1 The View from 1915 (or thereabouts)

1.1 Recovering the Old Formulas

Einstein’s Theory of Gravitation came into being before many of the tools of modern differential geometry. One way to appreciate what these tools have done for us is to work our way back to the point of view that prevailed in the early days of Einstein’s theory.

First of all, forget about metricity and torsion — they are set to zero. Second, assume that the only basis vectors are the holonomic ones associated with a coordinate system so that (in our modern notation)

\[ e_\alpha = \partial_\alpha = \frac{\partial}{\partial x^\alpha}. \]

All of these basis vectors commute, so the commutation coefficients \( c^\beta_{\alpha\beta} \) all vanish and the expression for the connection coefficients in terms of the metric tensor becomes

\[ \Gamma^\alpha_{\beta\delta} = \frac{1}{2} g^{\alpha\rho} \left( \partial_\beta g_{\rho\delta} + \partial_\delta g_{\rho\beta} - \partial_\rho g_{\beta\delta} \right). \]  

(1)

Older books used a funny curly bracket notation for these coefficients and called them “Christoffel symbols”.

Often the curvature tensor was introduced by just giving an expression for the curvature components in terms of the connection coefficients

\[ R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\rho\mu} \Gamma^\rho_{\beta\nu} - \Gamma^\alpha_{\rho\nu} \Gamma^\rho_{\beta\mu}. \]  

(2)

and observing that these transform like the components of a fourth rank tensor.

1.2 Guess the Metric Tensor

From the “Classic” point of view, Einstein’s equations are a set of partial differential equations for the ten independent functions \( g_{\mu\nu} \). Much of our accumulated wisdom about solving differential equations boils down to “guess a solution with some adjustable things (functions, constants, etc.), substitute it into the equation and see if it works.” Thus, the natural approach to this odd new set of differential equations was to start guessing simple forms for the metric tensor and the stress-energy tensor. For each guess, calculate the connection coefficients and then the curvature components, and then the Ricci tensor, and then the curvature scalar, and finally the Einstein tensor. Set the resulting Einstein tensor equal to the assumed stress-energy tensor and try to solve for the adjustable functions.

This procedure has two limitations: (1) It is not easy to decide what stress-energy tensor represents reasonable matter. (2) Many metric guesses turn out to represent very unphysical situations. A constructive way to deal with the first problem is to look for vacuum solutions where the stress-energy is zero or to look for solutions in which the stress-energy takes the well-known form of an isotropic perfect fluid. Moving beyond these situations to model real matter
with viscosity, finite heat conductivity, or worse yet, solids with various elastic properties becomes very messy. The second limitation cannot be dealt with by the pure guessing approach. One must understand something about the global geometry of the spacetimes that a particular "guess" describes.

1.3 Symmetries of the Metric Tensor

1.3.1 Isometries

The most successful way of guessing metric tensors that correspond to meaningful classes of spacetimes is to impose symmetries. A motion that preserves the metric tensor is called an isometry. A motion is described by a family of curves that fill the spacetime. It may also be described by giving the vector field \( k \) tangent to this family. This family of curves defines a map of the spacetime into itself — pick up each point and slide it along the curve that goes through it by a given amount of parameter-value. Any structure at one point — vector, tensor, whatever — is slid along by this map.

The derivative obtained by comparing a vector or tensor field after such a motion to the one before is called the Lie derivative. For a vector field, \( u \) and a connection with zero torsion, it is given by

\[
\mathcal{L}_k u = [k, u] = \nabla_k u - \nabla_u k
\]

or, in components

\[
\mathcal{L}_k u^\alpha = k^\delta \nabla_{\epsilon^\delta} u^\alpha - u^\rho \nabla_{\epsilon^\rho} k^\alpha = k^\delta u^\alpha_{\gamma\delta} - u^\rho k^\alpha_{,\rho}.
\]

Notice that this expression has a structure similar to that of a covariant derivative with the tensor \( k^{\alpha,\rho} \) "correcting" the index on the vector. Similarly, because the Lie derivative obeys the product rule, one finds similar corrections with the opposite sign for covariant (form) indexes and the Lie derivative of a second-rank covariant tensor is given by

\[
\mathcal{L}_k m_{\alpha\beta} = k^\delta m_{\alpha\beta,\delta} + m_{\rho\beta} k^\alpha_{,\alpha} + m_{\alpha\rho} k^\alpha_{,\beta}.
\]
For the Lie derivative of the metric tensor (using metric compatibility) one finds
\[ \mathcal{L}_k g_{\alpha\beta} = g_{\rho\beta} k^\rho ;\alpha + g_{\alpha\rho} k^\rho ;\beta = k_{\beta;\alpha} + k_{\alpha;\beta}. \]

Thus, if the motion described by the vector-field \( k \) is an isometry, the Lie derivative of the metric vanishes and the result is *Killing’s Equation*
\[ k_{\beta;\alpha} + k_{\alpha;\beta} = 0. \]

A vector which obeys this equation is called a *Killing vector*.

### 1.3.2 Stationary and Static Spacetimes

A simple example of a spacetime with a Killing vector is a *stationary spacetime*. Such a spacetime has a timelike Killing vector. To simplify the spacetime metric components in such a spacetime, choose a coordinate system such that the Killing vector is just \( \partial_0 \). In such a coordinate system, the most general stationary metric has components \( g_{\mu\nu}(x^i) \) which do not depend on the time coordinate \( x^0 \). It may or may not be possible to find spacelike hypersurfaces which are orthogonal to such a Killing vector field. If it is possible, then the coordinates can be chosen so that \( g_{0i} = 0 \) and the spacetime is said to be *static*.

### 1.3.3 Isometry group orbits

When there are several independent Killing vector fields, the situation can become more complex. An essential concept is the idea of an isometry group orbit. This is the set of all spacetime points which can result from carrying out all of the motions on a single point. For a stationary spacetime, the group orbits are all timelike curves.

When all of the Killing vectors commute, the situation is still fairly simple and the group orbits have the same dimensionality as the number of independent Killing vector fields. For example, a stationary, axisymmetric spacetime has two Killing vectors which can be represented in an adapted coordinate system.
The group orbits are then the two-dimensional surfaces at constant $r$ and constant $\theta$ in this coordinate system.

When the Killing vectors do not commute, then it becomes possible for the group orbits to have lower dimensionality than the group does. The most frequently encountered example is spherical symmetry. The spherical symmetry group is generated by three Killing vector fields with the commutation relations

$$[\eta_x, \eta_y] = \eta_z \quad (& \text{cyclic})$$

and at least one point which is not moved by any operation of the group. In a Cartesian coordinate system $(x, y, z)$ which is adapted to the symmetry, each vector field corresponds to a rotation about one of the coordinate axes and the vector fields are represented by

$$\eta_x = y\partial_z - z\partial_y \quad (& \text{cyclic}).$$

The orbits of this isometry group are two-dimensional spheres.

It is easy to see that a curve perpendicular to the group orbits will have extremal length which makes it a geodesic. The observation that these orbit-orthogonal curves are geodesics is important because it gives us a complete set of preferred basis vectors at each point, one set are killing vectors tangent to the group orbit and the rest are tangents to geodesics.
1.3.4 Homogeneity

If the isometry group orbits are spacelike three-surfaces, then the spacetime is said to be spatially homogeneous. More generally, if all of the points of a manifold can be connected by operations of a group, then the manifold is said to be homogeneous under the group. An obvious but key observation is that every group orbit is homogeneous under the group. This observation turns out to be extremely useful because there are only a few homogeneous metric spaces of low dimension and they are very well understood. For example, a connected two-dimensional group orbit of finite area (2-volume?) can always be regarded as either a torus or a 2-sphere. The only other homogeneous two-dimensional metric spaces (the Klein bottle and the projective geometry) are covered by these.

2 The Spacetime of a Static Isolated Star

2.1 Form of the metric tensor

Suppose we have a star that is not rotating or changing in any way. The star has no preferred orientation and we assume that it is alone in the universe, so the spacetime geometry must also have no preferred orientation. Under these conditions it is reasonable to expect the spacetime geometry to be spherically symmetric — there is an isometry group whose orbits are homogeneous 2-spheres. Use the usual spherical coordinates on the 2-sphere and note that the metric on the 2-sphere is

\[ ds^2 = R^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \]

where the scale parameter \( R \) determines the area of the sphere

\[ A = 4\pi R^2. \]

For a 2-sphere embedded in a flat space, \( R \) is the radius of the sphere. In a more general situation \( R \) will not necessarily be simply related to the radius but it will always be simply related to the area.

Because the situation is static, it can be reflected in time, which means that we can choose coordinates such that the metric components \( g_{0i} = 0 \) where \( \partial_0 = \frac{\partial}{\partial t} \) is the timelike Killing vector field. In other words, there is a family of spacelike hypersurfaces orthogonal to the timelike Killing vector field. Each
spacelike hypersurface can be labeled by a time coordinate \( t = x^0 \) that is also a group parameter for the time-translation group.

On each of these spacelike hypersurfaces, the curves that are orthogonal to the spherical symmetry group orbits are geodesics. Use these geodesics to map the spherical coordinates \( \theta, \varphi \) from one orbit to all of the others that can be reached by these orthogonal curves. Label each group orbit by a coordinate-value \( r \) as well as the time coordinate \( t \) that labels the spacelike hypersurface. Because of the way that these coordinates are constructed, all of the off-diagonal components of the metric tensor are zero and the most general form of the metric tensor is

\[
ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + R(r)^2 d\Omega^2.
\]

The coefficients of \( dt^2 \) and \( dr^2 \) are unknown functions of \( r \). The exponential form used here turns out to simplify the computations a bit but is not significant otherwise.

