CHAPTER 13

Cardinality of Sets

This chapter is all about cardinality of sets. At first this looks like a very simple concept. To find the cardinality of a set, just count its elements. If $A = \{a,b,c,d\}$, then $|A| = 4$; if $B = \{n \in \mathbb{Z} : -5 \le n \le 5\}$, then $|B| = 11$. In this case $|A| < |B|$. What could be simpler than that?

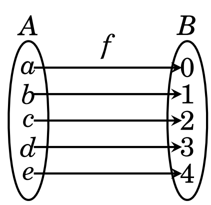
Actually, the idea of cardinality becomes quite subtle when the sets are infinite. The main point of this chapter is to show you that there are numerous different kinds of infinity, and some infinities are bigger than others. Two sets A and B can both have infinite cardinality, yet $|A| < |B|$.

13.1 Sets With Equal Cardinalities

We begin with a discussion of what it means for two sets to have the same cardinality. Up until this point we've said $|A| = |B|$ if A and B have the same number of elements: Count the elements of A, then count the elements of B. If you get the same number, then $|A| = |B|$.

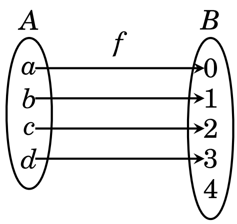
Although this is a fine strategy if the sets are finite (and not too big!), it doesn't apply to infinite sets because we'd never be done counting their elements. We need a new definition of cardinality that applies to both finite and infinite sets. Here it is.

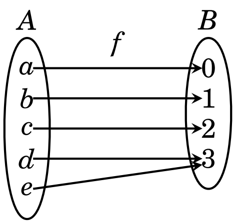
Definition 13.1 Two sets A and B have the **same cardinality**, written $|A| = |B|$, if there exists a bijective function $f : A \to B$. If no such bijective function exists, then A and B have **unequal cardinalities**, that is $|A| \ne |B|$.



The above picture illustrates our definition. There is a bijective function $f : A \to B$, so $|A| = |B|$. Function f matches up A with B. Think of f as describing how to overlay A onto B so that they fit together perfectly.

On the other hand, if A and B are as indicated in either of the following figures, then there can be no bijection $f : A \to B$. (The best we can do is a function that is either injective or surjective, but not both). Therefore the definition says $|A| \neq |B|$ in these cases.





Example 13.1 The sets $A = \{n \in \mathbb{Z} : 0 \leq n \leq 5\}$ and $B = \{n \in \mathbb{Z} : -5 \leq n \leq 0\}$ have the same cardinality because there is a bijective function $f : A \to B$ given by the rule $f(n) = -n$.

Several comments are in order. First, if $|A| = |B|$, there can be *lots* of bijective functions from A to B. We only need to find one of them in order to conclude $|A| = |B|$. Second, as bijective functions play such a big role here, we use the word **bijection** to mean *bijective function*. Thus the function $f(n) = -n$ from Example 13.1 is a bijection. Also, an injective function is called an **injection** and a surjective function is called a **surjection**.

We emphasize and reiterate that Definition 13.1 applies to finite as well as infinite sets. If A and B are infinite, then $|A| = |B|$ provided there exists a bijection $f : A \to B$. If no such bijection exists, then $|A| \neq |B|$.

Example 13.2 This example shows that $|\mathbb{N}| = |\mathbb{Z}|$. To see why this is true, notice that the following table describes a bijection $f : \mathbb{N} \to \mathbb{Z}$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
$f(n)$	0	1	−1	2	−2	3	−3	4	−4	5	−5	6	−6	7	−7	...

Notice that f is described in such a way that it is both injective and surjective. Every integer appears exactly once on the infinitely long second row. Thus, according to the table, given any $b \in \mathbb{Z}$ there is some natural number n with $f(n) = b$, so f is surjective. It is injective because the way the table is constructed forces $f(m) \neq f(n)$ whenever $m \neq n$. Because of this bijection $f : \mathbb{N} \to \mathbb{Z}$, we must conclude from Definition 13.1 that $|\mathbb{N}| = |\mathbb{Z}|$.

You may find Example 13.2 slightly unsettling. On one hand it makes sense that $|\mathbb{N}| = |\mathbb{Z}|$ because $\mathbb{N}$ and $\mathbb{Z}$ are both infinite, so their cardinalities are both "infinity." On the other hand, $\mathbb{Z}$ seems twice as large as $\mathbb{N}$ because

$\mathbb{Z}$ has all the negative integers as well as the positive ones. Definition 13.1 settles the issue by producing $|\mathbb{N}| = |\mathbb{Z}|$. We summarize this with a theorem.

