed* in the first type of list and *repetition is not allowed* in the second kind of list. (Often we call a list in which repetition is not allowed a **non-repetitive list**.) The following example illustrates the difference.

Example 3.2 Consider making lists from symbols *A, B, C, D, E, F, G*.

(a) How many length-4 lists are possible if repetition is allowed?

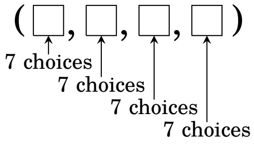
(b) How many length-4 lists are possible if repetition is **not** allowed?

(c) How many length-4 lists are possible if repetition is **not** allowed and the list must contain an *E*?

(d) How many length-4 lists are possible if repetition is allowed and the list must contain an *E*?

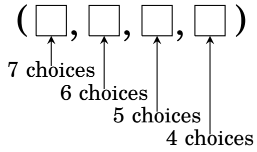
Solutions:

(a) Imagine the list as containing four boxes that we fill with selections from the letters *A,B,C,D,E,F* and *G*, as illustrated below.



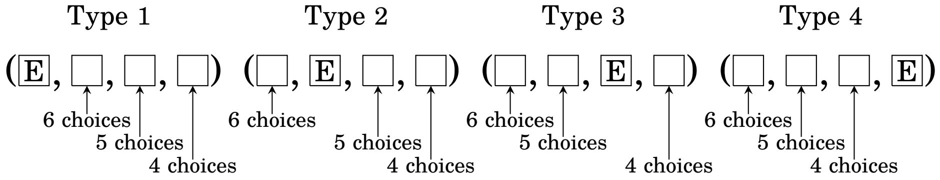
There are seven possibilities for the contents of each box, so the total number of lists that can be made this way is $7 \cdot 7 \cdot 7 \cdot 7 =$ **2401**.

(b) This problem is the same as the previous one except that repetition is not allowed. We have seven choices for the first box, but once it is filled we can no longer use the symbol that was placed in it. Hence there are only six possibilities for the second box. Once the second box has been filled we have used up two of our letters, and there are only five left to choose from in filling the third box. Finally, when the third box is filled we have only four possible letters for the last box.



Thus the answer to our question is that there are $7 \cdot 6 \cdot 5 \cdot 4 = \mathbf{840}$ lists in which repetition does not occur.

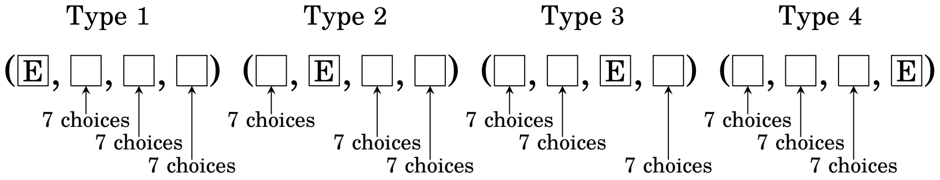
(c) We are asked to count the length-4 lists in which repetition is not allowed and the symbol E must appear somewhere in the list. Thus E occurs once and only once in each such list. Let us divide these lists into four categories depending on whether the E occurs as the first, second, third or fourth entry. These four types of lists are illustrated below.



Consider lists of the first type, in which the E appears in the first entry. We have six remaining choices (*A,B,C,D,F* or *G*) for the second entry, five choices for the third entry and four choices for the fourth entry. Hence there are $6 \cdot 5 \cdot 4 = 120$ lists having an E in the first entry. As indicated in the above diagram, there are also $6 \cdot 5 \cdot 4 = 120$ lists having an E in the second, third or fourth entry. Thus there are $120 + 120 + 120 + 120 = \mathbf{480}$ such lists all together.

(d) Now we must find the number of length-four lists where repetition is allowed and the list must contain an E. Our strategy is as follows. By Part (a) of this exercise there are $7 \cdot 7 \cdot 7 \cdot 7 = 7^4 = 2401$ lists where repetition is allowed. Obviously this is not the answer to our current question, for many of these lists contain no E. We will subtract from 2401 the number of lists that **do not** contain an E. In making a list that does not contain an E, we have six choices for each list entry (because we can choose any one of the six letters *A,B,C,D,F* or *G*). Thus there are $6 \cdot 6 \cdot 6 \cdot 6 = 6^4 = 1296$ lists that do not have an E. Therefore the final answer to our question is that there are $2401 - 1296 =$ **1105** lists with repetition allowed that contain at least one E.

Perhaps you wondered if Part (d) of Example 3.2 could be solved with a setup similar to that of Part (c). Let's try doing it that way. We want to count the length-4 lists (with repetition allowed) that contain at least one E. The following diagram is adapted from Part (c), the only difference being that there are now seven choices in each slot because we are allowed to repeat any of the seven letters.



This gives a total of $7^3 + 7^3 + 7^3 + 7^3 = 1372$ lists, an answer that is substantially larger than the (correct) value of 1105 that we got in our solution to Part (d) above. It is not hard to see what went wrong. The list (E,E,A,B) is of type 1 *and* type 2, so it got counted *twice*. Similarly (E,E,C,E) is of type 1, 3 and 4, so it got counted three times. In fact, you can find many similar lists that were counted multiple times.

In solving counting problems, we must always be careful to avoid this kind of double-counting or triple-counting, or worse.

Exercises for Section 3.1

Note: A calculator may be helpful for some of the exercises in this chapter. This is the only chapter for which a calculator may be helpful. (As for the exercises in the other chapters, a calculator makes them harder.)

1. Consider lists made from the letters *T,H,E,O,R,Y*, with repetition allowed.

  (a) How many length-4 lists are there?

  (b) How many length-4 lists are there that begin with T?

  (c) How many length-4 lists are there that do not begin with T?

2. Airports are identified with 3-letter codes. For example, the Richmond, Virginia airport has the code *RIC*, and Portland, Oregon has *PDX*. How many different 3-letter codes are possible?