This form of the metric can be specialized a bit more because we have not yet defined the radial coordinate \( r \). The original choice made by Schwarzchild is \( r = R(r) \) so that the radial coordinate actually measures the area of each sphere. Another popular choice is to choose \( r \) so that the space part of the metric is conformal to a flat metric. That choice corresponds to taking \( r = e^{\lambda(r)} R(r) \). In terms of Schwarzchild coordinates,

\[
ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2
\]

which corresponds to the metric components

\[
g_{00} = -e^{2\nu(r)}, \quad g_{11} = e^{2\lambda(r)}, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2 \theta
\]

where the coordinates are traditionally numbered according to

\[
x^0 = t, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \varphi.
\]

Because this metric is diagonal, it is not hard to invert it to find the components \( g^{\mu\nu} \).

\[
g^{00} = -e^{-2\nu(r)}, \quad g^{11} = e^{-2\lambda(r)}, \quad g^{22} = r^{-2}, \quad g^{33} = \frac{1}{r^2 \sin^2 \theta}
\]

### 2.2 Calculating the Connection Coefficients (Traditional)

The next task in the traditional approach is to use equation (1) to evaluate the connection coefficients one-by-one. Since we are using holonomic reference frames and the torsion is zero, these coefficients have the symmetry

\[
\Gamma^\rho_{\alpha\beta} = \Gamma^\rho_{\beta\alpha}
\]

which still leaves 32 of them to evaluate. As it happens, most of them will be zero. Modern techniques such as the use of the calculus of differential forms speed the calculation by letting us deal only with the non-zero terms. However,
in the traditional approach, we have to check them all. Here is one way to organize such a calculation:

\[ \Gamma^\alpha_{\beta\delta} = g^\alpha_{\beta\delta} \]

\[ \Gamma_{\rho\beta\delta} = \frac{1}{2} \left( \partial_\rho g_{\beta\delta} + \partial_\beta g_{\rho\delta} - \partial_\delta g_{\rho\beta} \right) \]

\[ \Gamma_{0\beta\delta} = \frac{1}{2} \left( \partial_0 g_{\beta\delta} + \partial_\beta g_{0\delta} - \partial_\delta g_{0\beta} \right) = \frac{1}{2} \left( \partial_0 g_{0\beta} + \partial_\beta g_{00} \right) \]

\[ \Gamma_{00\delta} = \frac{1}{2} \left( \partial_0 g_{0\delta} + \partial_\delta g_{00} \right) = \frac{1}{2} \partial_0 g_{00} \]

\[ \Gamma_{001} = \frac{1}{2} \partial_1 g_{00} = -\nu' e^{2\nu} \]  

(3)

\[ \Gamma_{002} = \Gamma_{003} = 0 \]

\[ \Gamma_{01\delta} = \frac{1}{2} \left( \partial_0 g_{0\delta} + \partial_1 g_{0\delta} \right) \]

\[ \Gamma_{010} = \Gamma_{001} \]

\[ \Gamma_{011} = \frac{1}{2} \left( \partial_0 g_{11} + \partial_1 g_{01} \right) = 0 \]

\[ \Gamma_{012} = \frac{1}{2} \left( \partial_0 g_{12} + \partial_1 g_{02} \right) = 0 \]

\[ \Gamma_{013} = \frac{1}{2} \left( \partial_0 g_{13} + \partial_1 g_{03} \right) = 0 \]

\[ \Gamma_{02\delta} = \frac{1}{2} \left( \partial_0 g_{2\delta} + \partial_2 g_{0\delta} \right) \]

\[ \Gamma_{022} = \frac{1}{2} \left( \partial_0 g_{22} + \partial_2 g_{02} \right) = 0 \]

\[ \Gamma_{023} = \frac{1}{2} \left( \partial_0 g_{23} + \partial_2 g_{03} \right) = 0 \]

\[ \Gamma_{03\delta} = \frac{1}{2} \left( \partial_0 g_{3\delta} + \partial_3 g_{0\delta} \right) = 0 \]

\[ \Gamma_{1\beta\delta} = \frac{1}{2} \left( \partial_0 g_{1\beta} + \partial_\beta g_{10} - \partial_\delta g_{10} \right) \]

\[ \Gamma_{10\delta} = \frac{1}{2} \left( \partial_0 g_{10} + \partial_0 g_{1\delta} - \partial_0 g_{10} \right) = -\frac{1}{2} \partial_1 g_{0\delta} \]

\[ \Gamma_{100} = -\frac{1}{2} \partial_1 g_{00} = \nu' e^{2\nu} \]

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\[ \Gamma_{101} = -\frac{1}{2} \partial_1 g_{01} = 0 \]

\[ \Gamma_{102} = -\frac{1}{2} \partial_1 g_{02} = 0 \]

\[ \Gamma_{103} = -\frac{1}{2} \partial_1 g_{03} = 0 \]

\[ \Gamma_{11\delta} = \frac{1}{2} \left( \partial_0 g_{11} + \partial_1 g_{11} - \partial_1 g_{01} \right) \]

\[ \Gamma_{110} = \frac{1}{2} \left( \partial_0 g_{10} + \partial_1 g_{10} - \partial_1 g_{01} \right) = 0 \]

\[ \Gamma_{111} = \frac{1}{2} \left( \partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11} \right) = \frac{1}{2} \partial_1 g_{11} \]

\[ \Gamma_{111} = \chi e^{2\lambda} \]

(5)

\[ \Gamma_{112} = \frac{1}{2} \left( \partial_1 g_{12} + \partial_1 g_{12} - \partial_1 g_{12} \right) = 0 \]

\[ \Gamma_{113} = \frac{1}{2} \left( \partial_1 g_{13} + \partial_1 g_{13} - \partial_1 g_{13} \right) = 0 \]

\[ \Gamma_{12\delta} = \frac{1}{2} \left( \partial_0 g_{12} + \partial_2 g_{12} - \partial_1 g_{2\delta} \right) \]

\[ \Gamma_{120} = \frac{1}{2} \left( \partial_2 g_{10} + \partial_2 g_{10} - \partial_1 g_{20} \right) = 0 \]

\[ \Gamma_{121} = \frac{1}{2} \left( \partial_2 g_{11} + \partial_2 g_{11} - \partial_1 g_{21} \right) = 0 \]

\[ \Gamma_{122} = \frac{1}{2} \left( \partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22} \right) = -\frac{1}{2} \partial_1 g_{12} \]

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\[ \Gamma_{123} = \frac{1}{2} \left( \partial_2 g_{13} + \partial_2 g_{13} - \partial_1 g_{23} \right) = 0 \]

\[ \Gamma_{13\delta} = \frac{1}{2} \left( \partial_0 g_{13} + \partial_3 g_{13} - \partial_1 g_{3\delta} \right) = -\frac{1}{2} \partial_1 g_{3\delta} \]

\[ \Gamma_{133} = -r \sin^2 \theta \]

(7)
\[ \Gamma_{2,3\delta} = \frac{1}{2} (\partial_{3} g_{2\delta} + \partial_{3} g_{2\delta} - \partial_{2} g_{3\delta}) \\
\Gamma_{2\delta 3} = \frac{1}{2} (\partial_{3} g_{2\delta} + \partial_{3} g_{2\delta} - \partial_{2} g_{3\delta}) = 0 \\
\Gamma_{21\delta} = \frac{1}{2} (\partial_{1} g_{2\delta} + \partial_{1} g_{2\delta} - \partial_{2} g_{1\delta}) \\
\Gamma_{210} = \frac{1}{2} (\partial_{1} g_{20} - \partial_{2} g_{10}) = 0 \\
\Gamma_{211} = \frac{1}{2} (\partial_{1} g_{21} - \partial_{2} g_{11}) = 0 \\
\Gamma_{212} = \frac{1}{2} (\partial_{1} g_{22} - \partial_{2} g_{12}) = \frac{1}{2} \partial_{1} g_{22} \\
\Gamma_{212} = r \\
\Gamma_{213} = \frac{1}{2} (\partial_{1} g_{23} - \partial_{2} g_{13}) = 0 \\
\Gamma_{22\delta} = \frac{1}{2} (\partial_{2} g_{2\delta} + \partial_{2} g_{2\delta} - \partial_{2} g_{2\delta}) \\
\Gamma_{220} = \frac{1}{2} (\partial_{0} g_{22} + \partial_{2} g_{20} - \partial_{2} g_{20}) = 0 \\
\Gamma_{221} = \frac{1}{2} (\partial_{1} g_{22} + \partial_{2} g_{21} - \partial_{2} g_{21}) = r \\
\Gamma_{222} = \frac{1}{2} (\partial_{2} g_{22} + \partial_{2} g_{22} - \partial_{2} g_{22}) = 0 \\
\Gamma_{223} = \frac{1}{2} (\partial_{3} g_{22} + \partial_{2} g_{23} - \partial_{2} g_{23}) = 0 \\
\Gamma_{23\delta} = \frac{1}{2} (\partial_{3} g_{2\delta} + \partial_{3} g_{2\delta} - \partial_{2} g_{3\delta}) = -\frac{1}{2} \partial_{2} g_{3\delta} \\
\Gamma_{230} = -\frac{1}{2} \partial_{2} g_{30} = 0 \\
\Gamma_{231} = -\frac{1}{2} \partial_{2} g_{31} = 0 \\
\Gamma_{232} = -\frac{1}{2} \partial_{2} g_{32} = 0 \\
\Gamma_{233} = -\frac{1}{2} \partial_{2} g_{33} \\
\Gamma_{233} = -r^2 \sin \theta \cos \theta \\
\Gamma_{3,3\delta} = \frac{1}{2} (\partial_{3} g_{3\delta} + \partial_{3} g_{3\delta} - \partial_{3} g_{3\delta}) \\
\Gamma_{3\delta 3} = \frac{1}{2} (\partial_{3} g_{3\delta} + \partial_{3} g_{3\delta} - \partial_{3} g_{3\delta}) = 0 \\
\Gamma_{31\delta} = \frac{1}{2} (\partial_{1} g_{3\delta} + \partial_{1} g_{3\delta} - \partial_{3} g_{1\delta}) \\
\Gamma_{313} = \frac{1}{2} \partial_{1} g_{33} \\
\Gamma_{313} = r \sin^2 \theta \\
\Gamma_{32\delta} = \frac{1}{2} (\partial_{2} g_{3\delta} + \partial_{2} g_{3\delta} - \partial_{3} g_{2\delta}) \\
\Gamma_{323} = \frac{1}{2} \partial_{2} g_{33} \\
\Gamma_{323} = r^2 \sin \theta \cos \theta \\
\Gamma_{33\delta} = \frac{1}{2} (\partial_{3} g_{3\delta} + \partial_{3} g_{3\delta} - \partial_{3} g_{3\delta}) \\
\Gamma_{333} = \frac{1}{2} \partial_{3} g_{33} \\
\Gamma_{333} = r \sin^2 \theta \\
\Gamma_{333} = r^2 \sin \theta \cos \theta \\
\text{Now collect all the results} \\
\Gamma_{001} = \Gamma_{010} = -\nu e^{2\nu}, \quad \Gamma_{100} = \nu e^{2\nu} \\
\Gamma_{111} = \chi e^{2\chi}, \quad \Gamma_{122} = -r, \quad \Gamma_{133} = -r \sin^2 \theta \\
\Gamma_{212} = \Gamma_{221} = r, \quad \Gamma_{233} = -r^2 \sin \theta \cos \theta \\
\Gamma_{313} = \Gamma_{331} = r \sin^2 \theta, \quad \Gamma_{323} = \Gamma_{332} = r^2 \sin \theta \cos \theta
Now find the connection coefficients $\Gamma^\alpha_{\beta\delta}$.