Theorem 13.1 There exists a bijection $f : \mathbb{N} \to \mathbb{Z}$. Therefore $|\mathbb{N}| = |\mathbb{Z}|$.

The fact that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality might prompt us to ask if other pairs of infinite sets have the same cardinality. How, for example, do $\mathbb{N}$ and $\mathbb{R}$ compare? Let's turn our attention to this issue.

In fact, $|\mathbb{N}| \neq |\mathbb{R}|$. This was first recognized by G. Cantor (1845–1918), who devised an ingenious argument to show that there are no surjective functions $f : \mathbb{N} \to \mathbb{R}$, which implies there are no bijections $f : \mathbb{N} \to \mathbb{R}$, so $|\mathbb{N}| \neq |\mathbb{R}|$ by Definition 13.1.

We will now describe Cantor's argument for why there are no surjections $f : \mathbb{N} \to \mathbb{R}$. We will reason informally, rather than writing out an exact proof. Take any arbitrary function $f : \mathbb{N} \to \mathbb{R}$. The following reasoning shows why f can't be surjective.

Imagine making a table for f, where values of n in $\mathbb{N}$ are in the left-hand column and the corresponding values $f(n)$ are on the right. The first few entries might look something as follows. In this table, the real numbers $f(n)$ are written with all their decimal places trailing off to the right. Thus, even though $f(1)$ happens to be the real number 0.4, we write it as 0.40000000…., etc.

n	$f(n)$
1	0 . 4 0 0 0 0 0 0 0 0 0 0 0 0 0...
2	8 . 5 6 0 6 0 7 0 8 6 6 6 9 0 0...
3	7 . 5 0 5 0 0 9 4 0 0 4 4 1 0 1...
4	5 . 5 0 7 0 4 0 0 8 0 4 8 0 5 0...
5	6 . 9 0 0 2 6 0 0 0 0 0 0 5 0 6...
6	6 . 8 2 8 0 9 5 8 2 0 5 0 0 2 0...
7	6 . 5 0 5 0 5 5 5 0 6 5 5 8 0 8...
8	8 . 7 2 0 8 0 6 4 0 0 0 0 4 4 8...
9	0 . 5 5 0 0 0 0 8 8 8 8 0 0 7 7...
10	0 . 5 0 0 2 0 7 2 2 0 7 8 0 5 1...
11	2 . 9 0 0 0 0 8 8 0 0 0 0 9 0 0...
12	6 . 5 0 2 8 0 0 0 8 0 0 9 6 7 1...
13	8 . 8 9 0 0 8 0 2 4 0 0 8 0 5 0...
14	8 . 5 0 0 0 8 7 4 2 0 8 0 2 2 6...
⋮	⋮

There is a diagonal shaded band in the table. For each $n \in \mathbb{N}$, this band covers the n^{th} decimal place of $f(n)$:

The 1^{st} decimal place of $f(1)$ is 4, and it's shaded.
The 2^{nd} decimal place of $f(2)$ is 6, and it's shaded.
The 3^{rd} decimal place of $f(3)$ is 5, and it's shaded.
The 4^{th} decimal place of $f(4)$ is 0, and it's shaded, etc.

This shaded diagonal shows why f cannot be surjective, for it implies that there is a real number b that does not equal any $f(n)$. Just let $b \in \mathbb{R}$ be a number whose n^{th} decimal place always differs from the n^{th} decimal place of $f(n)$. For definiteness, let's define b to be the number between 0 and 1 whose n^{th} decimal place is 6 if the n^{th} decimal place of $f(n)$ is not 6, and whose n^{th} decimal place is 2 if the n^{th} decimal place of $f(n)$ *is* 6. Thus, for the function f illustrated in the above table, we have

$$b = 0.62662666666262\ldots$$

and b has been defined so that, for any $n \in \mathbb{N}$, the n^{th} decimal place of b is unequal to the n^{th} decimal place of $f(n)$. Therefore $f(n) \neq b$ for every natural number n, meaning f is not surjective.

