1 Initial Value Structure of Einstein’s Equations

1.1 Standard System

The traditional form of Einstein’s field equations consists of the structure equations

\[ Q^{\alpha\beta\gamma} = 0, \quad T^\nu_{\alpha\beta} = 0 \]  

(1)

that are usually solved for the connection coefficients in terms of the metric tensor components and the curvature equation

\[ G^{\mu\nu} = 8\pi G T^{\mu\nu} \]  

(2)

where the trace-reversed Ricci tensor is given the special symbol

\[ G^{\mu\nu} = R^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \]

and is called the Einstein Tensor. In this form, the curvature equation becomes a system of ten partial differential equations for the ten metric tensor components \( g_{\mu\nu} \).

Early research into the structure of these equations was greatly hampered by the fact that the standard form of them is really quite complex. Once the solution of the structure equations is substituted into the curvature equation, the result is lengthy and involves both the metric components \( g_{\mu\nu} \) and their inverse matrix \( g^{\mu\nu} \). The presence of the inverse matrix components makes the equations especially unpleasant because they are not merely nonlinear but nonpolynomial as well.

One puzzle, which arises at once, is that there seems to be the wrong number of equations. With ten independent equations for ten metric tensor components, one would expect the metric tensor components to be fixed once the appropriate boundary conditions have been given. However, the curvature equation consists of components of a geometrical, coordinate-independent condition. Thus, the coordinates must remain as four arbitrary functions per spacetime point. That seems to contradict having the metric tensor components determined throughout spacetime once boundary conditions have been given.

1.2 General Initial Value Problem (Fails)

Suppose we wish to generate values for the functions \( g_{\mu\nu} (x) \) for each set \( x = \{x^a, x^b, x^c, x^*\} \) of values of a particular set of spacetime coordinates. Here, the coordinate \( x^* \) is the one that has been picked out for evolving a solution of the Einstein Field equations. Start by specifying the appropriate values and derivatives of the metric tensor on the surface \( x^* = 0 \) and then use the Einstein equations to evolve the metric tensor to other values of \( x^* \). The problem of setting up the appropriate values on the initial surface \( x^* = 0 \) is called the initial value problem. It can take several forms, depending on the nature of the initial surface. If that initial surface is spacelike, then \( x^* = x^0 \) is a time coordinate,
$x^a, x^b, x^c$ are space coordinates $x^1, x^2, x^3$ and the \textit{spacelike initial value problem} is the result. If that initial surface is null or lightlike, then $x^* = x^-$ is a null coordinate that might, for example, be retarded time, $x^c = x^+$ is another null coordinate that might be advanced time, and one usually combines the remaining (spacelike) coordinates into a complex coordinate $x^a + i x^b = z$. The result is called the \textit{characteristic initial value problem}, a name that comes from the general theory of second order partial differential equations.

Start with the (incorrect) assumption that the curvature equations provide a straightforward set of second order partial differential equations for the metric tensor components. Solve these equations for the ten second derivative functions $g_{\mu
u, **}$ with a result that we represent as

$$g_{\mu
u, **} = F_{\mu\nu} (g_{\alpha\beta}, g_{\alpha\beta, \gamma}, g_{\alpha\beta, q*}, g_{\alpha\beta, qs})$$

where the indexes $q, s$ take values $a, b, c$. To evolve a solution in the variable $x^*$ specify $g_{\alpha\beta}$ and $g_{\alpha\beta, *}$ everywhere on the initial surface $x^* = 0$. All of the other derivatives that appear in the function $F_{\mu\nu}$ can then be found by taking derivatives within the surface. At this point the system has been reduced to first order form. Define $p_{\mu\nu} = g_{\mu\nu, *}$ and write the first order system

$$\dot{g}_{\mu\nu} = p_{\mu\nu}$$
$$\dot{p}_{\mu\nu} = F_{\mu\nu} (g_{\alpha\beta}, p_{\alpha\beta}, g_{\alpha\beta, q}, p_{\alpha\beta, q}, g_{\alpha\beta, qs})$$

where dots are being used to represent derivatives with respect to the evolution variable $x^*$. Evolve this system on a computer by converting the derivatives to differences so that the above system becomes

$$g_{\mu\nu} (\Delta x^*) = g_{\mu\nu} (0) + p_{\mu\nu} (0) \Delta x^*$$
$$p_{\mu\nu} (\Delta x^*) = p_{\mu\nu} (0) + F_{\mu\nu} (0) \Delta x^*$$

The critical assumption here is that all ten of the equations contain independent second derivatives with respect to $x^*$ so that the solution $F_{\mu\nu}$ can be found. That assumption is wrong and we can find that out the hard way by writing out the equations in detail or we can take a short cut by exploiting the geometrical nature of the curvature equation.