$$
\Gamma^\alpha_{\beta\delta} = g^{\alpha\rho} \Gamma_{\rho\beta\delta}
$$

$$
\Gamma^0_{01} = g^{00} \Gamma_{001} = (-e^{-2\nu})(-\nu' e^{2\nu}) = \nu'
$$

$$
\Gamma^1_{00} = g^{11} \Gamma_{100} = (e^{-2\lambda})(\nu' e^{2\nu}) = \nu' e^{2(\nu - \lambda)}
$$

$$
\Gamma^1_{11} = g^{11} \Gamma_{111} = (e^{-2\lambda})(\nu' e^{2\lambda}) = \nu'
$$

$$
\Gamma^2_{22} = g^{11} \Gamma_{122} = (e^{-2\lambda})(-r) = -2e^{-2\lambda}
$$

$$
\Gamma^1_{33} = g^{11} \Gamma_{133} = (e^{-2\lambda})(-r \sin^2 \theta) = -re^{-2\lambda} \sin^2 \theta
$$

$$
\Gamma^2_{12} = g^{22} \Gamma_{212} = (r^{-2})(r) = r^{-1}
$$

$$
\Gamma^3_{33} = g^{22} \Gamma_{233} = (r^{-2})(-r^{-2} \sin \theta \cos \theta) = -\sin \theta \cos \theta
$$

$$
\Gamma^3_{13} = g^{33} \Gamma_{313} = \left(\frac{1}{r^2 \sin^2 \theta}\right)(r \sin^2 \theta) = \frac{r}{r^2 \sin^2 \theta}
$$

$$
\Gamma^3_{23} = g^{33} \Gamma_{323} = \left(\frac{1}{r^2 \sin^2 \theta}\right)(r \sin \theta \cos \theta) = \cot \theta
$$

2.3 Calculating the Connection Coefficients (Matrix Organization)

In an environment that supports matrix operations, it is useful to organize
the connection coefficient calculation in terms of matrices. It takes up a more
room but most of the computations can be done automatically. In Scientific
Workplace/Notebook, each matrix product is carried out by marking it with
the mouse and pressing CTRL-E. Derivatives of matrices are also automatic
but require a bit more care.

The basis vectors are arranged as a column matrix

$$
[\partial] = \\
\begin{bmatrix}
\partial_0 \\
\partial_1 \\
\partial_2 \\
\partial_3
\end{bmatrix}
$$

and can be recovered from the column matrix by using the row basis matrices:

$$
[E_0] = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}
$$

$$
[E_1] = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}
$$

$$
[E_2] = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}
$$

$$
[E_3] = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}
$$

in the expression

$$
\partial_\delta = [E_\delta] [\partial].
$$

The metric tensor components are arranged as a matrix

$$
[g] = \\
\begin{bmatrix}
g_{00} & g_{01} & g_{02} & g_{03} \\
g_{10} & g_{11} & g_{12} & g_{13} \\
g_{20} & g_{21} & g_{22} & g_{23} \\
g_{30} & g_{31} & g_{32} & g_{33}
\end{bmatrix} = \\
\begin{bmatrix}
-e^{2\nu} & 0 & 0 & 0 \\
0 & e^{2\lambda} & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta
\end{bmatrix}
$$
with inverse

\[
[g]^{-1} = \begin{bmatrix}
g^{00} & g^{01} & g^{02} & g^{03} \\
g^{10} & g^{11} & g^{12} & g^{13} \\
g^{20} & g^{21} & g^{22} & g^{23} \\
g^{30} & g^{31} & g^{32} & g^{33}
\end{bmatrix} = \begin{bmatrix}
-e^{-2\nu} & 0 & 0 & 0 \\
0 & e^{-2\lambda} & 0 & 0 \\
0 & 0 & 1/r^2 & 0 \\
0 & 0 & 0 & 1/r^2 \sin^2 \theta
\end{bmatrix}
\]

Notice that rows of metric tensor components can be extracted using the row basis matrices

\[
g_{\delta} = [E_{\delta}] [g] = [ g_{00} \ g_{01} \ g_{02} \ g_{03} ]
\]

and arrays of partial derivatives can be constructed like this:

\[
[\partial g_\delta] = [\partial] [g_\delta] = \begin{bmatrix}
\partial_0 g_{00} & \partial_0 g_{01} & \partial_0 g_{02} & \partial_0 g_{03} \\
\partial_1 g_{00} & \partial_1 g_{01} & \partial_1 g_{02} & \partial_1 g_{03} \\
\partial_2 g_{00} & \partial_2 g_{01} & \partial_2 g_{02} & \partial_2 g_{03} \\
\partial_3 g_{00} & \partial_3 g_{01} & \partial_3 g_{02} & \partial_3 g_{03}
\end{bmatrix}
\]

The connection coefficients, arranged as arrays

\[
[\Gamma_\delta] = \begin{bmatrix}
\Gamma^0_{0\delta} & \Gamma^0_{1\delta} & \Gamma^0_{2\delta} & \Gamma^0_{3\delta} \\
\Gamma^1_{0\delta} & \Gamma^1_{1\delta} & \Gamma^1_{2\delta} & \Gamma^1_{3\delta} \\
\Gamma^2_{0\delta} & \Gamma^2_{1\delta} & \Gamma^2_{2\delta} & \Gamma^2_{3\delta} \\
\Gamma^3_{0\delta} & \Gamma^3_{1\delta} & \Gamma^3_{2\delta} & \Gamma^3_{3\delta}
\end{bmatrix}
\]

can then be expressed as

\[
[\Gamma_\delta] = \frac{1}{2} [g]^{-1} \left( [\partial_\delta g] + [\partial g_\delta]^T - [\partial g_\delta] \right)
\]

Now work out what all of these arrays are.

For \(\delta = 0\):

\[
[\Gamma_0] = \frac{1}{2} [g]^{-1} \left( [\partial_0 g] + [\partial g_0]^T - [\partial g_0] \right)
\]

\[
[\partial g_0] = [\partial] [E_0] [g] = \begin{bmatrix}
\partial_0 \\
\partial_1 \\
\partial_2 \\
\partial_3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-\partial_0 e^{2\nu} & 0 & 0 & 0 \\
-\partial_1 e^{2\nu} & 0 & 0 & 0 \\
-\partial_2 e^{2\nu} & 0 & 0 & 0 \\
-\partial_3 e^{2\nu} & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\Rightarrow [\Gamma_0] = \frac{1}{2} \begin{bmatrix}
-e^{-2\nu} & 0 & 0 & 0 \\
0 & e^{-2\lambda} & 0 & 0 \\
0 & 0 & 1/r^2 & 0 \\
0 & 0 & 0 & 1/r^2 \sin^2 \theta
\end{bmatrix}
\]

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or

\[
\begin{bmatrix}
\Gamma_{00} & \Gamma_{01} & \Gamma_{02} & \Gamma_{03} \\
\Gamma_{10} & \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\
\Gamma_{20} & \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\
\Gamma_{30} & \Gamma_{31} & \Gamma_{32} & \Gamma_{33}
\end{bmatrix}
= \begin{bmatrix}
0 & \nu' & 0 & 0 \\
\nu e^{2(\nu-\lambda)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

For \( \delta = 1 \)

\[
[G_1] = \frac{1}{2} [g]^{-1} \left( \partial_1 [g] + [\partial g_1]^T - [\partial g_1] \right)
\]

\[
\partial_1 [g] = \frac{d}{dr}
\begin{bmatrix}
-\nu e^{2\nu} & 0 & 0 & 0 \\
0 & 2\lambda e^{2\lambda} & 0 & 0 \\
0 & 0 & 2r & 0 \\
0 & 0 & 0 & 2r \sin^2(\theta)
\end{bmatrix}
\]

If you mark this matrix with your mouse and select Compute|Calculus|Implicit Differentiation and say that the independent variable is \( r \), you get

Solution:

\[
\begin{bmatrix}
-2\nu' e^{2\nu} & 0 & 0 & 0 \\
0 & 2\lambda e^{2\lambda} & 0 & 0 \\
0 & 0 & 2r & 0 \\
0 & 0 & 0 & 2r \sin^2(\theta)
\end{bmatrix}
\]

In this “Solution,” The assumed “functions,” \( \nu, \lambda, \theta \) have been converted to a format that will not print and will prevent further matrix manipulations, so you need to convert them back to ordinary symbols at this point. Since \( \theta \) is an independent variable, discard the \( \theta' \) term and get

\[
\partial_1 [g] = \begin{bmatrix}
-2\nu' e^{2\nu} & 0 & 0 & 0 \\
0 & 2\lambda e^{2\lambda} & 0 & 0 \\
0 & 0 & 2r & 0 \\
0 & 0 & 0 & 2r \sin^2(\theta)
\end{bmatrix}
\]

Now work out

\[
[\partial g_1] = [\partial] [E_1] [g] = \begin{bmatrix}
\partial_0 \\
\partial_1 \\
\partial_2 \\
\partial_3
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-\nu e^{2\nu} & 0 & 0 & 0 \\
0 & 2\lambda e^{2\lambda} & 0 & 0 \\
0 & 0 & 2r & 0 \\
0 & 0 & 0 & 2r \sin^2(\theta)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & \partial_0 e^{2\lambda} & 0 & 0 \\
0 & \partial_1 e^{2\lambda} & 0 & 0 \\
0 & \partial_2 e^{2\lambda} & 0 & 0 \\
0 & \partial_3 e^{2\lambda} & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \partial_1 e^{2\lambda} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Since this is diagonal, we have

\[
[\partial g_1]^T - [\partial g_1] = 0
\]

and

\[
[G_1] = \frac{1}{2} [g]^{-1} (\partial_1 [g])
\]
or

\[
\begin{bmatrix}
\Gamma_{0\{1} \quad \Gamma_{0\{2} \quad \Gamma_{0\{3} \quad \Gamma_{0\{4}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
\frac{-1}{r^2} & 0 & 0 & 0 \\
0 & \frac{1}{r} & 0 & 0 \\
0 & 0 & \frac{1}{r^2} & 0 \\
0 & 0 & 0 & \frac{1}{r \sin \theta}
\end{bmatrix} \begin{bmatrix}
-2 \nu' e^{2\nu} & 0 & 0 & 0 \\
0 & 2 \lambda' e^{2\lambda} & 0 & 0 \\
0 & 0 & 2r & 0 \\
0 & 0 & 0 & 2r \sin^2 (\theta)
\end{bmatrix}
\]

\[
[\Gamma_1] = \begin{bmatrix}
\nu' & 0 & 0 & 0 \\
0 & \lambda' & 0 & 0 \\
0 & 0 & \frac{1}{r} & 0 \\
0 & 0 & 0 & \frac{1}{r}
\end{bmatrix}
\]