3. How many lists of length 3 can be made from the symbols *A,B,C,D,E,F* if...

- (**a**) ... repetition is allowed.
- (**b**) ... repetition is not allowed.
- (**c**) ... repetition is not allowed and the list must contain the letter A.
- (**d**) ... repetition is allowed and the list must contain the letter A.

4. Five cards are dealt off of a standard 52-card deck and lined up in a row. How many such line-ups are there in which all 5 cards are of the same suit?

5. Five cards are dealt off of a standard 52-card deck and lined up in a row. How many such line-ups are there in which all 5 cards are of the same color (i.e., all black or all red)?

6. Five cards are dealt off of a standard 52-card deck and lined up in a row. How many such line-ups are there in which exactly one of the 5 cards is a queen?

7. This problem involves 8-digit binary strings such as 10011011 or 00001010 (i.e., 8-digit numbers composed of 0's and 1's).

- (**a**) How many such strings are there?
- (**b**) How many such strings end in 0?
- (**c**) How many such strings have the property that their second and fourth digits are 1's?
- (**d**) How many such strings have the property that their second **or** fourth digits are 1's?

8. This problem concerns lists made from the symbols A,B,C,D,E.

- (**a**) How many such length-5 lists have at least one letter repeated?
- (**b**) How many such length-6 lists have at least one letter repeated?

9. This problem concerns 4-letter codes made from the letters $A,B,C,D,...,Z$.

- (**a**) How many such codes can be made?
- (**b**) How many such codes have no two consecutive letters the same?

10. This problem concerns lists made from the letters A,B,C,D,E,F,G,H,I,J.

- (**a**) How many length-5 lists can be made from these letters if repetition is not allowed and the list must begin with a vowel?
- (**b**) How many length-5 lists can be made from these letters if repetition is not allowed and the list must begin and end with a vowel?
- (**c**) How many length-5 lists can be made from these letters if repetition is not allowed and the list must contain exactly one A?

11. This problem concerns lists of length 6 made from the letters A,B,C,D,E,F,G,H. How many such lists are possible if repetition is not allowed and the list contains two consecutive vowels?

12. Consider the lists of length six made with the symbols P, R, O, F, S, where repetition is allowed. (For example, the following is such a list: (P,R,O,O,F,S).) How many such lists can be made if the list must end in an S and the symbol O is used more than once?

3.2 Factorials

In working the examples from Section 3.1, you may have noticed that often we need to count the number of non-repetitive lists of length n that are made from n symbols. In fact, this particular problem occurs with such frequency that a special idea, called a *factorial*, is introduced to handle it.

The table below motivates this idea. The first column lists successive integer values n (beginning with 0) and the second column contains a set $\{A,B,\cdots\}$ of n symbols. The third column contains all the possible non-repetitive lists of length n which can be made from these symbols. Finally, the last column tallies up how many lists there are of that type. Notice that when $n = 0$ there is only one list of length 0 that can be made from 0 symbols, namely the empty list $()$. Thus the value 1 is entered in the last column of that row.

n	Symbols	Non-repetitive lists of length n made from the symbols	$n!$
0	$\{\}$	$()$	1
1	$\{A\}$	(A)	1
2	$\{A,B\}$	$(A,B),(B,A)$	2
3	$\{A,B,C\}$	$(A,B,C),(A,C,B),(B,C,A),(B,A,C),(C,A,B),(C,B,A)$	6
4	$\{A,B,C,D\}$	$(A,B,C,D),(A,B,D,C),(A,C,B,D),(A,C,D,B),(A,D,B,C),(A,D,C,B)$ $(B,A,C,D),(B,A,D,C),(B,C,A,D),(B,C,D,A),(B,D,A,C),(B,D,C,A)$ $(C,A,B,D),(C,A,D,B),(C,B,A,D),(C,B,D,A),(C,D,A,B),(C,D,B,A)$ $(D,A,B,C),(D,A,C,B),(D,B,A,C),(D,B,C,A),(D,C,A,B),(D,C,B,A)$	24
$\vdots$	$\vdots$	$\vdots$	$\vdots$

For $n > 0$, the number that appears in the last column can be computed using the multiplication principle. The number of non-repetitive lists of length n that can be made from n symbols is $n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$. Thus, for instance, the number in the last column of the row for $n = 4$ is $4\cdot 3\cdot 2\cdot 1 = 24$.

The number that appears in the last column of Row n is called the **factorial** of n. It is denoted as $n!$ (read "n factorial"). Here is the definition:

Definition 3.1 If n is a non-negative integer, then the **factorial** of n, denoted $n!$, is the number of non-repetitive lists of length n that can be made from n symbols. Thus $0! = 1$ and $1! = 1$. If $n > 1$, then $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$.

It follows that

$$\begin{aligned}0! &= 1\\ 1! &= 1\\ 2! &= 2\cdot 1 = 2\\ 3! &= 3\cdot 2\cdot 1 = 6\\ 4! &= 4\cdot 3\cdot 2\cdot 1 = 24\\ 5! &= 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120\\ 6! &= 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720, \quad \text{and so on.}\end{aligned}$$

Students are often tempted to say $0! = 0$, but this is wrong. The correct value is $0! = 1$, as the above definition and table tell us. Here is another way to see that 0! must equal 1: Notice that $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5\cdot(4\cdot 3\cdot 2\cdot 1) = 5\cdot 4!$. Also $4! = 4\cdot 3\cdot 2\cdot 1 = 4\cdot(3\cdot 2\cdot 1) = 4\cdot 3!$. Generalizing this reasoning, we have the following formula.

$$n! = n\cdot(n-1)! \tag{3.1}$$

Plugging in $n = 1$ gives $1! = 1\cdot(1-1)! = 1\cdot 0!$, that is, $1! = 1\cdot 0!$. If we mistakenly thought 0! were 0, this would give the incorrect result $1! = 0$.

We round out our discussion of factorials with an example.