### 1.3 Implications of the Bianchi Identities

In the process of finding the curvature equations we noted that the Einstein tensor obeys the identities

$$G_{\mu
u, ;\nu} = 0$$
that can be obtained by contracting the curvature Bianchi identities twice. The important thing to remember is that these equations are identities — if one expands the curvature tensor in terms of connection coefficients and metric components and calculates the above expression, all of the terms will cancel. Now write out the sum in the above expression

$$G^\mu q_{,q} + G^{\mu*}_{,s} = 0$$

and be more explicit about the derivatives

$$\frac{\partial G^\mu q}{\partial x^q} + \Gamma^\mu_{pq} G^p q + \Gamma^q_{pq} G^{\mu p} + \frac{\partial G^{\mu*}}{\partial x^s} + \Gamma^\mu_{ps} G^{p*} + \Gamma^{*}_{ps} G^{\mu p} = 0$$

Pull one term to the left of the equation so that we can focus on it.

$$- \frac{\partial G^{\mu*}}{\partial x^s} = \frac{\partial G^\mu q}{\partial x^q} + \Gamma^\mu_{pq} G^p q + \Gamma^q_{pq} G^{\mu p} + \Gamma^\mu_{ps} G^{p*} + \Gamma^{*}_{ps} G^{\mu p}$$

The Einstein tensor contains up to second derivatives of the metric tensor components. The first term on the right in this last expression has third derivatives in it, but the extra derivative is not with respect to the evolution variable. Thus, the right-hand side of the equation has no third derivatives with respect to $x^*$. Since this expression is an identity, the left-hand side of the equation must also contain no third derivatives with respect to $x^*$. One derivative with respect to $x^*$ can be plainly seen, so the Einstein tensor components $G^{\mu*}$ cannot contain any second derivatives of the metric with respect to $x^*$.

The four curvature equations

$$G^{\mu*} = 8\pi T^{\mu*}$$

have a special character. They cannot be solved for $g_{\mu*,*}$ and can only depend on quantities $g_{\mu*}, g_{\mu*,*}, g_{\mu*,q*}, g_{\mu*,qs}$ that can all be obtained from $g_{\mu*}, g_{\mu*,*}$ on the initial surface. Thus, they contribute nothing to evolving the metric tensor off the surface and simply constrain the initial values that can be given for the functions $g_{\mu*}, g_{\mu*,*}$. For this reason they are called the initial value constraint equations.

The remaining six Einstein curvature equations can be solved for second derivatives and contribute to the evolution of solutions in the usual way. Thus, we have four constraints on the initial data that must be solved before the evolution begins and six dynamical equations that evolve the ten metric functions, leaving four metric functions arbitrary as demanded by the coordinate-independent nature of the formalism.

One more point remains to be noted. From the twice contracted Bianchi identity as we last wrote it above, the derivative $\frac{\partial G^{\mu*}}{\partial x^s}$ is expressed in terms of the components of the Einstein tensor and its derivatives with respect to coordinates other than $x^*$. Because the stress-energy tensor obeys an equation with the same structure as the twice contracted Bianchi identity, the difference tensor

$$E^{\mu\nu} = G^{\mu\nu} - 8\pi T^{\mu\nu}$$
will also obey the same sort of equation:

$$-\frac{\partial E^\mu x^\nu}{\partial x^\nu} = \frac{\partial E^\mu q}{\partial x^q} + \Gamma^\mu_{\rho q} E^{\rho q} + \Gamma^\mu_{\rho q} E^{\rho \nu} + \Gamma^\mu_{\rho \nu} E^{\rho q}$$

Thus, if we start with initial data that satisfies the initial value constraints $E^\mu x^\nu = 0$ and use the dynamical equations $E^{pq} = 0$ to determine the evolution of the metric tensor, then all of the terms on the right-hand side of the above equation will be zero. Consequently, for any solution constructed in this way,

$$\frac{\partial E^\mu x^\nu}{\partial x^\nu} = 0$$

so that the initial value constraints do not need to be reimposed — they are preserved by the dynamical equations.

### 1.4 The Spacelike Initial Value Problem

To generate a solution of Einstein’s equations from initial data on a spacelike hypersurface (which I will take as $t = 0$), follow these steps:

1. Solve the initial value equations $E^\mu 0 = 0$ on the initial spacelike hypersurface.

2. Choose four conditions on the spacetime metric functions in order to determine how the spacetime coordinates will evolve. For example, the three conditions $g_{m0} = 0$ cause the constant-position lines ($x^m = \text{const.}$) to be orthogonal to the $x^0$ = const. hypersurfaces while the condition $g_{00} = -1$ ensures that these lines will be geodesics with $x^0 = t$ the proper time along them. Another example of a set of coordinate conditions would be $\partial_0 g^{\mu \nu} = 0$ which turns out to be equivalent to using spacetime coordinates which obey the wave equation $\Delta (x^\mu) = 0$. (Show this.)

3. Combine the six dynamical Einstein equations $E^{ij} = 0$ with the four coordinate conditions to find a set of equations that can be integrated forward in time.

A useful trick, due to Arnowitt, Deser, and Misner, is to cast the combined coordinate conditions and dynamical equations into the form of Hamilton’s equations

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

so that, knowing the values of the variables $q, p$ everywhere at a given time $t$ lets one estimate their values at a later time $t + \Delta t$ from

$$q (t + \Delta t) = q (t) + \Delta \frac{\partial H}{\partial p} (q(t), p(t))$$

$$p (t + \Delta t) = p (t) - \Delta \frac{\partial H}{\partial q} (q(t), p(t))$$
The appropriate \( q \) variables to use in this formalism are the “three-metric” components \( ^3g_{ij} (x^k, t) \) on the \( x^0 = t = \text{constant} \) hypersurfaces. The momentum variables that are conjugate to the metric components are denoted \( \pi^{ij} (x^k, t) \) and are combinations of the time derivative of the three-metric with space derivatives of the mixed components \( g_{0i} \) of the spacetime metric. We will return to this scheme later in this chapter and again, when we discuss how to project geometrical structures onto surfaces.