For \( \delta = 2 \)

\[
[\Gamma_2] = \frac{1}{2} [g]^{-1} \left( \partial_2 [g] + [\partial g_2]^T - [\partial g_2] \right)
\]

\[
\partial_1 [g] = \frac{d}{d\theta} \begin{bmatrix}
-e^{2\nu} & 0 & 0 & 0 \\
0 & e^{2\lambda} & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2r^2 \cos \theta \sin \theta & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta
\end{bmatrix}
\]

\[
[\partial g_2] = [\partial] [E_2] [g] = \begin{bmatrix}
\partial_0 & \partial_1 & \partial_2 & \partial_3
\end{bmatrix} \begin{bmatrix}
0 & 0 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & \partial_0 r^2 & 0 \\
0 & 0 & \partial_1 r^2 & 0 \\
0 & 0 & \partial_2 r^2 & 0 \\
0 & 0 & \partial_3 r^2 & 0
\end{bmatrix}
\]

\[
[\Gamma_2] = \frac{1}{2} \begin{bmatrix}
\frac{-1}{r^2} & 0 & 0 & 0 \\
0 & \frac{1}{r} & 0 & 0 \\
0 & 0 & \frac{1}{r^2} & 0 \\
0 & 0 & 0 & \frac{1}{r \sin \theta}
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2r^2 \cos \theta \sin \theta & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -2r & 0 \\
0 & 2r & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
[\Gamma_2] = \begin{bmatrix}
\Gamma_{0,02} & \Gamma_{0,12} & \Gamma_{0,22} & \Gamma_{0,32} \\
\Gamma_{1,02} & \Gamma_{1,12} & \Gamma_{1,22} & \Gamma_{1,32} \\
\Gamma_{2,02} & \Gamma_{2,12} & \Gamma_{2,22} & \Gamma_{2,32} \\
\Gamma_{3,02} & \Gamma_{3,12} & \Gamma_{3,22} & \Gamma_{3,32}
\end{bmatrix}
\]

For \( \delta = 3 \)

\[
[\Gamma_3] = \frac{1}{2} [g]^{-1} \left( \partial_3 [g] + [\partial g_3]^T - [\partial g_3] \right)
\]

\[
= \frac{1}{2} [g]^{-1} \left( [\partial g_3]^T - [\partial g_3] \right)
\]

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\[ [\partial g_3] = [\partial] [E^3] [g] = \begin{bmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & e^{2\nu} & 0 & 0 \\ 0 & 0 & e^{2\lambda} & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix} \]

\[ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2r (\sin^2 \theta) \\ 0 & 0 & 0 & r^2 \frac{d}{d\theta} (\sin^2 \theta) \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} \partial g_3 \end{bmatrix}^T - [\partial g_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2r (\sin^2 \theta) \\ 0 & 0 & 0 & -2r^2 \sin \theta \cos \theta \\ 0 & 2r (\sin^2 \theta) & 2r^2 \sin \theta \cos \theta & 0 \end{bmatrix} \]

\[ [\Gamma_3] = \frac{1}{2} \begin{bmatrix} -\frac{1}{e^{2\nu}} & 0 & 0 & 0 \\ 0 & \frac{1}{e^{2\nu}} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2r (\sin^2 \theta) \\ 0 & 0 & 0 & -2r^2 \sin \theta \cos \theta \\ 0 & 2r (\sin^2 \theta) & 2r^2 \sin \theta \cos \theta & 0 \end{bmatrix} \]

Collecting the results:

\[ [\Gamma_0] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \nu' e^{2(\nu-\lambda)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad [\Gamma_1] = \begin{bmatrix} \nu' & 0 & 0 & 0 \\ 0 & \lambda' & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r} \end{bmatrix} \]

\[ [\Gamma_2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{r}{e^{2\nu}} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & \frac{\cos \theta}{\sin \theta} & 0 \end{bmatrix}, \quad [\Gamma_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r \frac{\sin^2 \theta}{e^{2\nu}} \\ 0 & 0 & 0 & -\cos \theta \sin \theta \\ 0 & \frac{1}{r} & \cos \theta & -\cos \theta \sin \theta \end{bmatrix} \]

2.4 Calculating the Riemann Tensor Components (Traditional Method)

Now plug all these results into the expression for the Riemann tensor components. The traditional method is to work out each coefficient separately.

\[ R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\rho\mu} \Gamma^\rho_{\beta\nu} - \Gamma^\alpha_{\rho\nu} \Gamma^\rho_{\beta\mu} \]
\[ R^\sigma_{0\mu\nu} = \partial_\mu \Gamma^\sigma_{0\nu} - \partial_\nu \Gamma^\sigma_{0\mu} + \Gamma^\sigma_{\rho\mu} \Gamma^\rho_{0\nu} - \Gamma^\sigma_{\rho\nu} \Gamma^\rho_{0\mu} \]
\[ R^1_{0\mu\nu} = \partial_\mu \Gamma^1_{0\nu} - \partial_\nu \Gamma^1_{0\mu} + \Gamma^1_{\rho\mu} \Gamma^\rho_{0\nu} - \Gamma^1_{\rho\nu} \Gamma^\rho_{0\mu} \]
\[ R^1_{00\nu} = \partial_\nu \Gamma^1_{00} - \partial_\nu \Gamma^1_{00} + \Gamma^1_{\rho\nu} \Gamma^\rho_{00} + \Gamma^1_{\rho\nu} \Gamma^\rho_{00} \]
\[ R^1_{001} = \partial_\nu \Gamma^1_{01} - \partial_1 \Gamma^1_{00} + \Gamma^1_{\rho\nu} \Gamma^\rho_{01} - \Gamma^1_{\rho1} \Gamma^\rho_{00} \]
\[ = - (\nu' e^{2(\nu-\lambda)})' + \Gamma^1_{\rho0} \Gamma^\rho_{01} - \Gamma^1_{\rho1} \Gamma^\rho_{00} \]
\[ = (\nu' e^{2(\nu-\lambda)})' + (\nu') (\nu' e^{2(\nu-\lambda)}) - (\nu' e^{2(\nu-\lambda)}) (\lambda') \]
\[ R^1_{001} = - (\nu'' + \nu^2 - \lambda \nu') e^{2(\nu-\lambda)} \] (12)

\[ R^1_{002} = \partial_\nu \Gamma^1_{02} - \partial_2 \Gamma^1_{00} + \Gamma^1_{\rho0} \Gamma^\rho_{02} - \Gamma^1_{\rho2} \Gamma^\rho_{00} = 0 \]
\[ R^1_{003} = \partial_\nu \Gamma^1_{03} - \partial_3 \Gamma^1_{00} + \Gamma^1_{\rho0} \Gamma^\rho_{03} - \Gamma^1_{\rho3} \Gamma^\rho_{00} = 0 \]
\[ R^1_{012} = \partial_1 \Gamma^1_{02} - \partial_2 \Gamma^1_{01} + \Gamma^1_{\rho0} \Gamma^\rho_{02} - \Gamma^1_{\rho2} \Gamma^\rho_{01} = 0 \]
\[ R^1_{023} = \partial_2 \Gamma^1_{03} - \partial_3 \Gamma^1_{02} + \Gamma^1_{\rho0} \Gamma^\rho_{03} - \Gamma^1_{\rho3} \Gamma^\rho_{02} = 0 \]

\[ R^2_{0\mu\nu} = \partial_\mu \Gamma^2_{0\nu} - \partial_\nu \Gamma^2_{0\mu} + \Gamma^2_{\rho\mu} \Gamma^\rho_{0\nu} - \Gamma^2_{\rho\nu} \Gamma^\rho_{0\mu} \]
\[ R^2_{00\nu} = \partial_\nu \Gamma^2_{00} - \partial_\nu \Gamma^2_{00} + \Gamma^2_{\rho\nu} \Gamma^\rho_{00} + \Gamma^2_{\rho\nu} \Gamma^\rho_{00} \]
\[ R^2_{002} = \partial_\nu \Gamma^2_{02} - \partial_2 \Gamma^2_{00} + \Gamma^2_{\rho\nu} \Gamma^\rho_{02} - \Gamma^2_{\rho2} \Gamma^\rho_{00} = 0 \]
\[ = -\Gamma^2_{12} \Gamma^1_{00} = - (\nu^{-1}) (\nu' e^{2(\nu-\lambda)}) \]
\[ R^2_{002} = - \nu' e^{2(\nu-\lambda)} \] (13)

\[ R^2_{003} = \partial_\nu \Gamma^2_{03} - \partial_3 \Gamma^2_{00} + \Gamma^2_{\rho\nu} \Gamma^\rho_{03} - \Gamma^2_{\rho3} \Gamma^\rho_{00} = 0 \]
\[ R^2_{012} = \partial_1 \Gamma^2_{02} - \partial_2 \Gamma^2_{01} + \Gamma^2_{\rho1} \Gamma^\rho_{02} - \Gamma^2_{\rho2} \Gamma^\rho_{01} = 0 \]
\[ R^2_{013} = \partial_1 \Gamma^2_{03} - \partial_3 \Gamma^2_{01} + \Gamma^2_{\rho1} \Gamma^\rho_{03} - \Gamma^2_{\rho3} \Gamma^\rho_{01} = 0 \]
\[ R^2_{023} = \partial_2 \Gamma^2_{03} - \partial_3 \Gamma^2_{02} + \Gamma^2_{\rho2} \Gamma^\rho_{03} - \Gamma^2_{\rho3} \Gamma^\rho_{02} = 0 \]

\[ R^3_{0\mu\nu} = \partial_\mu \Gamma^3_{0\nu} - \partial_\nu \Gamma^3_{0\mu} + \Gamma^3_{\rho\mu} \Gamma^\rho_{0\nu} - \Gamma^3_{\rho\nu} \Gamma^\rho_{0\mu} \]
\[ R^3_{00\nu} = \partial_\nu \Gamma^3_{00} - \partial_\nu \Gamma^3_{00} + \Gamma^3_{\rho\nu} \Gamma^\rho_{00} + \Gamma^3_{\rho\nu} \Gamma^\rho_{00} \]
\[+\Gamma^3_{\rho\mu\nu}\Gamma^\nu_{03} - \Gamma^3_{\rho\mu\nu}\Gamma^\nu_{01} = -\Gamma^3_{13}\Gamma^1_{00}\]
\[-(\nu' e^{2(\nu - \lambda)}) (r^{-1})\]

\[R^3_{003} = -\nu' e^{2(\nu - \lambda)} \quad (14)\]

\[R^3_{01\nu} = \partial_1 \Gamma^3_{\nu\rho\nu} - \partial_\nu \Gamma^3_{10} + \Gamma^3_{\rho\mu\nu}\Gamma^\rho_{00} - \Gamma^3_{\rho\mu\nu}\Gamma^\rho_{01},\]
\[R^2_{1\nu} = \partial_\nu \Gamma^2_{1\nu} - \partial_\nu \Gamma^3_{1\nu},\]
\[R^3_{12} = \partial_2 \Gamma^3_{12} - \partial_2 \Gamma^3_{11} + \Gamma^3_{\rho\mu\nu}\Gamma^\rho_{12} - \Gamma^3_{\rho\mu\nu}\Gamma^\rho_{11} = 0\]
\[R^3_{13} = \partial_1 \Gamma^3_{13} - \partial_3 \Gamma^3_{11} + \Gamma^3_{\rho\mu\nu}\Gamma^\rho_{13} - \Gamma^3_{\rho\mu\nu}\Gamma^\rho_{11} = 0\]
\[R^3_{23} = \partial_3 \Gamma^3_{23} - \partial_2 \Gamma^3_{23} + \Gamma^3_{\rho\mu\nu}\Gamma^\rho_{23} - \Gamma^3_{\rho\mu\nu}\Gamma^\rho_{22} = 0\]

\[R^3_{123} = \partial_2 \Gamma^3_{123} - \partial_3 \Gamma^3_{121} + \Gamma^3_{\rho\mu\nu}\Gamma^\rho_{123} - \Gamma^3_{\rho\mu\nu}\Gamma^\rho_{111} = 0\]
\[R^3_{113} = \partial_1 \Gamma^3_{113} - \partial_3 \Gamma^3_{111} + \Gamma^3_{\rho\mu\nu}\Gamma^\rho_{113} - \Gamma^3_{\rho\mu\nu}\Gamma^\rho_{111} = 0\]
\[R^3_{223} = \partial_3 \Gamma^3_{223} - \partial_3 \Gamma^3_{222} + \Gamma^3_{\rho\mu\nu}\Gamma^\rho_{223} - \Gamma^3_{\rho\mu\nu}\Gamma^\rho_{222} = 0\]

\[R^3_{113} = -\frac{\lambda'}{r} \quad (15)\]

\[R^3_{123} = -\frac{\lambda'}{r} \quad (16)\]
Now collect together the non-zero components of the Riemann tensor. Notice that, because we are not using an orthonormal frame, we need to go back and raise and lower indexes using the metric tensor in order to get the symmetric partners of the components that we found.