Example 3.3 This problem involves making lists of length seven from the symbols $0, 1, 2, 3, 4, 5$ and 6.

(a) How many such lists are there if repetition is not allowed?

(b) How many such lists are there if repetition is not allowed *and* the first three entries must be odd?

(c) How many such lists are there in which repetition is allowed, and the list must contain at least one repeated number?

To answer the first question, note that there are seven symbols, so the number of lists is $7! = \mathbf{5040}$. To answer the second question, notice that the set $\{0,1,2,3,4,5,6\}$ contains three odd numbers and four even numbers. Thus in making the list the first three entries must be filled by odd numbers and the final four must be filled with even numbers. By the multiplication principle, the number of such lists is $3\cdot 2\cdot 1\cdot 4\cdot 3\cdot 2\cdot 1 = 3!4! = \mathbf{144}$.

To answer the third question, notice that there are $7^7 = 823{,}543$ lists in which repetition is allowed. The set of all such lists includes lists that are non-repetitive (e.g., $(0,6,1,2,4,3,5)$) as well as lists that have some repetition (e.g., $(6,3,6,2,0,0,0)$). We want to compute the number of lists that have at least one repeated number. To find the answer we can subtract the number of non-repetitive lists of length seven from the total number of possible lists of length seven. Therefore the answer is $7^7 - 7! = 823{,}543 - 5040 = \mathbf{818{,}503}$.

We close this section with a formula that combines the ideas of the first and second sections of the present chapter. One of the main problems of Section 3.1 was as follows: Given n symbols, how many non-repetitive lists of length k can be made from the n symbols? We learned how to apply the multiplication principle to obtain the answer

$$n(n-1)(n-2)\cdots(n-k+1).$$

Notice that by cancellation this value can also be written as

$$\frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)(n-k-1)\cdots 3\cdot 2\cdot 1}{(n-k)(n-k-1)\cdots 3\cdot 2\cdot 1} = \frac{n!}{(n-k)!}.$$

We summarize this as follows:

Fact 3.2 The number of non-repetitive lists of length k whose entries are chosen from a set of n possible entries is $\frac{n!}{(n-k)!}$.

For example, consider finding the number of non-repetitive lists of length five that can be made from the symbols $1,2,3,4,5,6,7,8$. We will do this two ways. By the multiplication principle, the answer is $8\cdot 7\cdot 6\cdot 5\cdot 4 = 6720$. Using the formula from Fact 3.2, the answer is $\frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{40{,}320}{6} = 6720$.

The new formula isn't really necessary, but it is a nice repackaging of an old idea and will prove convenient in the next section.

Exercises for Section 3.2

1. What is the smallest n for which $n!$ has more than 10 digits?
2. For which values of n does $n!$ have n or fewer digits?
3. How many 5-digit positive integers are there in which there are no repeated digits and all digits are odd?
4. Using only pencil and paper, find the value of $\frac{100!}{95!}$.
5. Using only pencil and paper, find the value of $\frac{120!}{118!}$.
6. There are two 0's at the end of $10! = 3{,}628{,}800$. Using only pencil and paper, determine how many 0's are at the end of the number 100!.
7. Compute how many 9-digit numbers can be made from the digits $1,2,3,4,5,6,7,8,9$ if repetition is not allowed and all the odd digits occur first (on the left) followed by all the even digits (i.e. as in 137598264, but not 123456789).
8. Compute how many 7-digit numbers can be made from the digits $1,2,3,4,5,6,7$ if there is no repetition and the odd digits must appear in an unbroken sequence. (Examples: 3571264 or 2413576 or 2467531, etc., but **not** 7234615.)

9. There is a very interesting function $\Gamma : [0,\infty) \to \mathbb{R}$ called the **gamma function**. It is defined as $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. It has the remarkable property that if $x \in \mathbb{N}$, then $\Gamma(x) = (x-1)!$. Check that this is true for $x = 1,2,3,4$.
Notice that this function provides a way of extending factorials to numbers other than integers. Since $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$, we have the formula $n! = \Gamma(n+1)$. But Γ can be evaluated at any number in $[0,\infty)$, not just at integers, so we have a formula for $n!$ for any $n \in [0,\infty)$. Extra credit: Compute $\pi!$.

10. There is another significant function called **Stirling's formula** that provides an approximation to factorials. It states that $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. It is an approximation to $n!$ in the sense that $\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$ approaches 1 as n approaches ∞. Use Stirling's formula to find approximations to 5!, 10!, 20! and 50!.

3.3 Counting Subsets

The previous two sections were concerned with counting the number of lists that can be made by selecting k entries from a set of n possible entries. We turn now to a related question: How many *subsets* can be made by selecting k elements from a set with n elements?

To highlight the differences between these two problems, look at the set $A = \{a,b,c,d,e\}$. First, think of the non-repetitive lists that can be made from selecting two entries from A. By Fact 3.2 (on the previous page), there are $\frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{120}{6} = 20$ such lists. They are as follows.

$$\begin{array}{cccccccccc} (a,b), & (a,c), & (a,d), & (a,e), & (b,c), & (b,d), & (b,e), & (c,d), & (c,e) & (d,e) \\ (b,a), & (c,a), & (d,a), & (e,a), & (c,b), & (d,b), & (e,b), & (d,c), & (e,c) & (e,d) \end{array}$$

Next consider the *subsets* of A that can made from selecting two elements from A. There are only ten such subsets, as follows.

$$\{a,b\},\ \{a,c\},\ \{a,d\},\ \{a,e\},\ \{b,c\},\ \{b,d\},\ \{b,e\},\ \{c,d\},\ \{c,e\},\ \{d,e\}.$$

The reason that there are more lists than subsets is that changing the order of the entries of a list produces a different list, but changing the order of the elements of a set does not change the set. Using elements $a,b \in A$, we can make two lists (a,b) and (b,a), but only one subset $\{a,b\}$.