Since this argument applies to *any* function $f : \mathbb{N} \to \mathbb{R}$ (not just the one in the above example) we conclude that there exist no bijections $f : \mathbb{N} \to \mathbb{R}$, so $|\mathbb{N}| \neq |\mathbb{R}|$ by Definition 13.1. We summarize this as a theorem.

Theorem 13.2 There exist no bijections $f : \mathbb{N} \to \mathbb{R}$. Therefore $|\mathbb{N}| \neq |\mathbb{R}|$.

Exercises for Section 13.1

Show that the two given sets have equal cardinality by describing a bijection from one to the other. Describe your bijection with a formula (i.e. not as a table).

1. $\mathbb{R}$ and $(0,\infty)$
2. $\mathbb{R}$ and $(\sqrt{2},\infty)$
3. $\mathbb{R}$ and $(0,1)$
4. The set of even integers and the set of odd integers
5. $A = \{3k : k \in \mathbb{Z}\}$ and $B = \{7k : k \in \mathbb{Z}\}$
6. $\mathbb{N}$ and $S = \{\frac{\sqrt{2}}{n} : n \in \mathbb{N}\}$
7. $\mathbb{Z}$ and $S = \{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \ldots\}$
8. $\mathbb{Z}$ and $S = \{x \in \mathbb{R} : \sin x = 1\}$
9. $\{0,1\} \times \mathbb{N}$ and $\mathbb{N}$
10. $\{0,1\} \times \mathbb{N}$ and $\mathbb{Z}$

13.2 Countable and Uncountable Sets

Let's summarize the main points from the previous section.

1. $|A| = |B|$ if and only if there exists a bijection $f : A \to B$.
2. $|\mathbb{N}| = |\mathbb{Z}|$ because there exists a bijection $f : \mathbb{N} \to \mathbb{Z}$.
3. $|\mathbb{N}| \neq |\mathbb{R}|$ because there exists **no** bijection $f : \mathbb{N} \to \mathbb{R}$.

Thus, even though $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ are all infinite sets, their cardinalities are not all the same. Sets $\mathbb{N}$, $\mathbb{Z}$ have the same cardinality, but $\mathbb{R}$'s cardinality is different from that of both the other sets. This is our first indication of how infinite sets can have different sizes, and we will now make some definitions to put words and symbols to this phenomenon.

In a certain sense you can count the elements of $\mathbb{N}$; you can count its elements off as $1, 2, 3, 4, \ldots$, but you'd have to continue this process forever to count the whole set. Thus we will call $\mathbb{N}$ a *countably infinite set*, and the same term is used for any set whose cardinality equals that of $\mathbb{N}$.

Definition 13.2 Suppose A is a set. Then A is **countably infinite** if $|\mathbb{N}| = |A|$, that is if there exists a bijection $f : \mathbb{N} \to A$. The set A is **uncountable** if A is infinite and $|\mathbb{N}| \neq |A|$, that is, if A is infinite and there exist *no* bijections $f : \mathbb{N} \to A$.

Thus $\mathbb{Z}$ is countably infinite but $\mathbb{R}$ is uncountable. This section deals mainly with countably infinite sets. Uncountable sets are treated later.

If A is countably infinite, then $|\mathbb{N}| = |A|$, so there is a bijection $f : \mathbb{N} \to A$. You can think of f as "counting" the elements of A. The first element of A is $f(1)$, followed by $f(2)$, then $f(3)$, and so on. It makes sense to think of a countably infinite set as the smallest type of infinite set, because if the counting process stopped, the set would be finite, not infinite; a countably infinite set has the fewest number of elements that a set can have and still be infinite. It is common to reserve the special symbol $\aleph_0$ to stand for the cardinality of countably infinite sets.

Definition 13.3 The cardinality of the natural numbers is denoted as $|\mathbb{N}| = \aleph_0$. Thus any countably infinite set has cardinality $\aleph_0$.

(The symbol $\aleph$ is the first letter in the Hebrew alphabet, and is pronounced "aleph." The symbol $\aleph_0$ is pronounced "aleph naught.") The summary of facts at the beginning of this section shows $|\mathbb{Z}| = \aleph_0$ and $|\mathbb{R}| \neq \aleph_0$.

Example 13.3 Let $E = \{2k : k \in \mathbb{Z}\}$ be the set of even integers. The function $f : \mathbb{Z} \to E$ defined as $f(n) = 2n$ is easily seen to be a bijection, so we have $|\mathbb{Z}| = |E|$. Thus, as $|\mathbb{N}| = |\mathbb{Z}| = |E|$, the set E is countably infinite and $|E| = \aleph_0$.