This method of generating solutions of Einstein’s equations has been implemented on large computers. It turns out to have several unpleasant aspects.

1. One must start with a solution of the initial value equations and they are not trivial to solve.

2. It is not always obvious that a given solution to the initial value equations represents the physical situation that one wants it to represent.

3. Numerical methods are only approximate, so the initial value equations are only approximately preserved as one evolves a solution in time. If one is not careful, the numerical errors in the initial value equations will grow exponentially in time, invalidating the solution.

4. A particular choice of coordinate conditions will often give rise to coordinate singularities even though nothing singular is happening to the spacetime. The numerical evolution stops at that point, often well before anything of real interest has happened to the spacetime. For example, the simplest coordinates, geodesic hypersurface-orthogonal coordinates, \((g_{00} = -1, \ g_{0i} = 0)\) break down because their constant-position lines are actually in free-fall and will intersect each other in finite amounts of time. Near the Earth, such coordinates are good for about one hour.

1.5 Degrees of Freedom of the Gravitational Field

How many functions per spacetime point characterize the gravitational field? It is necessary to count carefully. We start by eliminating four spacetime metric tensor components through four coordinate conditions that fix the evolution of the spacetime coordinates once they are given on an initial surface. The remaining six components and their time derivatives amount to twelve functions for an apparent six dynamical degrees of freedom. These twelve functions are subject to four initial value constraints, leaving eight freely specifiable variables per spacetime point. We must not forget that four of these variables correspond to the freedom to specify the spacetime coordinates on the initial surface. Thus, only four freely specifiable, physically significant functions can be given on an initial hypersurface. Since these functions include both the variables and their conjugate momenta, the final count is two degrees of freedom per spacetime point.
1.6 Maxwell’s Equations Initial Value Problem

Maxwell’s equations turn out to have a structure very similar to the one that we have been exploring for Einstein’s equations. It is somewhat dangerous to use the properties of Maxwell’s equations as a guide to the properties of Einstein’s equations because Maxwell’s equations have many nice properties (linearity for one) that Einstein’s equations do not have. However, now that we have discussed the structure of Einstein’s equations, it is helpful to see that Maxwell’s equations have the same structure.

Maxwell’s equations have the covariant form

\[ F_{\mu\nu,\gamma} = 0 \]  

(3)

corresponding to the familiar equations

\[ \nabla \cdot B = 0, \quad \dot{B} - \nabla \times E = 0 \]

and

\[ g^{\rho\sigma} F_{\mu\sigma,\rho} = 4\pi j_\mu \]  

(4)

which corresponds to

\[ \nabla \cdot E = 4\pi \rho, \quad \dot{E} + \nabla \times B = 4\pi j \]

Equation (3) can be solved by introducing the vector potential \( A_\mu \)

\[ F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \]

and the remaining Maxwell equations become

\[ g^{\mu\nu} g^{\rho\sigma} (A_{\mu,\sigma} - A_{\sigma,\mu})_{,\rho} = 4\pi j^\nu \]

Just as with Einstein’s equations, there is an identity

\[ \left[ g^{\mu\nu} g^{\rho\sigma} (A_{\mu,\sigma} - A_{\sigma,\mu})_{,\rho} \right]_{,\nu} = 0 \]

which causes Maxwell’s equations to guarantee the conservation of the source current — \( g^{\mu\nu} \nabla_\nu j_\mu = 0 \). This identity can be used to argue that one of Maxwell’s equations is an initial value constraint, just as we used the twice contracted Bianchi identities to analyze the nature of Einstein’s equations. Here, however, the equations are simple enough so that we can see their structure explicitly.

Consider the \( \mu = 0 \) equation

\[ g^{\rho\sigma} (A_{0,\sigma} - A_{\sigma,0})_{,\rho} = 4\pi j_0 \]

and write out its components in a Lorentz frame in Minkowski spacetime

\[ -\partial_0 (\partial_0 A_0 - \partial_0 A_0) \\
+ \partial_1 (\partial_1 A_0 - \partial_0 A_1) \\
+ \partial_2 (\partial_2 A_0 - \partial_0 A_2) \\
+ \partial_3 (\partial_3 A_0 - \partial_0 A_3) = 4\pi j_0 \]
The second time derivatives in the first term clearly cancel so that this equation involves only the vector potential components \( A_\mu \) and their first time derivatives. It is a constraint on the initial value data.

Where Einstein’s theory has arbitrary coordinates, electromagnetic theory has an arbitrary gauge. The vector potential can be modified according to

\[ A'_\mu = A_\mu + \Lambda_\mu \]

without changing the physically significant field tensor \( F_{\mu\nu} \) at all. Thus, we need to impose a condition that fixes the evolution of the gauge as well as a condition on the initial data to fix the initial gauge. A typical gauge evolution constraint is \( A_0 = 0 \) (the radiation gauge) which is exactly analogous to the hypersurface orthogonal time condition \( g_{\mu0} = 0, g_{00} = -1 \) in Einstein’s theory.