\[ R^3_{223} = -1 + e^{-2\lambda} \quad (17) \]

\[ R_{0\,01} = (\nu'' + \nu'^2 - \lambda'\nu') e^{2(\nu-\lambda)} \]

\[ R_{1\,00} = g_{00} R_{1010} = g_{11} g_{00} R_{0\,10} \]

\[ R_{1\,00} = - (\nu'' + \nu'^2 - \lambda'\nu') \]

\[ R_{0\,20} = R_{0\,30} = \nu' e^{-2\lambda} \]

\[ R_{2\,00} = g_{22} g_{00} R_{0\,20} = -r^2 e^{-2\nu} R_{0\,20} \]

\[ R_{3\,00} = g_{33} g_{00} R_{0\,30} = -r^2 \sin^2 \theta e^{-2\nu} R_{0\,30} \]

\[ R_{1\,21} = g_{11} g_{22} R_{1\,21} \]

\[ R_{2\,11} = g_{11} g_{22} R_{1\,21} \]

\[ R_{3\,11} = g_{11} g_{33} R_{1\,31} = r^2 e^{-2\nu} \sin^2 \theta R_{1\,31} \]

\[ R_{3\,11} = r e^{-2\nu} \lambda' \sin^2 \theta \]

2.5 Calculating the Riemann Tensor (Matrix Organization)

The expression for the Riemann tensor components:

\[ R^{\alpha \beta \mu \nu} = \partial_\mu \Gamma^{\alpha \beta \nu} - \partial_\nu \Gamma^{\alpha \beta \mu} + \Gamma^{\alpha \gamma \mu} \Gamma_{\beta \gamma \nu} - \Gamma^{\alpha \beta \mu} \Gamma_{\beta \gamma \nu} \]

can be regarded as a set of matrix equations for the six independent curvature matrices

\[ \Theta_{\mu \nu} = \begin{bmatrix}
R^0_{00 & 0 \mu} & R^0_{01 & 0 \mu} & R^0_{02 & 0 \mu} & R^0_{03 & 0 \mu} \\
R^1_{00 & 0 \mu} & R^1_{01 & 0 \mu} & R^1_{02 & 0 \mu} & R^1_{03 & 0 \mu} \\
R^2_{00 & 0 \mu} & R^2_{01 & 0 \mu} & R^2_{02 & 0 \mu} & R^2_{03 & 0 \mu} \\
R^3_{00 & 0 \mu} & R^3_{01 & 0 \mu} & R^3_{02 & 0 \mu} & R^3_{03 & 0 \mu}
\end{bmatrix} \]
\[ \Theta_{\mu\nu} = \partial_\mu [\Gamma_\nu] - \partial_\nu [\Gamma_\mu] + [\Gamma_\mu] [\Gamma_\nu] - [\Gamma_\nu] [\Gamma_\mu] \]

Since we already have the connection matrices \([\Gamma_\mu]\), we can just plug those in and evaluate the results.

First \(\mu, \nu = 0, 1\)

\[ \Theta_{01} = \partial_0 [\Gamma_1] - \partial_1 [\Gamma_0] + [\Gamma_0] [\Gamma_1] - [\Gamma_1] [\Gamma_0] \]
\[ = -\partial_1 [\Gamma_0] + [\Gamma_0] [\Gamma_1] - [\Gamma_1] [\Gamma_0] \]
\[ = -\frac{d}{dr} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = - \begin{bmatrix} \nu'' + \nu' (\nu' - \lambda') e^{2(\nu - \lambda)} & 0 & 0 \\ 0 & \nu' & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ \Theta_{01} = - \begin{bmatrix} 0 & 0 & 0 \\ 0 & \nu' e^{2(\nu - \lambda)} & 0 \\ 0 & 0 & 0 \end{bmatrix} = - \begin{bmatrix} -\nu'' + \lambda' \nu' - (\nu')^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

Next, \(\mu, \nu = 0, 2\)

\[ \Theta_{02} = \partial_0 [\Gamma_2] - \partial_2 [\Gamma_0] + [\Gamma_0] [\Gamma_2] - [\Gamma_2] [\Gamma_0] = [\Gamma_0] [\Gamma_2] - [\Gamma_2] [\Gamma_0] \]
\[ = -\frac{d}{dr} \begin{bmatrix} \nu' e^{2(\nu - \lambda)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & -\nu' e^{2(\nu - \lambda)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

Next, \(\mu, \nu = 0, 3\)

\[ \Theta_{03} = \partial_0 [\Gamma_3] - \partial_3 [\Gamma_0] + [\Gamma_0] [\Gamma_3] - [\Gamma_3] [\Gamma_0] = [\Gamma_0] [\Gamma_3] - [\Gamma_3] [\Gamma_0] \]
\[ = -\frac{d}{dr} \begin{bmatrix} \nu' e^{2(\nu - \lambda)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\nu' e^{-2 \lambda + 2 \nu} & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

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Next, $\mu, \nu = 1, 2$

$$\Theta_{03} = \begin{bmatrix} R_{003}^0 & R_{013}^0 & R_{023}^0 & R_{033}^0 \\ R_{103}^1 & R_{113}^1 & R_{123}^1 & R_{133}^1 \\ R_{203}^2 & R_{213}^2 & R_{223}^2 & R_{233}^2 \\ R_{303}^3 & R_{313}^3 & R_{323}^3 & R_{333}^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\frac{r \nu' \sin^2 \theta}{\sin \theta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{r} \nu' e^{-2\lambda + 2\nu} & 0 & 0 & 0 \end{bmatrix}$$

$$\Theta_{12} = \partial_t [\Gamma_2] - \partial_\Gamma [\Gamma_1] + [\Gamma_1] [\Gamma_2] - [\Gamma_2] [\Gamma_1] = [\Gamma_2]' + [\Gamma_1] [\Gamma_2] - [\Gamma_2] [\Gamma_1]$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{r}{c^2} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & \cos \theta \sin \theta \end{bmatrix} + \begin{bmatrix} \nu' & 0 & 0 & 0 \\ 0 & \lambda' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{c^2} + 2\frac{r \lambda'}{c^2} & 0 \\ 0 & -\frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \end{bmatrix} + \begin{bmatrix} \nu' & 0 & 0 & 0 \\ 0 & \lambda' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Next, $\mu, \nu = 1, 3$


$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{r \sin^2 \theta}{\sin \theta} \\ 0 & \frac{1}{r} & -\cos \theta \sin \theta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \nu' & 0 & 0 & 0 \\ 0 & \lambda' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
Finally, $\mu, \nu = 2, 3$

$$\Theta_{23} = \partial_{\rho} [\Gamma_{23} - \partial_{\theta} \Gamma_{2}] + [\Gamma_{2} | \Gamma_{3}] - [\Gamma_{3} | \Gamma_{2}] = \frac{\partial}{\partial \theta} [\Gamma_{3} + \Gamma_{2}] | \Gamma_{2} - [\Gamma_{3} | \Gamma_{2}]$$

$$= \frac{\partial}{\partial \theta} \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{r} \sin \theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r} \sin \theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{r} \sin \theta \\ 0 & -\frac{1}{r} \sin \theta & 0 \\ 0 & 0 & -\frac{1}{r} \sin \theta & 0 \\ 0 & 0 & 0 & -\frac{1}{r} \sin \theta \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{r} \sin \theta & 0 \\ 0 & -\frac{1}{r} \sin \theta & 0 \\ 0 & 0 & 0 & -\frac{1}{r} \sin \theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Theta_{23} = \begin{bmatrix} R_{00}^{\rho} & R_{01}^{\rho} & R_{02}^{\rho} & R_{03}^{\rho} \\ R_{10}^{\rho} & R_{11}^{\rho} & R_{12}^{\rho} & R_{13}^{\rho} \\ R_{20}^{\rho} & R_{21}^{\rho} & R_{22}^{\rho} & R_{23}^{\rho} \\ R_{30}^{\rho} & R_{31}^{\rho} & R_{32}^{\rho} & R_{33}^{\rho} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2.6 Calculating the Ricci Tensor Components and Trace

The next step in evaluating Einstein’s equation for this spherically symmetric spacetime is to construct the Ricci tensor.

$$R_{\mu\nu} = R_{\rho}^{\sigma} g_{\mu\rho} g_{\nu\sigma}$$

Notice that this tensor cannot have off-diagonal terms because the Riemann tensor is diagonal in index-pairs. Thus, we have just four components to calculate.

$$R_{00} = R_{00}^{\rho} g_{\rho\rho} = R_{01}^{\rho} + R_{02}^{\rho} + R_{03}^{\rho} = (\nu'' + \nu^2 - \lambda' \nu') e^{2(\nu - \lambda)} + 2 \nu' e^{2(\nu - \lambda)}$$

$$R_{00} = \left( \nu'' + \nu^2 - \lambda' \nu' + 2 \frac{\nu'}{r} \right) e^{2(\nu - \lambda)} \quad (18)$$

$$R_{11} = R_{11}^{\rho} g_{\rho\rho} = R_{10}^{\rho} + R_{12}^{\rho} + R_{13}^{\rho}$$

$$R_{11} = -\nu'' - \nu^2 + \lambda' \nu' + 2 \frac{\lambda'}{r} \quad (19)$$

$$R_{22} = R_{22}^{\rho} g_{\rho\rho} = R_{20}^{\rho} + R_{21}^{\rho} + R_{23}^{\rho}$$

$$R_{22} = -\nu' e^{-2\lambda} + re^{-2\nu} \lambda' + 1 - e^{-2\lambda} \quad (20)$$

$$R_{33} = R_{33}^{\rho} g_{\rho\rho} = R_{30}^{\rho} + R_{31}^{\rho} + R_{32}^{\rho}$$

$$R_{33} = -r e^{-2\lambda} \sin^2 \theta + re^{-2\nu} \lambda' \sin^2 \theta + \sin^2 \theta (1 - e^{-2\lambda})$$