In this section we are concerned not with counting lists, but with counting subsets. As was noted above, the basic question is this: How many subsets can be made by choosing k elements from an n-element set? We begin with some notation that gives a name to the answer to this question.

Definition 3.2 If n and k are integers, then $\binom{n}{k}$ denotes the number of subsets that can be made by choosing k elements from a set with n elements. The symbol $\binom{n}{k}$ is read "n choose k." (Some textbooks write $C(n,k)$ instead of $\binom{n}{k}$.)

To illustrate this definition, the following table computes the values of $\binom{4}{k}$ for various values of k by actually listing all the subsets of the 4-element set $A = \{a,b,c,d\}$ that have cardinality k. The values of k appear in the far-left column. To the right of each k are all of the subsets (if any) of A of size k. For example, when $k = 1$, set A has four subsets of size k, namely $\{a\}$, $\{b\}$, $\{c\}$ and $\{d\}$. Therefore $\binom{4}{1} = 4$. Similarly, when $k = 2$ there are six subsets of size k so $\binom{4}{2} = 6$.

k	k-element subsets of $\{a,b,c,d\}$	$\binom{4}{k}$
-1		$\binom{4}{-1} = 0$
0	$\emptyset$	$\binom{4}{0} = 1$
1	$\{a\},\{b\},\{c\},\{d\}$	$\binom{4}{1} = 4$
2	$\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}$	$\binom{4}{2} = 6$
3	$\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}$	$\binom{4}{3} = 4$
4	$\{a,b,c,d\}$	$\binom{4}{4} = 1$
5		$\binom{4}{5} = 0$
6		$\binom{4}{6} = 0$

When $k = 0$, there is only one subset of A that has cardinality k, namely the empty set, $\emptyset$. Therefore $\binom{4}{0} = 1$.

Notice that if k is negative or greater than $|A|$, then A has no subsets of cardinality k, so $\binom{4}{k} = 0$ in these cases. In general $\binom{n}{k} = 0$ whenever $k < 0$ or $k > n$. In particular this means $\binom{n}{k} = 0$ if n is negative.

Although it was not hard to work out the values of $\binom{4}{k}$ by writing out subsets in the above table, this method of actually listing sets would not be practical for computing $\binom{n}{k}$ when n and k are large. We need a formula. To find one, we will now carefully work out the value of $\binom{5}{3}$ in such a way that a pattern will emerge that points the way to a formula for any $\binom{n}{k}$.

To begin, note that $\binom{5}{3}$ is the number of 3-element subsets of $\{a,b,c,d,e\}$. These are listed in the following table. We see that in fact $\binom{5}{3} = 10$.

$$\longleftarrow \binom{5}{3} \longrightarrow$$

$\{a,b,c\}$	$\{a,b,d\}$	$\{a,b,e\}$	$\{a,c,d\}$	$\{a,c,e\}$	$\{a,d,e\}$	$\{b,c,d\}$	$\{b,c,e\}$	$\{b,d,e\}$	$\{c,d,e\}$

The formula will emerge when we expand this table as follows. Taking any one of the ten 3-element sets above, we can make 3! different non-repetitive lists from its elements. For example, consider the first set $\{a,b,c\}$. The first column of the following table tallies the $3! = 6$ different lists that can be the letters $\{a,b,c\}$. The second column tallies the lists that can be made from $\{a,b,d\}$, and so on.

$$\longleftarrow \binom{5}{3} \longrightarrow$$

$$\updownarrow\ 3!$$

abc	*abd*	*abe*	*acd*	*ace*	*ade*	*bcd*	*bce*	*bde*	*cde*
acb	*adb*	*aeb*	*adc*	*aec*	*aed*	*bdc*	*bec*	*bed*	*ced*
bac	*bad*	*bae*	*cad*	*cae*	*dae*	*cbd*	*cbe*	*dbe*	*dce*
bca	*bda*	*bea*	*cda*	*cea*	*dea*	*cdb*	*ceb*	*deb*	*dec*
cba	*dba*	*eba*	*dca*	*eca*	*eda*	*dcb*	*ecb*	*edb*	*edc*
cab	*dab*	*eab*	*dac*	*eac*	*ead*	*dbc*	*ebc*	*ebd*	*ecd*

This table has $\binom{5}{3}$ columns and 3! rows, so it has a total of $3!\binom{5}{3}$ lists. But notice also that the table consists of every non-repetitive length-3 list that can be made from the symbols $\{a,b,c,d,e\}$. We know from Fact 3.2 that there are $\frac{5!}{(5-3)!}$ such lists. Thus the total number of lists in the table is $3!\binom{5}{3} = \frac{5!}{(5-3)!}$. Dividing both sides of this equation by 3!, we get

$$\binom{5}{3} = \frac{5!}{3!(5-3)!}.$$

Working this out, you will find that it does give the correct value of 10.

But there was nothing special about the values 5 and 3. We could do the above analysis for any $\binom{n}{k}$ instead of $\binom{5}{3}$. The table would have $\binom{n}{k}$ columns and $k!$ rows. We would get

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

We summarize this as follows:

Fact 3.3 If $n,k \in \mathbb{Z}$ and $0 \le k \le n$, then $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Otherwise $\binom{n}{k} = 0$.

Let's now use our new knowledge to work some exercises.

Example 3.4 How many 4-element subsets does $\{1,2,3,4,5,6,7,8,9\}$ have? The answer is $\binom{9}{4} = \frac{9!}{4!(9-4)!} = \frac{9!}{4!5!} = \frac{9\cdot8\cdot7\cdot6\cdot5!}{4!5!} = \frac{9\cdot8\cdot7\cdot6}{4!} = \frac{9\cdot8\cdot7\cdot6}{24} = \mathbf{126}$.

Example 3.5 A single 5-card hand is dealt off of a standard 52-card deck. How many different 5-card hands are possible?