Here is a significant fact. The elements of any countably infinite set A can be written in an infinitely long list $a_1, a_2, a_3, a_4, \dots$ that begins with some element $a_1 \in A$ and includes every element of A. For example, the set E in the above example can be written in list form as $0, 2, -2, 4, -4, 6, -6, 8, -8, \dots$. The reason that this can be done is as follows. Since A is countably infinite, Definition 13.2 says there is a bijection $f : \mathbb{N} \to A$. This allows us to list out the set A as an infinite list $f(1), f(2), f(3), f(4), \dots$. Conversely, if the elements of a A can be written in list form as $a_1, a_2, a_3, \dots$, then the function $f : \mathbb{N} \to A$ defined as $f(n) = a_n$ is a bijection, so A is countably infinite. We summarize this as follows.

Theorem 13.3 A set A is countably infinite if and only if its elements can be arranged in an infinite list $a_1, a_2, a_3, a_4, \dots$.

As an example of how this theorem might be used, let P denote the set of all prime numbers. Since we can list its elements as $2, 3, 5, 7, 11, 13, \dots$, it follows that the set P is countably infinite.

As another consequence of Theorem 13.3, note that we can interpret the fact that the set $\mathbb{R}$ is not countably infinite as meaning that it is impossible to write out all the elements of $\mathbb{R}$ in an infinite list.

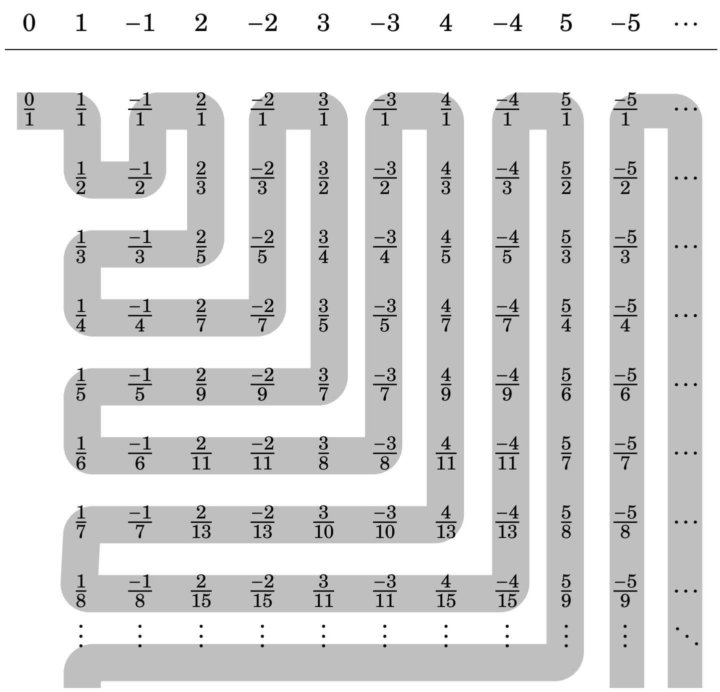
This begs a question. Is it also impossible to write out all the elements of $\mathbb{Q}$ in an infinite list? In other words, is the set $\mathbb{Q}$ of rational numbers countably infinite or uncountable? If you start plotting the rational numbers on the number line, they seem to mostly fill up $\mathbb{R}$. Sure, some numbers such as $\sqrt{2}$, π and e will not be plotted, but the dots representing rational numbers seem to predominate. We might thus expect $\mathbb{Q}$ to be uncountable. However it is a surprising fact that $\mathbb{Q}$ is countable. The proof of this fact works by showing how to write out all the rational numbers in an infinitely long list.

Theorem 13.4 The set $\mathbb{Q}$ of rational numbers is countable.

Proof. To prove this, we just need to show how to write the set $\mathbb{Q}$ in list form. Begin by arranging all rational numbers in an infinite array. This is done by making the following chart. The top row has a list of all integers, beginning with 0, then alternating signs as they increase. Each column headed by an integer k contains all the fractions (in reduced form) whose numerator is k. For example, the column headed by 2 contains the fractions $\frac{2}{1}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \dots$. It does not contain $\frac{2}{2}$, $\frac{2}{4}$, $\frac{2}{6}$, and so on, because those fractions are not reduced, and in fact their reduced forms appear in the column headed by 1. You should examine this table and convince yourself that it contains all rational numbers in $\mathbb{Q}$.