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\[ R_{33} = (-\nu' e^{-2\lambda} + re^{-2\nu} \lambda + 1 - e^{-2\lambda}) \sin^2 \theta \]

The next step is to use the metric to form the trace of the Ricci tensor. Recall the inverse metric components

\[ g^{00} = -e^{-2\nu}, \quad g^{11} = e^{-2\lambda}, \quad g^{22} = r^{-2}, \]

\[ g^{33} = \frac{1}{r^2 \sin^2 \theta} \]

\[ R = g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33} \]

\[ = \left(-e^{-2\nu}\right) \left(\nu'' + \nu'^2 - \lambda' \nu' + 2\frac{\nu'}{r} \right) e^{2(\nu - \lambda)} \]

\[ + \left(e^{-2\lambda}\right) \left(-\nu'' - \nu'^2 + \lambda' \nu' + 2\frac{\nu'}{r} \right) e^{2(\nu - \lambda)} \]

\[ + \left(r^{-2}\right) \left(-\nu' e^{-2\lambda} + re^{-2\nu} \lambda' + 1 - e^{-2\lambda} \right) \]

\[ + \left(r^{-2}\right) \left(-r \nu e^{-2\lambda} + re^{-2\nu} \lambda' + 1 - e^{-2\lambda} \right) \sin^2 \theta \]

\[ R = -\left(\nu'' + \nu'^2 - \lambda' \nu' + 2\frac{\nu'}{r} \right) e^{-2\lambda} \]

\[ + \left(-\nu'' - \nu'^2 + \lambda' \nu' + 2\frac{\nu'}{r} \right) e^{-2\lambda} \]

\[ + \left(r^{-2}\right) \left(-\nu' e^{-2\lambda} + re^{-2\nu} \lambda' + 1 - e^{-2\lambda} \right) \]

\[ + \left(r^{-2}\right) \left(-r \nu e^{-2\lambda} + re^{-2\nu} \lambda' + 1 - e^{-2\lambda} \right) \]

As a check on the accuracy of this result, notice that all of the angle dependence has gone away. Furthermore, if we specialize the metric to the flat case by taking the functions \( \lambda \) and \( \nu \) to be constants with \( e^{-2\lambda} = 1 \), we find that the scalar curvature vanishes everywhere just as it should.

### 2.7 Calculating the Einstein Tensor Components

The Einstein tensor components can now be calculated. Since the Ricci tensor and the metric are both diagonal, we only have the diagonal components to worry about.

\[ G_{00} = R_{00} - \frac{1}{2} g_{00} R \]

\[ = \left(\nu'' + \nu'^2 - \lambda' \nu' + 2\frac{\nu'}{r} \right) e^{2(\nu - \lambda)} \]

\[ - \frac{1}{2} \left(-e^{2\nu}\right) \left(-2 (\nu'' + \nu'^2 - 2\lambda' \nu') e^{-2\lambda} + 2 \frac{1 - e^{-2\lambda}}{r^2} - 4 \nu' e^{-2\lambda} + 4 \frac{\lambda'}{r} e^{-2\lambda} \right) \]

\[ = \left(\nu'' + \nu'^2 - \lambda' \nu' + 2\frac{\nu'}{r} \right) e^{2(\nu - \lambda)} \]

\[ - (\nu'' + \nu'^2 - \lambda' \nu') e^{2(\nu - \lambda)} + \frac{1 - e^{-2\lambda}}{r^2} e^{2\nu} - 2 \nu' e^{2(\nu - \lambda)} + 2 \frac{\lambda'}{r} e^{2(\nu - \lambda)} \]

\[ G_{00} = \frac{1 - e^{-2\lambda}}{r^2} e^{2\nu} + 2 \frac{\lambda'}{r} e^{2(\nu - \lambda)} \]
For a uid with four-velocity $u^\mu$

### 3.1.1 Projection Tensors

Collect the results for Einstein’s equations

$$G_{11} = R_{11} - \frac{1}{r^2} g_{11} R = -\nu'' - \nu'^2 + \lambda' + 2\lambda' r$$

$$= -\nu'' - \nu'^2 + \lambda' + 2\lambda' r + \nu'' + \nu'^2 - \lambda' \nu' \left( -2(\nu'' + \nu'^2 - \lambda' \nu') e^{-2\lambda} + 2 \frac{1 - e^{-2\lambda}}{r} - 4 \frac{\nu'}{r} e^{-2\lambda} + 4 \frac{\lambda'}{r} e^{-2\lambda} \right) + \nu'' + \nu'^2 - \lambda' \nu' \left( - \frac{1 - e^{-2\lambda}}{r} e^{2\lambda} + 2 \frac{\nu'}{r} + 2 \frac{\lambda'}{r} - 2 \frac{\lambda'}{r} \right)$$

$$= -\frac{1 - e^{-2\lambda}}{r^2} e^{2\lambda} + 2 \frac{\nu'}{r}$$

$$G_{22} = R_{22} - \frac{1}{r^2} g_{22} R$$

$$= -r \nu' e^{-2\lambda} + r e^{-2\nu' \lambda'} + 1 - e^{-2\lambda} - \frac{1}{2} r^2 \left( -2(\nu'' + \nu'^2 - \lambda' \nu') e^{-2\lambda} + 2 \frac{1 - e^{-2\lambda}}{r} - 4 \frac{\nu'}{r} e^{-2\lambda} + 4 \frac{\lambda'}{r} e^{-2\lambda} \right)$$

$$G_{33} = (-r \nu' e^{-2\lambda} + r e^{-2\nu' \lambda'} + 1 - e^{-2\lambda} \right) \sin^2 \theta$$

$$-\frac{1}{2} \left( r^2 \sin^2 \theta \right) \left( -2(\nu'' + \nu'^2 - \lambda' \nu') e^{-2\lambda} + 2 \frac{1 - e^{-2\lambda}}{r} - 4 \frac{\nu'}{r} e^{-2\lambda} + 4 \frac{\lambda'}{r} e^{-2\lambda} \right)$$

Collect the results for Einstein’s equations

$$\frac{1 - e^{-2\lambda}}{r^2} e^{2\nu} + 2 \frac{\lambda'}{r} e^{2(\nu - \lambda)} = 8 \pi T_{00}$$

$$-\frac{1 - e^{-2\lambda}}{r^2} e^{2\lambda} + 2 \frac{\nu'}{r} = 8 \pi T_{11}$$

$$r^2 (\nu'' + \nu'^2 - \lambda' \nu') e^{-2\lambda} + r (\nu' - \lambda') e^{-2\lambda} = 8 \pi T_{22}$$

$$r^2 (\nu'' + \nu'^2 - \lambda' \nu') e^{-2\lambda} + r (\nu' - \lambda') e^{-2\lambda} = 8 \pi \frac{T_{33}}{\sin^2 \theta}$$

### 3 The Matter Equations

#### 3.1 Stress-energy Tensor for a Perfect Fluid

##### 3.1.1 Projection Tensors

For a fluid with four-velocity $u^\mu$, it is natural to define the projection tensor

$$V^\alpha_{\beta} = -u^\alpha u_\beta$$

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that maps any four-vector $v^\beta$ into a vector

$$v^\mu = V^\mu_\beta v^\beta$$

that is tangent to the fluid flow lines. Notice that if a vector is already tangent to the fluid flow lines, this tensor does not change it:

$$v^\beta = au^\beta$$

$$V^\alpha_\beta v^\beta = a(-u^\alpha u_\beta) u^\beta = -au^\alpha (u_\beta u^\beta) = au^\alpha = v^\alpha$$

Thus, when this tensor is considered as a mapping of the tangent space into itself $V : T_P \to T_P$ it obeys the identity

$$V^2 = V$$

Any linear mapping that obeys this identity will be called a projection tensor.

Associated with the projection tensor $V$ is the tensor $H = I - V$ or

$$H^\alpha_\beta = \delta^\alpha_\beta - V^\alpha_\beta$$

which is called the complement of $V$. It is easy to check that this tensor corresponds to a mapping that obeys the defining equation of a projection tensor:

$$H^2 = H.$$

For a fluid, this map removes the component of a vector that is tangent to the fluid flow lines. In other words, it produces the part of the vector that is tangent to a constant-time surface in the local rest frame of the fluid — the rest-frame spacelike part of the vector.

### 3.1.2 Perfect Fluid

The stress-energy tensor of a perfect fluid has the form

$$T = p_V V + p_H H$$

where $p_V, p_H$ are functions. In terms of indexes, with one index raised using the metric tensor,

$$T^{\mu\nu} = p_V V^{\mu\nu} + p_H H^{\mu\nu}. \quad (21)$$
In the rest-frame of the fluid, the components of this tensor are

\[ T^{00} = p V^{00} = -p V^0 u^0 = -p \nu \]

and

\[ T^{mn} = p H^{mn} \]

so that we identify the mass-density and pressure as

\[ \rho = -p \nu, \quad p = p H \]

and the stress-energy tensor is given by

\[ T_{\alpha \beta} = -\rho V^{\alpha \beta} + p H^{\alpha \beta} \]

### 3.1.3 Stress-energy Components

To complete Einstein’s equations in the form that we have obtained, we need to lower an index in a non-orthonormal frame

\[ T_{\mu \nu} = g_{\mu \rho} T^{\rho \nu} = -\rho g_{\mu \nu} V^\rho \nu + p g_{\mu \rho} H^{\rho \nu}. \]

Since the projection tensors act as identity operators on their respective subspaces, their components in this rest-frame coordinate system are just

\[ V^0_0 = 1, \quad H^j_\ell = \delta^i_j. \]

The operation of lowering the index just multiplies these diagonal matrices by the diagonal metric tensor yielding

\[ T_{00} = -\rho g_{00} V^0_0 = -\rho g_{00} = \rho e^{2\nu} \]

\[ T_{ij} = p g_{ij}: \quad T_{11} = \rho e^{2\lambda}, \quad T_{22} = pr^2, \quad T_{33} = pr^2 \sin^2 \theta \]