Now count the degrees of freedom of an electromagnetic field. From now on, I use a local Lorentz frame in Minkowski spacetime for all equations. Of the four vector potential components, one is eliminated by a gauge evolution constraint such as \( A_0 = 0 \). Six functions per spacetime point — three components \( A_j \) and their three time derivatives \( \partial_0 A_j \) are evolved by the three dynamical Maxwell equations. These six functions must satisfy the constraint

\[ \sum_{j=1}^{3} (\partial_j A_0 - \partial_0 A_j)_{,j} = 4\pi j_0 \]

which corresponds to \( \nabla \cdot E = 4\pi \rho \). Furthermore, some condition is needed to eliminate the gauge freedom on the initial surface. One good condition for specifying the initial gauge is

\[ \sum_{j=1}^{3} \partial_j A_j = 0 \]

which means that the six functions \( A_j, A_{j;0} \) are subject to two conditions, leaving four freely specifiable variables per spacetime point or two degrees of freedom, exactly as in the gravitational case.

2 The Imperfect Analogy

2.1 Maxwell’s Theory

The structure of Einstein’s Theory of Gravity bears strong similarities to the structure of Maxwell’s theory of electromagnetism but there are problems in deciding just how to compare these theories. In Maxwell’s theory, the actual forces on charged particles are governed by the field tensor \( F_{\mu\nu} \). This tensor is the physically significant electromagnetic field. The vector potential that, in a local Lorentz frame, is related to the field tensor by

\[ F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} \]

includes some physically irrelevant gauge information. Now consider Einstein’s theory and ask “What fields govern the actual forces on particles?”
2.2 Gravity: The Force Analogy

The apparent force in a particular coordinate system is given by the connection coefficients $\Gamma^\rho_{\sigma\delta}$. If we decide that these are analogous to the electromagnetic field tensor components $F_{\mu\nu}$, then the equation

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\sigma}(g_{\sigma\beta,\gamma} + g_{\sigma\gamma,\beta} - g_{\beta\gamma,\sigma})$$

suggests that the metric tensor components are analogous to the vector potentials. Notice, however, that the structure of this last equation is not much like that of equation (5)

Now write out the curvature tensor components

$$[R]_{\mu\nu} = \nabla_\mu[\Gamma]_{\nu} - \nabla_\nu[\Gamma]_{\mu} + [\Gamma]_{\mu}[\Gamma]_{\nu} - [\Gamma]_{\nu}[\Gamma]_{\mu} - c_{\mu\nu}[\Gamma]_{\sigma}$$

and assume a local Lorentz frame (where the connection coefficients vanish but their derivatives do not) so that we get just

$$[R]_{\mu\nu} = [\Gamma]_{\nu,\mu} - [\Gamma]_{\mu,\nu}$$

and see if the structure of Einstein’s equations looks like Maxwell’s equations in terms of this analogy. or

$$R^\alpha_{\beta\mu\nu} = -2\Gamma^\alpha_{\beta[\mu,\nu]}$$

(6)

where I have used the $\Gamma = 0$ property of the local Lorentz frame. The Einstein tensor is then

$$G_{\mu\nu} = R^\rho_{\mu\rho\nu} - \frac{1}{2}g_{\mu\nu}R^\rho_{\alpha\rho\beta}g^{\alpha\beta}$$

$$= -2\Gamma^\rho_{\mu[\rho,\nu]} + g_{\mu
u}\Gamma^\rho_{\alpha[\rho,\beta]}g^{\alpha\beta}$$

and Einstein’s Field Equations become

$$2\Gamma^\lambda_{\mu[\nu,\lambda]} - g^{\rho\sigma}\Gamma^\lambda_{\rho[\sigma,\lambda]}g_{\mu\nu} = 8\pi GT_{\mu\nu}.$$ 

Does this look like Maxwell’s equations in terms of $F_{\mu\nu}$? If we use harmonic coordinates, that obey the conditions

$$\Gamma^\lambda_{\mu\lambda} = 0$$

and are compatible with our local Lorentz frame conditions, the Einstein equations take the form

$$\Gamma^\lambda_{\mu\nu,\lambda} - \frac{1}{2}g^{\rho\sigma}\Gamma^\lambda_{\rho\sigma,\lambda}g_{\mu\nu} = 8\pi GT_{\mu\nu}$$

or

$$\Gamma^\lambda_{\mu\nu,\lambda} = 8\pi GT_{\mu\nu}$$

which is not too different from Maxwell’s equation.
One problem with interpreting the connection coefficients as analogous to the electromagnetic field tensor is that the connection coefficients can be made to vanish at any point by choosing a freely falling inertial coordinate system. The freely falling observer has no local way to find out what the connection coefficients are. The only thing that such an observer can measure is the tidal force tensor that corresponds to components of the curvature.

### 2.3 Gravity: The Tidal Force Analogy

Tidal forces are described by the Riemann tensor $R^{\alpha}_{\beta \mu \nu}$ which cannot be made to vanish at a point by any coordinate transformation. Equation (6) then corresponds to the electromagnetic vector potential equation (5) which can be rewritten as

$$F_{\mu \nu} = 2A_{[\mu, \nu]}.$$  

These two equations really do look a lot alike. The gravitational equation just has a few extra indexes. Thus, the connection coefficients are analogous to the vector potential components. Recall that vector potential was introduced to solve the Maxwell equations

$$F_{[\mu \nu]} = 0.$$  

The curvature tensor constructed from the connection coefficients for a zero-torsion connection obeys the curvature Bianchi identity that can be written in the form

$$R^{\alpha}_{\beta \mu \nu ; \gamma} = 0.$$  

However, this analogy leaves the Einstein equations looking nothing at all like the remaining Maxwell equation.