0	1	−1	2	−2	3	−3	4	−4	5	−5	⋯
$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$	$\frac{5}{1}$	$\frac{-5}{1}$	⋯
	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{3}$	$\frac{-4}{3}$	$\frac{5}{2}$	$\frac{-5}{2}$	⋯
	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{5}$	$\frac{-4}{5}$	$\frac{5}{3}$	$\frac{-5}{3}$	⋯
	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{7}$	$\frac{-2}{7}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{7}$	$\frac{-4}{7}$	$\frac{5}{4}$	$\frac{-5}{4}$	⋯
	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{9}$	$\frac{-2}{9}$	$\frac{3}{7}$	$\frac{-3}{7}$	$\frac{4}{9}$	$\frac{-4}{9}$	$\frac{5}{6}$	$\frac{-5}{6}$	⋯
	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{11}$	$\frac{-2}{11}$	$\frac{3}{8}$	$\frac{-3}{8}$	$\frac{4}{11}$	$\frac{-4}{11}$	$\frac{5}{7}$	$\frac{-5}{7}$	⋯
	$\frac{1}{7}$	$\frac{-1}{7}$	$\frac{2}{13}$	$\frac{-2}{13}$	$\frac{3}{10}$	$\frac{-3}{10}$	$\frac{4}{13}$	$\frac{-4}{13}$	$\frac{5}{8}$	$\frac{-5}{8}$	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Next, draw an infinite path in this array, beginning at $\frac{0}{1}$ and snaking back and forth as indicated below. Every rational number is on this path.



Beginning at $\frac{0}{1}$ and following the path, we get an infinite list of all rational numbers:

$$0, 1, \frac{1}{2}, -\frac{1}{2}, -1, 2, \frac{2}{3}, \frac{2}{5}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{2}{7}, -\frac{2}{7}, -\frac{2}{5}, -\frac{2}{3}, -\frac{2}{3}, -2, 3, \frac{3}{2}, \ldots$$

Therefore, by Theorem 13.3, it follows that $\mathbb{Q}$ is countably infinite. ■

It is also true that the Cartesian product of two countably infinite sets is itself countably infinite, as our next theorem states.

Theorem 13.5 If A and B are both countably infinite, then $A \times B$ is countably infinite.

Proof. Suppose A and B are both countably infinite. By Theorem 13.3, we know we can write A and B in list form as

$$\begin{aligned} A &= \{a_1, a_2, a_3, a_4, \ldots\}, \\ B &= \{b_1, b_2, b_3, b_4, \ldots\}. \end{aligned}$$

Figure 13.1 shows how to form an infinite path winding through all of $A \times B$. Therefore $A \times B$ can be written in list form, so it is countably infinite. ■

As an example of a consequence of this theorem, notice that since $\mathbb{Q}$ is countably infinite, the set $\mathbb{Q} \times \mathbb{Q}$ is also countably infinite.

Recall that the word "corollary" means a result that follows easily from some other result. We have the following corollary of Theorem 13.5.

Corollary 13.1 Given n countably infinite sets $A_1, A_2, A_3, \ldots, A_n$, with $n \geq 2$, the Cartesian product $A_1 \times A_2 \times A_3 \times \cdots \times A_n$ is also countably infinite.

Proof. The proof is by induction. For the basis step, notice that when $n = 2$ the statement asserts that for countably infinite sets A_1 and A_2, the product $A_1 \times A_2$ is countably infinite, and this is true by Theorem 13.5.

Now assume that for $k \geq 2$, any product $A_1 \times A_2 \times A_3 \times \cdots \times A_k$ of countably infinite sets is countably infinite. Now consider a product $A_1 \times A_2 \times A_3 \times \cdots \times A_{k+1}$ of countably infinite sets. Observe that

$$\begin{aligned} A_1 \times A_2 \times A_3 \times \cdots \times A_{k+1} &= A_1 \times A_2 \times A_3 \times \cdots \times A_k \times A_{k+1} \\ &= (A_1 \times A_2 \times A_3 \times \cdots \times A_k) \times A_{k+1}. \end{aligned}$$

By the induction hypothesis, this is a product of two countably infinite sets, so it is countably infinite by Theorem 13.5. ■