### 3.2 Solving Einstein’s Equations

#### 3.2.1 Simplifying the Equations

Einstein’s equations then become

\[ \frac{1 - e^{-\lambda}}{r^2} e^{2\nu} + 2 \frac{\lambda'}{r} e^{2(\nu - \lambda)} = 8\pi \rho e^{2\nu} \]

\[ \frac{1 - e^{-\lambda}}{r^2} e^{2\lambda} + 2 \frac{\nu'}{r} = 8\pi \rho e^{2\lambda} \]

\[ r^2 \left( \nu'' + \nu'^2 - \lambda' \nu' \right) e^{-2\lambda} + r \left( \nu' - \lambda' \right) e^{-2\lambda} = 8\pi pr^2 \]

\[ r^2 \left( \nu'' + \nu'^2 - \lambda' \nu' \right) e^{-2\lambda} + r \left( \nu' - \lambda' \right) e^{-2\lambda} = 8\pi pr^2 \]
Notice that the $G_{22}$ and $G_{33}$ equations are the same and the $G_{00}$ and $G_{11}$ equations are very similar. Multiply the first and last equation by $e^{2\nu}$:

\[
\frac{e^{2\lambda}}{r^2} - e^{2\nu} + \frac{2\lambda'}{r} e^{2\nu} = 8\pi r e^{2(\nu+\lambda)}
\]

\[-\frac{e^{2\lambda}}{r^2} - e^{2\nu} + \frac{\nu'}{r} e^{2\nu} = 8\pi r e^{2(\nu+\lambda)}
\]

The sum and difference of these two equations is

\[
2\frac{\nu'}{r} e^{2\nu} + 2\frac{\lambda'}{r} e^{2\nu} = 8\pi (\rho + p) e^{2(\nu+\lambda)}
\]

\[2\frac{e^{2\lambda}}{r^2} - 1 e^{2\nu} + 2\frac{\lambda' - \nu'}{r} e^{2\nu} = 8\pi (\rho - p) e^{2(\nu+\lambda)}
\]

The sum equation simplifies to just

\[(\nu + \lambda)' = 4\pi r (\rho + p) e^{2\lambda} \quad (22)\]

while the difference equation becomes

\[r (\lambda' - \nu') + e^{2\lambda} - 1 = 4\pi r^2 (\rho - p) e^{2\lambda} \quad (23)\]

The remaining equation that must be satisfied is

\[\nu'' + \nu'^2 - \lambda'\nu' + \nu' - \lambda' \frac{\nu'}{r} = 8\pi r e^{2\lambda} \quad (24)\]

### 3.2.2 The External (Vacuum) Solution

Outside of the star we will assume that there is a negligible amount of matter so that the stress-energy tensor is zero there. The Einstein equations then become

\[(\nu + \lambda)' = 0\]

\[r (\lambda' - \nu') + e^{2\lambda} - 1 = 0\]

and

\[\nu'' + \nu'^2 - \lambda'\nu' + \nu' - \lambda' \frac{\nu'}{r} = 0\]

The first of these equations is easily solved:

\[\nu = -\lambda + b\]

where $b$ is a constant. Substitute this solution into the other two equations

\[2r\lambda' + e^{2\lambda} - 1 = 0\]

\[-\lambda'' + 2\lambda'^2 - 2\frac{\lambda'}{r} = 0\]
Since we only have one function which must satisfy both these equations, the equations had better be related to one another. Multiply the first equation by $e^{-2\lambda}$ and differentiate it

\[ 2r\lambda e^{-2\lambda} + 1 - e^{-2\lambda} = 0 \]

\[ 2\lambda' e^{-2\lambda} + 2r\lambda'' e^{-2\lambda} - 4r\lambda'^2 e^{-2\lambda} + 2\lambda' e^{-2\lambda} = 0 \]

which is equivalent to

\[ \lambda'' - 2\lambda'^2 + 2\frac{\lambda'}{r} = 0 \]

which is the second equation. Thus, one equation is just the derivative of the other.

We now have just one equation left to solve.

\[ 2r\lambda' + e^{2\lambda} - 1 = 0 \]

Rewrite this equation a few times:

\[ 2r \frac{d\lambda}{dr} = 1 - e^{2\lambda} \]

\[ \frac{2d\lambda}{1 - e^{2\lambda}} = \frac{dr}{r} \]

Define a new variable

\[ f = e^{2\lambda}, \quad df = 2f d\lambda \]

\[ \frac{df}{f - f^2} = \frac{dr}{r} \]

Do a little algebra to reduce the left-hand side to something integrable

\[ \frac{1}{f - f^2} = \frac{1}{f(1 - f)} = \frac{1}{f} + \frac{1}{(1 - f)} \]

\[ \frac{df}{f} + \frac{df}{(1 - f)} = \frac{dr}{r} \]

Integrate this equation

\[ \ln f - \ln (1 - f) = \ln r - \ln r_0 \]

where $r_0$ is a constant. Exponentiate this expression to get

\[ \frac{f}{1 - f} = \frac{r}{r_0} \]

and solve for $f$

\[ f = \frac{r}{r_0} (1 - f) = \frac{r}{r_0} - \frac{r}{r_0} f \]

\[ f \left( 1 + \frac{r}{r_0} \right) = \frac{r}{r_0} \]
\[ f = \frac{r}{r_0} = \frac{1}{1 + \frac{r_0}{r}} \]

In terms of our original variables,
\[ e^{2\lambda} = \frac{1}{1 + r_0^2}, \quad e^{2\nu} = e^{2b}e^{-2\lambda} = e^{2b} \left( 1 + \frac{r_0}{r} \right) \]

so that the metric tensor has the form
\[ ds^2 = -e^{2b} \left( 1 + \frac{r_0}{r} \right) dt^2 + \frac{1}{1 + \frac{r_0}{r}} dr^2 + r^2 d\Omega^2. \]

### 3.2.3 Fixing the Constants in the Vacuum Solution

The constant \( b \) just corresponds to the freedom to rescale the time coordinate. If we want the time coordinate to agree with proper time at infinite \( r \) then the only possible choice is \( b = 0 \).

If we want the time coordinate to agree with proper time at infinite \( r \) then the only possible choice is \( b = 0 \).

Fixing the value of the constant \( r_0 \) requires a bit more work. Remember that the gravitational acceleration \( g \) in the radial direction is given by the connection coefficient \(-\Gamma^1_{00}\) which we found to be
\[ \Gamma^1_{00} = \nu^2 e^{2(\nu-\lambda)}. \]

Rewriting this expression
\[ -g = -\nu^2 e^{-4\lambda} = \frac{1}{2} \left( e^{-2\lambda} \right)' \left( e^{-2\lambda} \right) = \frac{1}{2} \left( 1 + \frac{r_0}{r} \right)' \left( 1 + \frac{r_0}{r} \right) \]

so that the radial gravitational acceleration is
\[ g_{\text{Einstein}} = \frac{r_0}{2r^2} \left( 1 + \frac{r_0}{r} \right) = \frac{r_0}{2r^2} + \frac{r_0^2}{2r^3}. \]

Newton’s theory of gravitation gives the expression
\[ g_{\text{Newton}} = -\frac{GM}{r^2}. \]

These expressions will agree for sufficiently large values of \( r \) if the constant \( r_0 \) is chosen to be
\[ r_0 = -2GM. \]

In geometrical units where \( G = 1 \), mass is measured in the same units as length and we have the standard expression for the Schwarzschild metric
\[ ds^2 = -\left( 1 - \frac{2M}{r} \right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\Omega^2. \]

The expression for the radial acceleration of a freely falling body in this gravitational field is given by
\[ g = \frac{M}{r^2} - \frac{4M^2}{2r^3} = \frac{M}{r^2} \left( 1 - \frac{2M}{r} \right). \]
Evidently the deviations from Newton’s theory will be small so long as
\[ r >> 2M. \]

The characteristic radius \(2GM\) is called the Schwarzchild Radius. For the Earth it is about a centimeter, for the Sun, about a kilometer. Thus, the extra term should be extremely small outside these bodies.

### 3.3 Finding Interior Solutions

#### 3.3.1 Boundary Conditions on the Metric

The metric tensor inside a spherically symmetric, static star can have the same form as was assumed outside.

\[ ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2d\Omega^2 \]

Inside of a star, the metric tensor must satisfy a boundary condition at \( r = 0 \) where the metric tensor has to give the flat-space spherically symmetric line element. Without such a condition, the ratio of the circumference to the radius of a very small circle around the origin will deviate from \(2\pi\) no matter how small the circle is. That condition would correspond to a conical singularity and would violate the assumption that the spacetime is locally just like Minkowski spacetime. Thus, we need the coefficient of \(dr^2\) to be equal to one at the origin which implies the boundary condition

\[ e^{2\lambda(0)} = 1. \]

At the surface of the star, we want the spacetime metric tensor to match up with the exterior solution.

\[ ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2d\Omega^2 \]

The areas of spheres should agree there, so the surface of a star with surface radius \(R\) in external Schwarzchild coordinates should occur at radius \(r = R\) in the internal metric. Also, clocks just below the surface should be keeping the same time as clocks just above the surface. Thus, we need \(g_{00}\) to match up. If the star has an apparent mass of \(M\), and a radius \(R\) (calculated by dividing its radius by \(4\pi\) and taking the square root) then the external value of \(g_{00}\) is just 
\[ -1 + \frac{2M}{R} \]
and we have the boundary condition

\[ e^{2\nu(R)} = 1 - \frac{2M}{R}. \]

Similarly, the relationship between small intervals in the radial coordinate and distances in spacetime should not change abruptly at the star’s surface, so we also require the boundary condition

\[ e^{-2\lambda(R)} = 1 - \frac{2M}{R} \]
radius. Ordinarily we do not start out knowing the value of the apparent mass \( M \), so we use one of the boundary conditions to find it.

\[
M = \frac{1}{2} R \left( 1 - e^{-2\lambda(R)} \right)
\]

and impose the relationship

\[
\nu(R) = -\lambda(R)
\]

on the solution.

### 3.3.2 Equation of State

With matter present, we need some relation between the density and the pressure. General relativity becomes important when the pressures are very high and the processes which supply such very high pressures usually involve much more energy per particle than the kinetic energies of the particles. Thus a simple connection of the form \( p = P(\rho) \) for some function \( P \) is usually a good description. The form of the function \( P \) is governed by the detailed physical processes which are supplying the pressure. In a realistic model, this function will be given by a numerical table which is itself the result of a numerical model of various nuclear and elementary particle processes.

The possible equations of state range from the softest possible one

\[
p = 0
\]

which corresponds to "dust" to the stiffest one that makes any physical sense

\[
p = \rho.
\]

In this last case, usually called the "stiff matter equation of state" the speed of sound \( \sqrt{\frac{p}{\rho}} \) is equal to the speed of light.