To answer this, think of the deck as being a set D of 52 cards. Then a 5-card hand is just a 5-element subset of D. For example, here is one of many different 5-card hands that might be dealt from the deck.

$$\left\{ \boxed{\begin{smallmatrix} 7 \\ \clubsuit \end{smallmatrix}}, \boxed{\begin{smallmatrix} 2 \\ \clubsuit \end{smallmatrix}}, \boxed{\begin{smallmatrix} 3 \\ \heartsuit \end{smallmatrix}}, \boxed{\begin{smallmatrix} A \\ \spadesuit \end{smallmatrix}}, \boxed{\begin{smallmatrix} 5 \\ \diamondsuit \end{smallmatrix}} \right\}$$

The total number of possible hands equals the number of 5-element subsets of D, that is

$$\binom{52}{5} = \frac{52!}{5!\cdot47!} = \frac{52\cdot51\cdot50\cdot49\cdot48\cdot47!}{5!\cdot47!} = \frac{52\cdot51\cdot50\cdot49\cdot48}{5!} = 2,598,960.$$

Thus the answer to our question is that there are 2,598,960 different five-card hands that can be dealt from a deck of 52 cards.

Example 3.6 This problem concerns 5-card hands that can be dealt off of a 52-card deck. How many such hands are there in which two of the cards are clubs and three are hearts?

Solution: Think of such a hand as being described by a list of length two of the form

$$\left(\left\{ \boxed{\begin{smallmatrix} * \\ \clubsuit \end{smallmatrix}}, \boxed{\begin{smallmatrix} * \\ \clubsuit \end{smallmatrix}} \right\}, \left\{ \boxed{\begin{smallmatrix} * \\ \heartsuit \end{smallmatrix}}, \boxed{\begin{smallmatrix} * \\ \heartsuit \end{smallmatrix}}, \boxed{\begin{smallmatrix} * \\ \heartsuit \end{smallmatrix}} \right\} \right),$$

where the first entry is a 2-element subset of the set of 13 club cards, and the second entry is a 3-element subset of the set of 13 heart cards. There are $\binom{13}{2}$ choices for the first entry and $\binom{13}{3}$ choices for the second entry, so by the multiplication principle there are $\binom{13}{2}\binom{13}{3} = \frac{13!}{2!11!}\frac{13!}{3!10!} = 22,308$ such lists. Answer: There are **22,308 possible 5-card hands with two clubs and three hearts.**

Example 3.7 Imagine a lottery that works as follows. A bucket contains 36 balls numbered 1,2,3,4,...,36. Six of these balls will be drawn randomly. For \$1 you buy a ticket that has six blanks: $\boxed{\square\square\square\square\square\square}$. You fill in the blanks with six different numbers between 1 and 36. You win \$1,000,000 if you chose the same numbers that are drawn, regardless of order. What are your chances of winning?

Solution: In filling out the ticket you are choosing six numbers from a set of 36 numbers. Thus there are $\binom{36}{6} = \frac{36!}{6!(36-6)!} = 1,947,792$ different combinations of numbers you might write. Only one of these will be a winner. **Your chances of winning are one in 1,947,792.**

Exercises for Section 3.3

1. Suppose a set A has 37 elements. How many subsets of A have 10 elements? How many subsets have 30 elements? How many have 0 elements?
2. Suppose A is a set for which $|A| = 100$. How many subsets of A have 5 elements? How many subsets have 10 elements? How many have 99 elements?
3. A set X has exactly 56 subsets with 3 elements. What is the cardinality of X?
4. Suppose a set B has the property that $\left|\{X : X \in \mathscr{P}(B), |X| = 6\}\right| = 28$. Find $|B|$.
5. How many 16-digit binary strings contain exactly seven 1's? (Examples of such strings include 0111000011110000 and 0011001100110010, etc.)
6. $\left|\{X \in \mathscr{P}(\{0,1,2,3,4,5,6,7,8,9\}) : |X| = 4\}\right| =$
7. $\left|\{X \in \mathscr{P}(\{0,1,2,3,4,5,6,7,8,9\}) : |X| < 4\}\right| =$
8. This problem concerns lists made from the symbols *A,B,C,D,E,F,G,H,I*.
   (a) How many length-5 lists can be made if repetition is not allowed and the list is in alphabetical order? (Example: *BDEFI* or *ABCGH*, but not *BACGH*.)
   (b) How many length-5 lists can be made if repetition is not allowed and the list is **not** in alphabetical order?
9. This problem concerns lists of length 6 made from the letters *A,B,C,D,E,F*, without repetition. How many such lists have the property that the *D* occurs before the *A*?
10. A department consists of 5 men and 7 women. From this department you select a committee with 3 men and 2 women. In how many ways can you do this?
11. How many positive 10-digit integers contain no 0's and exactly three 6's?
12. Twenty-one people are to be divided into two teams, the Red Team and the Blue Team. There will be 10 people on Red Team and 11 people on Blue Team. In how many ways can this be done?
13. Suppose n and k are integers for which $0 \le k \le n$. Use the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to show that $\binom{n}{k} = \binom{n}{n-k}$.
14. Suppose $n,k \in \mathbb{Z}$, and $0 \le k \le n$. Use Definition 3.2 alone (without using Fact 3.3) to show that $\binom{n}{k} = \binom{n}{n-k}$.

3.4 Pascal's Triangle and the Binomial Theorem

There are some beautiful and significant patterns among the numbers $\binom{n}{k}$. This section investigates a pattern based on one equation in particular. It happens that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \tag{3.2}$$

for any integers n and k with $1 \le k \le n$.