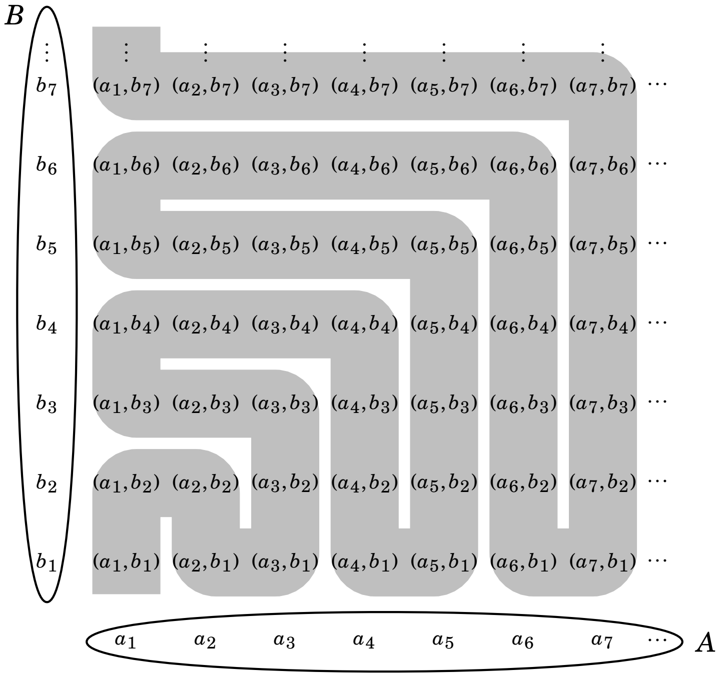


Figure 13.1. The product of countably infinite sets is countably infinite

Theorem 13.6 If A and B are both countably infinite, then $A \cup B$ is countably infinite.

Proof. Suppose A and B are both countably infinite. By Theorem 13.3, we know we can write A and B in list form as

$$\begin{aligned} A &= \{a_1, a_2, a_3, a_4, \ldots\}, \\ B &= \{b_1, b_2, b_3, b_4, \ldots\}. \end{aligned}$$

We can "shuffle" A and B into one infinite list for $A \cup B$ as follows.

$$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, \ldots\}.$$

(We agree not to list an element twice if it belongs to both A and B.) Therefore, by Theorem 13.3, it follows that $A \cup B$ is countably infinite. ■

Exercises for Section 13.2

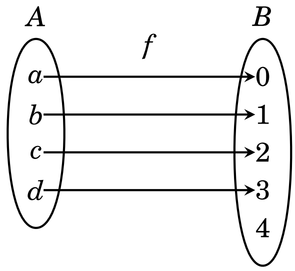
1. Prove that the set $A = \{\ln(n) : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is countably infinite.
2. Prove that the set $A = \{(m,n) \in \mathbb{N} \times \mathbb{N} : m \leq n\}$ is countably infinite.
3. Prove that the set $A = \{(5n,-3n) : n \in \mathbb{Z}\}$ is countably infinite.
4. Prove that the set of all irrational numbers is uncountable. (Suggestion: Consider proof by contradiction using theorems 13.4 and 13.6.)
5. Prove or disprove: There exists a countably infinite subset of the set of irrational numbers.
6. Prove or disprove: There exists a bijective function $f : \mathbb{Q} \to \mathbb{R}$.
7. Prove or disprove: The set $\mathbb{Q}^{100}$ is countably infinite.
8. Prove or disprove: The set $\mathbb{Z} \times \mathbb{Q}$ is countably infinite.
9. Prove or disprove: The set $\{0,1\} \times \mathbb{N}$ is countably infinite.
10. Prove or disprove: The set $A = \{\frac{\sqrt{2}}{n} : n \in \mathbb{N}\}$ countably infinite.
11. Describe a partition of $\mathbb{N}$ that divides $\mathbb{N}$ into eight countably infinite subsets.
12. Describe a partition of $\mathbb{N}$ that divides $\mathbb{N}$ into $\aleph_0$ countably infinite subsets.
13. Prove or disprove: If $A = \{X \subseteq \mathbb{N} : X \text{ is finite}\}$, then $|A| = \aleph_0$.
14. Suppose $A = \{(m,n) \in \mathbb{N} \times \mathbb{R} : n = \pi m\}$. Is it true that $|\mathbb{N}| = |A|$?