Sometimes it is reasonable to consider a situation in which matter is stiffer than possible — incompressible in fact. That situation is characterized by a constant density

\[
\rho = \rho_0
\]

and pressures which become whatever they have to be to keep the fluid from compressing. The interior solution to Einstein’s equations for a star which is made of incompressible fluid was found by Schwarzschild in 1916. He found that such stars can exist only when

\[
M < \frac{4R}{9}
\]

in contrast to the result from Newtonian theory that such stars can exist for any combination of total mass and radius.
3.3.3 The Mass Integral Solution of the Initial Value Equation

When matter is present it is useful to note that the $G_{00} = 8\pi T_{00}$ equation, which we identified earlier as one of the initial value constraints, can be integrated simply and directly no matter what the equation of state. Multiply that equation by $r^2e^{-2\lambda}$ and obtain

$$1 - e^{-2\lambda} + 2r\lambda e^{2\lambda} = 8\pi r^2 \rho$$

which is the same as

$$\frac{d}{dr} \left[r \left(1 - e^{-2\lambda}\right)\right] = 8\pi r^2 \rho.$$

Integrate this directly

$$\left[ r' \left(1 - e^{-2\lambda(r')}\right) \right]_{r'=r} - \left[ r' \left(1 - e^{-2\lambda(r')}\right) \right]_{r'=0} = 8\pi \int_0^r r^2 \rho(r') \, dr'.$$

The boundary condition $\lambda(0) = 0$ eliminates any possible contribution from the lower limit of the integral. Define the mass-integral

$$m(r) = 4\pi \int_0^r r^2 \rho(r') \, dr'$$

and write the resulting solution

$$r \left(1 - e^{-2\lambda}\right) = 2m(r)$$

or

$$\left(1 - e^{-2\lambda(r)}\right) = \frac{2m(r)}{r}$$

with the result

$$e^{2\lambda(r)} = g_{11}(r) = \frac{1}{1 - \frac{2m(r)}{r}}.$$

The total mass of the star, as seen from the outside is then given by

$$M = m(R) = 4\pi \int_0^R r^2 \rho(r') \, dr'.$$

This result seems curious because it is exactly the formula that Newton would have written. However, it is misleading because the integral does not use the volume element which goes with our assumed metric. The sum of the proper masses of all the individual molecules which went into making the star is given by the integral with the correct volume element

$$\omega^1 \land \omega^2 \land \omega^3 = \sqrt{\text{det}g} \, drd\theta d\phi = e^{\lambda} r^2 \sin^2 \theta \, drd\theta d\phi$$

$$M_{\text{proper}} = \iint \rho \omega^1 \land \omega^2 \land \omega^3 = 4\pi \int_0^R r^2 \rho(r) \frac{dr}{\sqrt{1 - \frac{2m(r)}{r}}}.$$ 

Notice that every part of the integrand is larger than the integrand in the integral for the external mass $M$. Thus, we always have

$$M_{\text{proper}} > M.$$

The difference between these two masses is just the binding energy of the star.
3.3.4 Solving the Rest of Einstein’s Equations

Now turn to the next Einstein equation \( G_{11} = 8\pi T_{11} \) or

\[
-\frac{1 - e^{-2\lambda}}{r^2} e^{2\lambda} + 2 \frac{\nu'}{r} = 8\pi pe^{2\lambda}
\]

and solve it for \( \nu' \)

\[
\nu' = \frac{1 - e^{-2\lambda}}{2r} e^{2\lambda} + 4\pi rp e^{2\lambda}
\]

where

\[
e^{-2\lambda} = 1 - \frac{2m(r)}{r}
\]

so that

\[
\nu' = \frac{2m(r)}{r} \frac{1}{1 - \frac{2m(r)}{r}} + \frac{4\pi rp}{1 - \frac{2m(r)}{r}}
\]

or

\[
\nu' = \frac{m(r)}{r^2} + 4\pi rp \frac{1}{1 - \frac{2m(r)}{r}} = \frac{1}{r} \frac{m(r) + 4\pi rp}{r - 2m(r)}
\]

(25)

3.3.5 Tolman-Openheimer-Volkoff Equation of Equilibrium

We could now turn to the last Einstein equation — equation (24) and substitute this last result into it as well as the solution for \( \lambda \). The result would be an equation for \( \frac{dp}{dr} \) but only after a lot of algebra. It is simpler to use the conservation law

\[
\nabla_{\mu} T^{\mu}_{\nu} = 0
\]

which is, after all, implied by Einstein’s equations. For a perfect fluid, we get

\[
\nabla_{\mu} T^{\mu}_{\nu} = \nabla_{\mu} (\rho V^{\mu}_{\nu} + p H^{\mu}_{\nu}) = -V^{\mu}_{\nu} \partial_{\mu} \rho - \rho \nabla_{\mu} V^{\mu}_{\nu} + H^{\mu}_{\nu} \partial_{\mu} p + p \nabla_{\mu} H^{\mu}_{\nu}
\]

I will use this calculation as an excuse for a bit of practice with projection tensors. Because the star is static, the derivative projected along the fluid world-lines will vanish and the conservation law becomes.

\[-\rho \nabla_{\mu} V^{\mu}_{\nu} + H^{\mu}_{\nu} \partial_{\mu} p + p \nabla_{\mu} H^{\mu}_{\nu} = 0\]

Now consider projecting this vector equation. Projecting with \( V \) would eliminate the \( \partial_{\mu} p \) term that we want to find an equation for. Thus, we want the complementary projection \( H \) that takes the spacelike component of the equation in the local rest frame.

\[
(-\rho \nabla_{\mu} V^{\mu}_{\nu} + H^{\mu}_{\nu} \partial_{\mu} p + p \nabla_{\mu} H^{\mu}_{\nu}) H^{\nu}_{\gamma} = 0
\]

which is equivalent to

\[
(-\rho \nabla_{\mu} V^{\mu}_{\nu} + \partial_{\nu} p + p \nabla_{\mu} H^{\mu}_{\nu}) H^{\nu}_{\gamma} = 0
\]
and it remains only to evaluate
\[
(\nabla_{\mu} V^{\mu}_{\nu}) H^{\nu}_{\gamma} = -H^{\nu}_{\gamma} \nabla_{\mu} (u^{\mu} u_{\nu}) \\
= - (\nabla_{\mu} u^{\mu}) H^{\nu}_{\gamma} u_{\nu} - (u^{\mu} \nabla_{\mu} u_{\nu}) H^{\nu}_{\gamma} \\
= - (u^{\mu} \nabla_{\mu} u_{\nu}) H^{\nu}_{\gamma}
\]
and
\[
(\nabla_{\mu} H^{\mu}_{\nu}) H^{\nu}_{\gamma} = (\nabla_{\mu} (V^{\mu}_{\nu})) H^{\nu}_{\gamma} = (u^{\mu} \nabla_{\mu} u_{\nu}) H^{\nu}_{\gamma}
\]
so that the equation we want becomes
\[
[\(\rho + p\) \(u^{\mu} \nabla_{\mu} u_{\nu}\) + \partial_{\nu} p] H^{\nu}_{\gamma} = 0
\]
Choose the spacelike direction to be the radial direction so that \(\gamma = 1\) and use the stationary nature of this star \(u^{\mu} = e^{-\nu} \delta_{0}^{\mu}\) obtain the result
\[
(\rho + p) e^{-\nu} \nabla_{0} u_{1} + \partial_{1} p = 0
\]
where
\[
\nabla_{0} u_{1} = \partial_{0} u_{1} - u_{\rho} \Gamma^{\rho}_{10} = -u_{0} \Gamma^{0}_{10} = e^{2\nu} u^{0} \Gamma^{0}_{10} = e^{\nu} \Gamma^{0}_{10} = e^{\nu} \Gamma^{0}_{01}
\]
But we found earlier that \(\Gamma^{0}_{01} = \nu'\) so the desired equation becomes
\[
(\rho + p) \nu' + p' = 0
\]
From equation (25) we obtain the desired condition on the pressure and density of the star:
\[
\frac{dp}{dr} = -\frac{1}{r} \frac{m(r) + 4\pi r^{3} p}{r - 2m(r)} (\rho + p)
\]
This is the pressure gradient that is needed to support the star in static equilibrium. Notice that the required gradient blows up for \(r\) near the value \(2m(r)\).

### 3.3.6 Relativistic Stellar Models

We now have a system of equations which can be integrated on a computer once an equation of state \(p(\nu)\) has been given. From the definition of \(m\) and the Tolman-Oppenheimer-Volkoff equation above, we get a first order system with two dependent variables \(m, \rho\) all set for numerical integration.

\[
\frac{dm}{dr} = 4\pi r^{2} \rho (r)
\]

\[
\frac{d\rho}{dr} = -\frac{1}{r} \frac{1}{d\rho/d\rho} \frac{m(r) + 4\pi r^{3} p(\rho (r))}{r - 2m(r)} (\rho (r) + p(\rho (r)))
\]

Start at \(m = 0, \rho = \rho_{c}\) and integrate outward until the pressure drops to zero. For each choice of central density \(\rho_{c}\) we get a value for the radius \(R\) of the star and the total externally observed mass \(M = m(R)\) of the star. The functions
$R(\rho_c)$ and $M(\rho_c)$ are the essential results of the calculation. These results are usually presented as a parameterized curve in the $M-R$ plane.

Equation (25) can be added to the first order system to obtain the remaining unknown function $\nu(r)$. Start at $\nu = 0$ and then add a constant to the result (rescale the internal time coordinate) so that the result agrees with the external metric value of $\nu(R)$. The result is a complete spacetime metric to go with the stellar model. However, it is often not necessary to add this step since the critical information about stars is contained in the $M$ vs. $R$ plot.
For low values of the central density, one expects both the mass and the radius of the star to rise as the central density rises. Imagine dropping more and more cold matter onto a star. After each increase in total mass, the star must seek a new equilibrium state consistent with that total mass. So long as the star is on a part of the curve where $M$ is rising with central density, it just increases its central density and settles down again. Suppose, however, that $M$ goes through a local maximum at the present configuration and we add some mass. Now there is no nearby stable configuration for the star to go to and the star becomes unstable, increasing its central density and possibly shedding mass in an explosion until it reaches a part of the curve where $M$ rises to the value that it needs.

The usual equation of state — a composite of many processes — shows local maxima corresponding to white dwarf stars which are supported by electron degeneracy pressure and neutron stars which are supported by neutron degeneracy pressure. The white dwarf maximum is about 1.4 solar masses at a radius about equal to the radius of the Earth. The neutron star maximum depends upon the equation of state used and is usually less than 3 solar masses at a radius of a few kilometers. When a white dwarf accumulates matter beyond its maximum (called the Chandrasekhar limit), it undergoes a nearly free-fall collapse to the much smaller white-dwarf stage. The enormous gravitational potential energy released during this collapse causes an outward shock wave when the material “hits bottom” and the result is called a “Type I supernova”. The aftermath of such an explosion is a small, rapidly rotating neutron star. For some equations of state (like the one pictured above) it is possible for the matter to fall right past all of the possible neutron star configurations and collapse without end. Alternatively, if matter slowly falls onto the neutron star until it reaches a maximum $M$, the star again becomes unstable and collapses further. Since there is no further maximum of $M$, the collapse does not stop. The result of an endless collapse is called a “black hole” and will be discussed later.