To see why this is true, recall that $\binom{n+1}{k}$ equals the number of k-element subsets of a set with $n+1$ elements. Now, the set $A = \{0,1,2,3,\ldots,n\}$ has $n+1$ elements, so $\binom{n+1}{k}$ equals the number of k-element subsets of A. Such subsets can be divided into two types: those that contain 0 and those that do not contain 0. To make a k-element subset that contains 0 we can start with $\{0\}$ and then append to this set an additional $k-1$ numbers selected from $\{1,2,3,\ldots,n\}$. There are $\binom{n}{k-1}$ ways to make this selection, so there are $\binom{n}{k-1}$ k-element subsets of A that contain 0. Concerning the k-element subsets of A that do not contain 0, there are $\binom{n}{k}$ of these sets, for we can form them by selecting k elements from the n-element set $\{1,2,3,\ldots,n\}$. In light of all this, Equation (3.2) just expresses the obvious fact that the number of k-element subsets of A equals the number of k-element subsets that contain 0 plus the number of k-element subsets that do not contain 0.

$$\begin{array}{c}
\binom{0}{0}\\
\binom{1}{0}\quad\binom{1}{1}\\
\binom{2}{0}\quad\binom{2}{1}\quad\binom{2}{2}\\
\binom{3}{0}\quad\binom{3}{1}\quad\binom{3}{2}\quad\binom{3}{3}\\
\binom{4}{0}\quad\binom{4}{1}\quad\binom{4}{2}\quad\binom{4}{3}\quad\binom{4}{4}\\
\binom{5}{0}\quad\binom{5}{1}\quad\binom{5}{2}\quad\binom{5}{3}\quad\binom{5}{4}\quad\binom{5}{5}\\
\binom{6}{0}\quad\binom{6}{1}\quad\binom{6}{2}\quad\binom{6}{3}\quad\binom{6}{4}\quad\binom{6}{5}\quad\binom{6}{6}\\
\binom{7}{0}\quad\binom{7}{1}\quad\binom{7}{2}\quad\binom{7}{3}\quad\binom{7}{4}\quad\binom{7}{5}\quad\binom{7}{6}\quad\binom{7}{7}\\
\cdot^{\cdot^{\cdot}}\qquad\qquad\qquad\vdots\qquad\qquad\qquad\ddots
\end{array}
\qquad
\begin{array}{c}
1\\
1\quad 1\\
1\quad 2\quad 1\\
1\quad 3\quad 3\quad 1\\
1\quad 4\quad 6\quad 4\quad 1\\
1\quad 5\quad 10\quad 10\quad 5\quad 1\\
1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1\\
1\quad 7\quad 21\quad 35\quad 35\quad 21\quad 7\quad 1\\
\cdot^{\cdot^{\cdot}}\qquad\qquad\qquad\vdots\qquad\qquad\qquad\ddots
\end{array}$$

Figure 3.2. Pascal's triangle

Now that we have seen why Equation (3.2) is true, we are going to arrange the numbers $\binom{n}{k}$ in a triangular pattern that highlights various relationships among them. The left-hand side of Figure 3.2 shows numbers $\binom{n}{k}$ arranged in a pyramid with $\binom{0}{0}$ at the apex, just above a row containing $\binom{1}{k}$ with $k=0$ and $k=1$. Below *this* is a row listing the values of $\binom{2}{k}$ for $k=0,1,2$. In general, each row listing the numbers $\binom{n}{k}$ is just above a row listing the numbers $\binom{n+1}{k}$.

Any number $\binom{n+1}{k}$ for $0 < k < n$ in this pyramid is immediately below and between the the two numbers $\binom{n}{k-1}$ and $\binom{n}{k}$ in the previous row. But Equation 3.2 says $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$, and therefore any number (other than 1) in the pyramid is the sum of the two numbers immediately above it.

This pattern is especially evident on the right of Figure 3.2, where each $\binom{n}{k}$ is worked out. Notice how 21 is the sum of the numbers 6 and 15 above it. Similarly, 5 is the sum of the 1 and 4 above *it* and so on.

The arrangement on the right of Figure 3.2 is called **Pascal's triangle**. (It is named after Blaise Pascal, 1623–1662, a French mathematician and philosopher who discovered many of its properties.) Although we have written only the first eight rows of Pascal's triangle (beginning with Row 0 at the apex), it obviously could be extended downward indefinitely. We could add an additional row at the bottom by placing a 1 at each end and obtaining each remaining number by adding the two numbers above its position. Doing this would give the following row:

$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

This row consists of the numbers $\binom{8}{k}$ for $0 \le k \le 8$, and we have computed them without the formula $\binom{8}{k} = \frac{8!}{k!(8-k)!}$. Any $\binom{n}{k}$ can be computed this way.

The very top row (containing only 1) is called *Row* 0. Row 1 is the next down, followed by Row 2, then Row 3, etc. With this labeling, Row n consists of the numbers $\binom{n}{k}$ for $0 \le k \le n$.

Notice that Row n appears to be a list of the coefficients of $(x+y)^n$. For example $(x+y)^2 = \mathbf{1}x^2 + \mathbf{2}xy + \mathbf{1}y^2$, and Row 2 lists the coefficients 1 2 1. Similarly $(x+y)^3 = \mathbf{1}x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + \mathbf{1}y^3$, and Row 3 is 1 3 3 1. Pascal's triangle is shown on the left of Figure 3.3 and on the right are the expansions of $(x+y)^n$ for $0 \le n \le 5$. In every case (at least as far as you care to check) the numbers in Row n match up with the coefficients of $(x+y)^n$.

$$\begin{array}{ccccccccccc}
&&&&&1&&&&& \\
&&&&1&&1&&&& \\
&&&1&&2&&1&&& \\
&&1&&3&&3&&1&& \\
&1&&4&&6&&4&&1& \\
1&&5&&10&&10&&5&&1 \\
\iddots&&&&&\vdots&&&&&\ddots
\end{array}
\qquad
\begin{array}{c}
1 \\
1x + 1y \\
1x^2 + 2xy + 1y^2 \\
1x^3 + 3x^2y + 3xy^2 + 1y^3 \\
1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 \\
1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5 \\
\iddots \qquad\qquad \vdots \qquad\qquad \ddots
\end{array}$$

Figure 3.3. The n^{th} row of Pascal's triangle lists the coefficients of $(x+y)^n$

In fact this turns out to be true for every n. This result is known as the binomial theorem, and it is worth mentioning here. It tells how to raise a binomial $x+y$ to a non-negative integer power n.