13.3 Comparing Cardinalities

At this point we know that there are at least two different kinds of infinity. On one hand, there are countably infinite sets such as $\mathbb{N}$, of cardinality $\aleph_0$. Then there is the uncountable set $\mathbb{R}$. Are there other kinds of infinity beyond these two kinds? The answer is "yes," but to see why we first need to introduce some new definitions and theorems.

Our first task will be to formulate a definition for what we mean by $|A| < |B|$. Of course if A and B are finite we know exactly what this means: $|A| < |B|$ means that when the elements of A and B are counted, A is found to have fewer elements than B. But this process breaks down if A or B is infinite, for then the elements can't be counted.

The language of functions helps us overcome this difficulty. Notice that for finite sets A and B it is intuitively clear that $|A| < |B|$ if and only if there exists an injective function $f : A \to B$ but there are no surjective functions $f : A \to B$. The following diagram illustrates this.



We will use this idea to define what is meant by $|A| < |B|$. For emphasis, the following definition also restates what is meant by $|A| = |B|$.

Definition 13.4 Suppose A and B are sets.
(1) $|A| = |B|$ means there is a bijective function $f : A \to B$.
(2) $|A| < |B|$ means there is an injective function $f : A \to B$, but no surjective $f : A \to B$.

For example, consider $\mathbb{N}$ and $\mathbb{R}$. The function $f : \mathbb{N} \to \mathbb{R}$ defined as $f(n) = n$ is clearly injective, but it is not surjective because given the element $\frac{1}{2} \in \mathbb{R}$, we have$f(n) \neq \frac{1}{2}$ for every $n \in \mathbb{N}$. In fact, recall that we proved in Section 13.1 that there exist no surjective functions $\mathbb{N} \to \mathbb{R}$. Therefore Definition 13.4 implies $|\mathbb{N}| < |\mathbb{R}|$. Said differently, $\aleph_0 < |\mathbb{R}|$.

Is there a set X for which $|\mathbb{R}| < |X|$? The answer is "yes," and the next theorem is a major key in understanding why. Recall that $\mathscr{P}(A)$ denotes the power set of A.

Theorem 13.7 If A is any set, then $|A| < |\mathscr{P}(A)|$.

Proof. Before beginning the proof, we remark that this statement is obvious if A is finite, for then $|A| < 2^{|A|} = |\mathscr{P}(A)|$. But our proof must apply to *all* sets A, both finite and infinite, so it must use Definition 13.4.

We will prove the theorem with direct proof. Suppose A is an arbitrary set. According to Definition 13.4, to prove $|A| < |\mathscr{P}(A)|$ we must show that there exists an injective function $f : A \to \mathscr{P}(A)$, but that there exist no surjective functions $f : A \to \mathscr{P}(A)$.

To see that there is an injective $f : A \to \mathscr{P}(A)$, define f by the rule $f(x) = \{x\}$. In words, f sends any element x of A to the one-element set $\{x\} \in \mathscr{P}(A)$. Then $f : A \to \mathscr{P}(A)$ is injective, because if $f(x) = f(y)$, then $\{x\} = \{y\}$. Now, the only way that $\{x\}$ and $\{y\}$ can be equal is if $x = y$, so it follows that $x = y$. Thus f is injective.

Next we need to show that there exist no surjections $f : A \to \mathscr{P}(A)$. Suppose for the sake of contradiction that there does exist a surjective function $f : A \to \mathscr{P}(A)$. Notice that for any element $x \in A$, we have $f(x) \in$

$\mathscr{P}(A)$, so $f(x)$ is a subset of A. Thus f is a function that sends elements of A to subsets of A. It follows that for any $x \in A$, either x is an element of the subset $f(x)$ or it is not. We use this idea to define the following subset B of A.

$$B = \{x \in A : x \notin f(x)\} \subseteq A$$

Now since $B \subseteq A$ we have $B \in \mathscr{P}(A)$, and since f is surjective there must be some element $a \in A$ for which $f(a) = B$. Now, either $a \in B$ or $a \notin B$. We will consider these two cases separately, and show that each leads to a contradiction.

Case 1. If $a \in B$, then the definition of B implies $a \notin f(a)$, and since $f(a) = B$ we have $a \notin B$, which is a contradiction.
Case 2. If $a \notin B$, then the definition of B implies $a \in f(a)$, and since $f(a) = B$ we have $a \in B$, again a contradiction.

Since the assumption that there is a surjective function $f : A \to \mathscr{P}(A)$ leads to a contradiction, we conclude that there are no such surjective functions.