### 2.4 An Extra Layer of Potentials

Einstein’s theory has three layers of variables, each a set of potentials for the next: $g_{\mu \nu} \rightarrow \Gamma^{\alpha}_{\beta \delta} \rightarrow R^{\alpha}_{\beta \mu \nu}$. Maxwell’s theory has only two such layers of variables, $A_{\mu} \rightarrow F_{\mu \nu}$. In each case, only the last layer corresponds to observables that are well-defined as soon as one picks a local set of basis vectors. Also, in each case, only the first layer corresponds to freely specifiable variables. Each way of comparing the Maxwell Theory’s two layers to a pair of the Einstein Theory’s three layers has some good points. These days most physicists favor the analogy

$$A_{\mu} \rightarrow F_{\mu \nu}$$

$$\Gamma^{\alpha}_{\beta \delta} \rightarrow R^{\alpha}_{\beta \mu \nu}$$

so that the connection coefficients are analogous to the vector potential. The main reason for favoring this analogy is that the vector potential can be interpreted as a connection on a fibre bundle over spacetime and the field tensor is the curvature of this fibre bundle. Nevertheless, the usual formulation of Einstein’s theory in terms of metric, connection, and curvature, is not a very comfortable match to Maxwell’s theory or to any other field theory in physics.
3 The 3+1 Split

3.1 Undoing Spacetime Covariance

A long struggle was needed to cast gravitation theory into a form that unifies space and time and does not single out any particular reference frame. By the late 1950s this struggle was over and the formulation of general relativity in terms of spacetime geometry was well understood. Thus it came as a bit of a shock to veteran relativists when the path to understanding the initial value problem involved splitting space and time in a preferred reference frame.

The spacelike initial value problem asks one to specify data on an initial spacelike hypersurface and then use Einstein’s equations to evolve the rest of the spacetime. The difficulty of the problem comes from the fact that some of Einstein’s equations constrain the initial data that can be given. Arnowitt, Deser, and Misner realized that this problem involves a preferred reference frame right from the start, the frame in which the initial hypersurface is at constant time. Their program was as follows: Take full advantage of that preferred frame. Express the full spacetime metric tensor in terms of the metric tensor on the initial hypersurface. Express spacetime covariant derivatives in terms of covariant derivatives within the hypersurface. Express the spacetime curvature tensor in terms of the three-dimensional curvature tensor of the hypersurface.

3.2 The ADM Variables

The best way to reexpress spacetime objects in terms of the geometry of a hypersurface is to use the tensor that projects spacetime vectors onto the hypersurface. We will discuss this projection tensor approach later. Here, I will simply introduce the new hypersurface-covariant variables much as Arnowitt, Deser, and Misner did. The metric tensor $h$ on the surface is easiest to find from the contravariant components which express dot products of basis one-forms. The unit one-form $d$ normal to the surface, defined by

$$d \cdot d = -1$$

expresses changes in the proper time $\tau$ measured along curves perpendicular to the surface and is related to changes in coordinate time by

$$d\tau = N dx^0$$

where $N$ is called the lapse function. In terms of the spacetime metric components,

$$N^2 dx^0 \cdot dx^0 = -1$$

or

$$g^{00} = -N^{-2}.$$  

For the remaining components, find the rates of change of space coordinates with respect to proper time $\tau$ along a curve perpendicular to the surface. The
The spacetime metric supplies the tangent vector to this curve:

$$\frac{\partial}{\partial \tau} = g^{-1} (d\tau)$$

so

$$\frac{\partial x^i}{\partial \tau} = g^{-1} (d\tau, dx^i) = N g^{-1} (dx^0, dx^i) = N g^{0i}$$

and the rate of change of the space coordinates with respect to coordinate time along such a curve is

$$N^i = \frac{\partial x^i}{\partial \tau} = N_i^i = N^2 g^{0i}.$$ These three components are referred to as the *shift vector*. They describe how the space coordinates shift relative to hypersurface-orthogonal time lines as one moves from one value of the time coordinate $x^0$ to a neighboring value. The spacetime metric tensor $g$ and its inverse $g^{-1}$ can now be expressed in terms of the lapse and shift function and the three-dimensional metric $h$ that is induced on the constant-time surfaces. The result is:

$$[g] = \begin{pmatrix}
|\bar{N}|^2 - N^2 & h (\bar{N})^T \\
N & [h]
\end{pmatrix}$$

$$[g^{-1}] = \begin{pmatrix}
-1/N^2 & \bar{N}^T/N^2 \\
N/N^2 & [h]^{-1} - \bar{N}^T N^2/N^2
\end{pmatrix}$$

where $\bar{N}$ is the set of shift-vector components $N^i$ arranged as a column matrix. Verify by direct matrix multiplication that $[g^{-1}] = [g]^{-1}$. Notice that the invariant four dimensional volume element is connected to the invariant three-volume by

$$\sqrt{|g|} d^4x = N \sqrt{|h|} dx^0 d^3x$$

The lapse and shift functions are clearly coordinate-dependent and say nothing directly about the actual spacetime geometry. Thus, we do not expect these functions to be dynamical variables. The three-dimensional metric tensor $h$ specifies part of the initial data that we need by giving the geometry of the initial hypersurface. The rate of change of this geometry must also be given in order to have a full set of initial data. The time derivatives $\partial_{\tau^0} h_{ij}$ would specify this rate of change but in a way that depends on the evolution of the coordinate system. Fortunately, the dependence can be separated out in the expression