Theorem 3.1 (Binomial Theorem) If n is a non-negative integer, then $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$.

For now we will be content to accept the binomial theorem without proof. (You will be asked to prove it in an exercise in Chapter 10.) You may find it useful from time to time. For instance, you can apply it if you ever need to expand an expression such as $(x+y)^7$. To do this, look at Row 7 of Pascal's triangle in Figure 3.2 and apply the binomial theorem to get

$$(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7.$$

For another example,

$$\begin{aligned}(2a-b)^4 &= ((2a)+(-b))^4 \\ &= (2a)^4 + 4(2a)^3(-b) + 6(2a)^2(-b)^2 + 4(2a)(-b)^3 + (-b)^4 \\ &= 16a^4 - 32a^3b + 24a^2b^2 - 8ab^3 + b^4.\end{aligned}$$

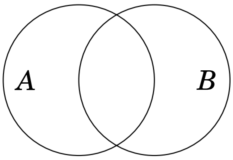
Exercises for Section 3.4

1. Write out Row 11 of Pascal's triangle.
2. Use the binomial theorem to find the coefficient of x^8y^5 in $(x+y)^{13}$.
3. Use the binomial theorem to find the coefficient of x^8 in $(x+2)^{13}$.
4. Use the binomial theorem to find the coefficient of x^6y^3 in $(3x-2y)^9$.
5. Use the binomial theorem to show $\sum_{k=0}^{n}\binom{n}{k} = 2^n$.
6. Use Definition 3.2 (page 74) and Fact 1.3 (page 12) to show $\sum_{k=0}^{n}\binom{n}{k} = 2^n$.
7. Use the binomial theorem to show $\sum_{k=0}^{n} 3^k\binom{n}{k} = 4^n$.
8. Use Fact 3.3 (page 76) to derive Equation 3.2 (page 78).
9. Use the binomial theorem to show $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \cdots + (-1)^n\binom{n}{n} = 0$.
10. Show that the formula $k\binom{n}{k} = n\binom{n-1}{k-1}$ is true for all integers n,k with $0 \le k \le n$.
11. Use the binomial theorem to show $9^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}10^{n-k}$.
12. Show that $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.
13. Show that $\binom{n}{3} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{n-1}{2}$.
14. The first five rows of Pascal's triangle appear in the digits of powers of 11: $11^0 = 1$, $11^1 = 11$, $11^2 = 121$, $11^3 = 1331$ and $11^4 = 14641$. Why is this so? Why does the pattern not continue with 11^5?

3.5 Inclusion-Exclusion

Many counting problems involve computing the cardinality of a union $A \cup B$ of two finite sets. We examine this kind of problem now.

First we develop a formula for $|A \cup B|$. It is tempting to say that $|A \cup B|$ must equal $|A| + |B|$, but that is not quite right. If we count the elements of A and then count the elements of B and add the two figures together, we get $|A| + |B|$. But if A and B have some elements in common, then we have counted each element in $A \cap B$ *twice*.



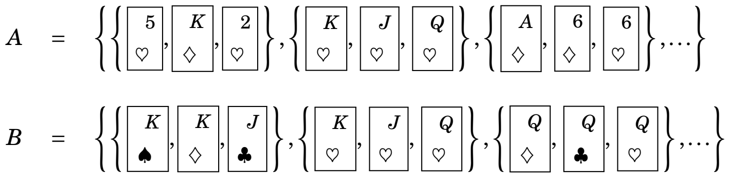
Therefore $|A| + |B|$ exceeds $|A \cup B|$ by $|A \cap B|$, and consequently $|A \cup B| = |A| + |B| - |A \cap B|$. This can be a useful equation.

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{3.3}$$

Notice that the sets A, B and $A \cap B$ are all generally smaller than $A \cup B$, so Equation (3.3) has the potential of reducing the problem of determining $|A \cup B|$ to three simpler counting problems. It is sometimes called an *inclusion-exclusion* formula because elements in $A \cap B$ are included (twice) in $|A|+|B|$, then excluded when $|A \cap B|$ is subtracted. Notice that if $A \cap B = \emptyset$, then we do in fact get $|A \cup B| = |A| + |B|$; conversely if $|A \cup B| = |A| + |B|$, then it must be that $A \cap B = \emptyset$.

Example 3.8 A 3-card hand is dealt off of a standard 52-card deck. How many different such hands are there for which all 3 cards are red or all three cards are face cards?

Solution: Let A be the set of 3-card hands where all three cards are red (i.e., either $\heartsuit$ or $\diamondsuit$). Let B be the set of 3-card hands in which all three cards are face cards (i.e., *J,K* or *Q* of any suit). These sets are illustrated below.



$$A = \left\{ \left\{ \overset{5}{\heartsuit}, \overset{K}{\diamondsuit}, \overset{2}{\heartsuit} \right\}, \left\{ \overset{K}{\heartsuit}, \overset{J}{\heartsuit}, \overset{Q}{\heartsuit} \right\}, \left\{ \overset{A}{\diamondsuit}, \overset{6}{\diamondsuit}, \overset{6}{\heartsuit} \right\}, \ldots \right\} \quad \text{(Red cards)}$$

$$B = \left\{ \left\{ \overset{K}{\spadesuit}, \overset{K}{\diamondsuit}, \overset{J}{\clubsuit} \right\}, \left\{ \overset{K}{\heartsuit}, \overset{J}{\heartsuit}, \overset{Q}{\heartsuit} \right\}, \left\{ \overset{Q}{\diamondsuit}, \overset{Q}{\clubsuit}, \overset{Q}{\heartsuit} \right\}, \ldots \right\} \quad \text{(Face cards)}$$