In conclusion, we have seen that there exists an injective function $A \to \mathscr{P}(A)$ but no surjective function $A \to \mathscr{P}(A)$, so definition 13.4 implies that $|A| < |\mathscr{P}(A)|$. ■

Beginning with the set the set $A = \mathbb{N}$ and applying Theorem 13.7 over and over again, we get the following chain of infinite cardinalities.

$$\aleph_0 = |\mathbb{N}| < |\mathscr{P}(\mathbb{N})| < |\mathscr{P}(\mathscr{P}(\mathbb{N}))| < |\mathscr{P}(\mathscr{P}(\mathscr{P}(\mathbb{N})))| < \cdots$$

Thus there is an infinite sequence of different types of infinity, starting with $\aleph_0$ and becoming ever larger. The set $\mathbb{N}$ is countable, and all the sets $\mathscr{P}(\mathbb{N})$, $\mathscr{P}(\mathscr{P}(\mathbb{N}))$, etc. are uncountable.

Although we shall not do it here, it is not hard to prove that $|\mathbb{R}| = |\mathscr{P}(\mathbb{N})|$, so $|\mathbb{N}|$ and $|\mathbb{R}|$ are just two relatively tame infinities in a long list of other wild and exotic infinities.

Unless you plan on studying advanced set theory or the foundations of mathematics, you are unlikely to ever encounter any types of infinity beyond $\aleph_0$ and $|\mathbb{R}|$. Still you will in future mathematics courses need to distinguish between countably infinite and uncountable sets, so we close with two final theorems that can help you do this.

Theorem 13.8 If A is an infinite subset of a countably infinite set, then A is countably infinite.

Proof. Suppose A is an infinite subset of the countably infinite set B. Since B is countably infinite, its elements can be written in a list $b_1, b_2, b_3, b_4, \ldots$. Then we can also write A's elements in list form by proceeding through the elements of B, in order, and selecting those that belong to A. Thus A can be written in list form, and since A is infinite, its list will be infinite. Consequently A is countably infinite. ■

Theorem 13.9 If $U \subseteq A$, and U is uncountable, then A is uncountable too.

Proof. Suppose for the sake of contradiction that $U \subseteq A$, and U is uncountable but A is not uncountable. Then since $U \subseteq A$ and U is infinite, then A must be infinite too. Since A is infinite, and not uncountable, it must be countably infinite. Then U is an infinite subset of a countably infinite set A, so U is countably infinite by Theorem 13.8. Thus U is both uncountable and countably infinite, a contradiction. ■

Theorems 13.8 and 13.9 are often useful when we need to decide whether a set A is countably infinite or uncountable. The theorems sometimes allow us to decide its cardinality by comparing it to a set whose cardinality is known.

For example, suppose we want to decide whether or not the set $A = \mathbb{R}^2$ is uncountable. Since the uncountable set $U = \mathbb{R}$ can be regarded as the x-axis of the plane $\mathbb{R}^2$ (and thus a subset of $\mathbb{R}^2$), Theorem 13.9 implies that $\mathbb{R}^2$ is uncountable. Other examples can be found in the exercises.

Exercises for Section 13.3

1. Suppose B is an uncountable set and A is a set. Given that there is a surjective function $f : A \to B$, what can be said about the cardinality of A?
2. Prove that the set $\mathbb{C}$ of complex numbers is uncountable.
3. Prove or disprove: If A is uncountable, then $|A| = |\mathbb{R}|$.
4. Prove or disprove: If $A \subseteq B \subseteq C$ and A and C are countably infinite, then B is countably infinite.
5. Prove or disprove: The set $\{0,1\} \times \mathbb{R}$ is uncountable.
6. Prove or disprove: Every infinite set is a subset of a countably infinite set.
7. Prove or disprove: If $A \subseteq B$ and A is countably infinite and B is uncountable, then $B - A$ is uncountable.

Conclusion

If you have internalized the ideas in this book, then you have a set of rhetorical tools for deciphering and communicating mathematics. These tools are indispensable at any level. But of course it takes more than mere tools to build something. Planning, creativity, inspiration, skill, talent and passion are also vitally important. It is safe to say that if you have come this far, then you probably possess a sufficient measure of these traits.

The quest to understand mathematics has no end, but you are well equipped for the journey. It is my hope that the things you have learned from this book will lead you to a higher plane of understanding, creativity and expression.

Good luck and best wishes.

R.H