$$\frac{\partial}{\partial x^0} h_{ij} = 2N K_{ij} + N_{i|j} + N_{j|i}$$

where the subscripts after `|` denote covariant derivatives with respect to a connection compatible with the hypersurface metric $h$. The invariant part of this quantity, $K_{ij}$, is called the extrinsic curvature or second fundamental form.
of the hypersurface and is explained in the appendix to this section. It turns out to be convenient to use a related quantity

$$\pi^{ij} = -\sqrt{h} \left( K^{ij} - h^{ij} K \right)$$

where $$\sqrt{h} = \sqrt{|h|}$$ is the square root of the determinant of $$h_{ij}$$, latin indexes are raised using the inverse hypersurface metric $$h^{ij}$$ and $$K = K^{rr}$$ is the trace. In terms of this quantity, Einstein’s vacuum field equations take the form of four constraints on the initial data

$$\sqrt{h} \left[ 3 R + h^{-1} \left( \frac{1}{2} \pi^2 - \pi^{ij} \pi_{ij} \right) \right] = 0$$

$$\pi^{ij} = 0$$

and a set of equations for evolving the data which take the form

$$\frac{\partial}{\partial x^0} \pi^{ij} = F^{ij} \left( h, \pi, N, \dot{N} \right).$$

For future reference, note that $$\pi^{ij}$$ represents a tensor-density rather than a simple tensor so that the covariant derivative in eq. 8 has an extra term and can be written in terms of tensor components as.

$$(K^{ij} - h^{ij} K)_{ij} = 0.$$

This system of equations and variations of it can be used by a computer to produce a numerically evolved solution to Einstein’s equations. To start the evolution, one needs a solution to the four constraints and coordinate conditions that determine the lapse and shift functions.

### 3.3 The Thin Sandwich Idea (Fails)

Since there are four constraint equations to be solved and four coordinate-dependent variables to be determined, it is tempting to simply put those two things together. Specify both the space metric $$h_{ij}$$ and its time derivative $$\frac{\partial}{\partial x^0} h_{ij} = \dot{h}_{ij}$$ so that

$$\pi^{ij} = -\sqrt{h} \left( K^{ij} - h^{ij} K \right)$$

$$= -\sqrt{h} \frac{1}{2N} \left[ (\dot{h}^{ij} - N^{|ij} - N^{j|i}) - h^{ij} h^{rs} \left( \dot{h}_{rs} - N_{r|s} - N_{s|r} \right) \right]$$

$$= -\sqrt{h} \frac{1}{2N} \left[ (\dot{h}^{ij} - h^{ij} h^{rs} \dot{h}_{rs}) + (N^{j|i} + N^{i|j} - 2h^{ij} h^{rs} N_{r|s}) \right]$$

and attempt to solve the four constraints for the lapse and shift functions $$N$$ and $$N^i$$. Equation 8 then takes the form of a system of linear second order differential equations for the shift components

$$- \left[ \frac{1}{2N} \left( \dot{h}^{ij} - h^{ij} h^{rs} \dot{h}_{rs} \right) \right]_{ij} - \left[ \frac{1}{2N} \left( D^i N^j + D^j N^i - 2h^{ij} h^{rs} D_r N_s \right) \right]_{ij} = 0$$

12
The ‘thin sandwich’ idea is to solve this linear system for the shift functions \( N^i \) and then solve the Hamiltonian constraint for the lapse function \( \bar{N} \). The ‘thin sandwich’ name comes from thinking of the space metric \( h_{ij} \) and its time derivative \( \partial_t h_{ij} = \dot{h}_{ij} \) as describing a pair of neighboring constant-time hypersurfaces and the lapse and shift functions as connecting them and ‘filling’ the sandwich.

At first, this system looks promising. It takes the form

\[
\mathcal{D} (\bar{N}) = \tilde{j}
\]

where

\[
j^i = -\left[ \frac{1}{2\bar{N}} \left( h^{ij} - h^{ij} \dot{h}_{rs} \right) \right]_{ij}
\]

and

\[
\mathcal{D}^i (\bar{N}) = \left[ \frac{1}{2\bar{N}} \left( h^{ir} N^j_{|r} + h^{jr} N^i_{|r} - 2h^{ij} N^r_{|r} \right) \right]_{ij}
\]

All one has to do is invert the second order operator \( \mathcal{D} \) and obtain the solution

\[
\bar{N} = \mathcal{D}^{-1} \tilde{j}.
\]

That program does not seem too different from what one does to invert the Laplacian operator to solve Poisson’s equation

\[
D^2 \Phi = h^{rs} \Phi_{|sr} = q \quad \Phi = D^{-2} q.
\]

The key to success is to establish that, with the right boundary conditions, the solution of the equation is unique.