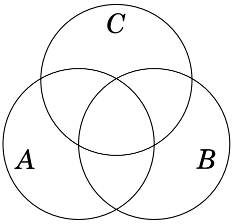
We seek the number of 3-card hands that are all red or all face cards, and this number is $|A\cup B|$. By Formula (3.3), $|A\cup B| = |A|+|B|-|A\cap B|$. Let's examine $|A|, |B|$ and $|A\cap B|$ separately. Any hand in A is formed by selecting three cards from the 26 red cards in the deck, so $|A| = \binom{26}{3}$. Similarly, any hand in B is formed by selecting three cards from the 12 face cards in the deck, so $|B| = \binom{12}{3}$. Now think about $A\cap B$. It contains all the 3-card hands made up of cards that are red face cards.

$$A\cap B \;=\; \left\{ \left\{ \boxed{K \atop \heartsuit}, \boxed{K \atop \diamondsuit}, \boxed{J \atop \heartsuit} \right\}, \left\{ \boxed{K \atop \heartsuit}, \boxed{J \atop \heartsuit}, \boxed{Q \atop \heartsuit} \right\}, \left\{ \boxed{Q \atop \diamondsuit}, \boxed{J \atop \diamondsuit}, \boxed{Q \atop \heartsuit} \right\}, \ldots \right\} \qquad \text{(Red face cards)}$$

The deck has only 6 red face cards, so $|A\cap B| = \binom{6}{3}$.

Now we can answer our question. The number of 3-card hands that are all red or all face cards is $|A\cup B| = |A|+|B|-|A\cap B| = \binom{26}{3}+\binom{12}{3}-\binom{6}{3} = 2600+220-20 = \mathbf{2800}$.

There is an analogue to Equation (3.3) that involves three sets. Consider three sets A, B and C, as represented in the following Venn Diagram.



Using the same kind of reasoning that resulted in Equation (3.3), you can convince yourself that

$$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|. \qquad (3.4)$$

There's probably not much harm in ignoring this one for now, but if you find this kind of thing intriguing you should definitely take a course in combinatorics. (Ask your instructor!)

As we've noted, Equation (3.3) becomes $|A\cup B| = |A|+|B|$ if it happens that $A\cap B = \emptyset$. Also, in Equation (3.4), note that if $A\cap B = \emptyset$, $A\cap C = \emptyset$ and $B\cap C = \emptyset$, we get the simple formula $|A\cup B\cup C| = |A|+|B|+|C|$. In general, we have the following formula for n sets, none of which overlap. It is sometimes called the **addition principle.**

Fact 3.4 (Addition Principle) If $A_1, A_2, \ldots, A_n$ are sets with $A_i\cap A_j = \emptyset$ whenever $i \neq j$, then $|A_1\cup A_2\cup\cdots\cup A_n| = |A_1|+|A_2|+\cdots+|A_n|$.

Example 3.9 How many 7-digit binary strings (0010100, 1101011, etc.) have an odd number of 1's?

Solution: Let A be the set of all 7-digit binary strings with an odd number of 1's, so the answer to the question will be $|A|$. To compute $|A|$, we break A up into smaller parts. Notice any string in A will have either one, three, five or seven 1's. Let A_1 be the set of 7-digit binary strings with only one 1. Let A_3 be the set of 7-digit binary strings with three 1's. Let A_5 be the set of 7-digit binary strings with five 1's, and let A_7 be the set of 7-digit binary strings with seven 1's. Therefore $A = A_1 \cup A_3 \cup A_5 \cup A_7$. Notice that any two of the sets A_i have empty intersection, so Fact 3.4 gives $|A| = |A_1| + |A_3| + |A_5| + |A_7|$.

Now the problem is to find the values of the individual terms of this sum. For instance take A_3, the set of 7-digit binary strings with three 1's. Such a string can be formed by selecting three out of seven positions for the 1's and putting 0's in the other spaces. Therefore $|A_3| = \binom{7}{3}$. Similarly $|A_1| = \binom{7}{1}$, $|A_5| = \binom{7}{5}$, and $|A_7| = \binom{7}{7}$. Finally the answer to our question is $|A| = |A_1| + |A_3| + |A_5| + |A_7| = \binom{7}{1} + \binom{7}{3} + \binom{7}{5} + \binom{7}{7} = 7 + 35 + 21 + 1 = 64$. **There are 64 seven-digit binary strings with an odd number of 1's.**

You may already have been using the Addition Principle intuitively, without thinking of it as a free-standing result. For instance, we used it in Example 3.2(c) when we divided lists into four types and computed the number of lists of each type.

Exercises for Section 3.5

1. At a certain university 523 of the seniors are history majors or math majors (or both). There are 100 senior math majors, and 33 seniors are majoring in both history and math. How many seniors are majoring in history?

2. How many 4-digit positive integers are there for which there are no repeated digits, or for which there may be repeated digits, but all are odd?

3. How many 4-digit positive integers are there that are even or contain no 0's?

4. This problem involves lists made from the letters *T,H,E,O,R,Y*, with repetition allowed.

(**a**) How many 4-letter lists are there that don't begin with T, or don't end in Y?

(**b**) How many 4-letter lists are there in which the sequence of letters *T,H,E* appears consecutively?

(**c**) How many 5-letter lists are there in which the sequence of letters *T,H,E* appears consecutively?

5. How many 7-digit binary strings begin in 1 or end in 1 or have exactly four 1's?
6. Is the following statement true or false? Explain. If $A_1 \cap A_2 \cap A_3 = \emptyset$, then $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$.
7. This problem concerns 4-card hands dealt off of a standard 52-card deck. How many 4-card hands are there for which all 4 cards are of the same suit or all 4 cards are red?
8. This problem concerns 4-card hands dealt off of a standard 52-card deck. How many 4-card hands are there for which all 4 cards are of different suits or all 4 cards are red?
9. A 4-letter list is made from the letters *L,I,S,T,E,D* according to the following rule: Repetition is allowed, and the first two letters on the list are vowels or the list ends in D. How many such lists are possible?
10. A 5-card poker hand is called a *flush* if all cards are the same suit. How many different flushes are there?