First, review how the uniqueness proof works for Poisson’s equation. Let \( \phi \) be the difference between two distinct solutions for \( \Phi \). That difference must obey the homogeneous equation:

\[
h^{rs} \phi_{|sr} = 0.
\]

Now consider the integral over the constant-time hypersurface \( \Sigma \)

\[
\int_{\Sigma} d^3 x \sqrt{h} \phi h^{rs} \phi_{|sr} = 0.
\]

Pull out a total divergence from the integrand and write this integral in the form

\[
\int_{\Sigma} d^3 x \sqrt{h} \left[ \left( \phi h^{rs} \phi_{|s} \right)_{|r} - \left( \phi_{|r} \right) h^{rs} \left( \phi_{|s} \right) \right] = 0
\]

From a familiar theorem of differential geometry, the total divergence term gives rise to a surface integral over the boundary of the hypersurface \( \Sigma \).

\[
\int_{\partial \Sigma} d a n^s \phi \phi_{|s} = \int_{\Sigma} d^3 x \sqrt{h} \phi_{|r} h^{rs} \phi_{|s} = 0
\]
Here $n^s$ represents the normal to the boundary and $da$ is an area element on the boundary. If the value of either $\Phi$ or the normal derivative $n^s\Phi |_s$ is fixed on the boundary, then the corresponding values for the difference $\phi$ will be zero and one obtains

$$\int_\Sigma d^3x \sqrt{h} \sqrt{h} \phi_{ijr} h^{rs} \phi |_s = 0$$

Since the integrand is positive somewhere for any non-zero function $\phi$, the integral will be positive unless $\phi$ is zero everywhere on the hypersurface. As a result, the solution of the original Poisson equation is unique and it is OK to represent it symbolically as $\Phi = D^2 q$.

For the shift-vector equation, the difference $\vec{\phi}$ between two solutions will obey the integral equation

$$\int_\Sigma d^3x \sqrt{h} \vec{\phi} \cdot \vec{\nabla} \left( \vec{\phi} \right) = 0$$

or, in detail

$$\int_\Sigma d^3x \sqrt{h} \phi_i \left[ \frac{1}{2N} \left( h^{ir} \phi^j_{|r} + h^{jr} \phi^i_{|r} - 2h^{ij} \phi^r_{|r} \right) \right]_{ij} = 0$$

which, after extracting a divergence, yields the relation

$$\int_{\Sigma} \int da \, n_j \phi_i \left[ \frac{1}{2N} \left( h^{ir} \phi^j_{|r} + h^{jr} \phi^i_{|r} - 2h^{ij} \phi^r_{|r} \right) \right] - \int_\Sigma d^3x \sqrt{h} \phi_{ij} \left[ \frac{1}{2N} \left( h^{ir} \phi^j_{|r} + h^{jr} \phi^i_{|r} - 2h^{ij} \phi^r_{|r} \right) \right] = 0$$

So long as the shift vector is fixed on the boundary of the region, the difference vector $\vec{\phi}$ will vanish there and the surface integral will be zero. However, when we inspect the resulting integral over the constant-time hypersurface,

$$\int_\Sigma d^3x \frac{\sqrt{h}}{2N} \left[ h^{ir} \phi^j_{|r} \phi_{ij} + h^{jr} \phi^i_{|r} \phi_{ij} - 2h^{ij} \phi^r_{|r} \phi_{ij} \right] = 0$$

or

$$\int_\Sigma d^3x \frac{\sqrt{h}}{2N} \left[ \phi^j_{|i} \phi^i_{|j} + h^{jr} h_{is} \phi^i_{|r} \phi^s_{|j} - 2 \left( \phi^r_{|r} \right)^2 \right] = 0$$

the integrand is simply not positive definite. Thus, one cannot conclude that $\vec{\phi}$ is zero and the uniqueness of solutions for the shift vector is not established. It appears that the thin sandwich approach to solving the constraint equations is not well posed.

4 Conformal Transformations

4.1 The Time-Symmetric Case

Suppose that a spacetime has a constant-time hypersurface that represents an instant of time symmetry. In that case, it should be possible to arrange for
all rates of change with respect to time to be zero at that instant. Since the shift vector is essentially a rate of change of the space coordinates with respect to hypersurface-orthogonal time, it too can be chosen to be zero. The rate of change of the space metric can also be zero which means that the second fundamental form $K_{ij}$ and therefore the quantities $\pi_{ij}$ must be zero in this case. The vacuum constraints then reduce to just one simple condition on the geometry of the constant-time hypersurface:

$$3R = 0$$

To solve this constraint, notice that there is a simple way to deform a Riemannian geometry: Multiply the three-dimensional metric tensor by a function.

$$h_{ij} = \phi^4 \tilde{h}_{ij}$$

The scalar curvatures associated with the related metrics $h_{ij}$ and $\tilde{h}_{ij}$ are connected by a remarkable relation:

$$3R = -8\phi^{-5} \tilde{\nabla}^2 \phi + \phi^{-4} \tilde{R}$$

where

$$\tilde{\nabla}^2 \phi = \tilde{h}^{rs} \tilde{D}_e \tilde{D}_r \phi = \tilde{h}^{rs} \left( \epsilon_s (\epsilon_r (\phi)) - \epsilon_a (\phi) \tilde{\Gamma}^a_{rs} \right)$$

is the laplacian operator associated with the reference metric $\tilde{h}_{ij}$. The one remaining vacuum constraint then takes the form of a linear wave equation:

$$\tilde{\nabla}^2 \phi + \frac{\tilde{R}}{8\phi} = 0.$$  

### 4.2 Wormholes (Lots of them)

A particularly interesting special case is a flat reference geometry. In that case, the constraint becomes just

$$\tilde{\nabla}^2 \phi = 0$$

where $\tilde{\nabla}^2$ is the familiar flat-space laplacian operator. Look for solutions with $\phi > 0$ everywhere and $\phi \to 1$ at infinity. These are asymptotically flat, regular space geometries. If the constraint equation must hold everywhere, then the only solution is $\phi = 1$ which corresponds to flat space. If we remove one point from the space and permit $\phi$ to diverge near that point, then we get the solutions

$$\phi = 1 + \frac{2m}{\tilde{r}}$$

that correspond to placing a ‘charge’ $+2m$ at the missing point. The area of a sphere at constant radial coordinate $\tilde{r}$ is given by the usual formula $4\pi \tilde{r}^2$ multiplied by the conformal factor $\phi^4$ so that

$$A(\tilde{r}) = 4\pi \left( 1 + \frac{2m}{\tilde{r}} \right)^4 \tilde{r}^2$$
This area goes through a minimum at $r = 2m$. As we will discuss shortly, this is the initial data for a \textit{wormhole} which is the maximally extended spacetime that matches the exterior geometry of a black hole.

The minimum area radius is called the wormhole throat and coincides with the event horizon of the black hole.

One can continue this discussion and consider solutions of the form

$$\phi = 1 + \frac{2m_1}{r_1} + \frac{2m_2}{r_2} + \frac{2m_3}{r_3} + \ldots$$

where $r_1, r_2, r_3, \ldots$ are distances in the flat metric from a set of points. The result is the initial data for a collection of wormholes interacting with one another at an instant of time symmetry.

### 4.3 Conformal Weight Games

This sort of game, due originally to James W. York, begins with a three-dimensional “reference metric” $\tilde{h}_{ij}$ that generates the actual three-dimensional metric on a hypersurface by the conformal transformation $h_{ij} = \phi^4 \tilde{h}_{ij}$. Just as in the time-symmetric case, the goal is to solve the scalar constraint (eq. 7) for the conformal factor $\phi$. However, the general case does not let us dispose of the second fundamental form $K^{ij}$ by setting it equal to zero. Instead, split out the ‘scalar part’ of this tensor by defining a trace-part

$$K = h_{ij}K^{ij}$$

and a trace-free part

$$A^{ij} = K^{ij} - \frac{1}{3}Kh^{ij}.$$  

Besides the second fundamental form, the general case also includes a mass-energy density $\rho$ and a momentum current vector with components $j^i$ that also enter into the initial value constraints.

The object of the game is to start with a reference metric $\tilde{h}_{ij}$ and associated reference objects $\tilde{A}^{ij}$, $\tilde{K}$, $\tilde{\rho}$ and $\tilde{j}^i$ and produce a metric $h_{ij} = \phi^4 \tilde{h}_{ij}$ and associated objects $A^{ij}, K, \rho, j^i$ that makes it solve the initial value constraint.
the following calculation and some intermediate results: the initial value constraints in terms of these variables: and work out the resulting form of the constraint equations (7, 8). Start with the initial value constraints in terms of these variables:

\[ 3R + K^2 - K^{ij}K_{ij} = 16\pi G\rho \]
\[ D_{e_j}(K^{ij} - h^{ij}K) = 8\pi G j^i \]

and some intermediate results:
\[ h = \det \left( \phi^4 \left[ \hat{h} \right] \right) = \phi^{16} \det \left( \left[ \hat{h} \right] \right) = \phi^{16} \hat{h}, \quad h^{ij} = \phi^{-4} \hat{h}^{ij} \]
\[ K^{ij} = A^{ij} + \frac{1}{4} K h^{ij} = \phi^{-10} \hat{A}^{ij} + \frac{1}{4} K \phi^{-4} \hat{h}^{ij} \]
\[ K_{ij} = \phi^{-2} \hat{A}_{ij} + \frac{1}{3} K \phi^{4} \hat{h}_{ij} \]
\[ K^{ij}K_{ij} = \left( \phi^{-10} \hat{A}^{ij} + \frac{1}{3} K \phi^{-4} \hat{h}^{ij} \right) \left( \phi^{-2} \hat{A}_{ij} + \frac{1}{3} K \phi^{4} \hat{h}_{ij} \right) \]
\[ = \phi^{-12} \hat{A}^{ij} \hat{A}_{ij} + \frac{1}{3} K^2 \]
\[ \Gamma^i_{jk} - \hat{\Gamma}^i_{jk} = 2\phi^{-1} \left( \delta^i_j \hat{D}_{e_k} \phi + \delta^i_k \hat{D}_{e_j} \phi - \hat{h}_{jk} \hat{h}^{is} \hat{D}_{e_s} \phi \right) \]
\[ D_{e_j} A^{ij} - \hat{D}_{e_j} A^{ij} = \left( \Gamma^i_{rj} - \hat{\Gamma}^i_{rj} \right) A^{ij} + \left( \Gamma^j_{rj} - \hat{\Gamma}^j_{rj} \right) A^{ir} \]
\[ = \left( \Gamma^i_{rj} - \hat{\Gamma}^i_{rj} \right) A^{ij} + \left( \Gamma^j_{rj} - \hat{\Gamma}^j_{rj} \right) A^{ir} \]
\[ = 2\phi^{-1} \left[ \left( \delta^i_j \hat{D}_{e_r} \phi + \delta^i_r \hat{D}_{e_j} \phi - \hat{h}_{jr} \hat{h}^{is} \hat{D}_{e_s} \phi \right) A^{ij} + \left( \delta^j_i \hat{D}_{e_r} \phi + \delta^j_r \hat{D}_{e_i} \phi - \hat{h}_{ir} \hat{h}^{js} \hat{D}_{e_s} \phi \right) A^{ir} \right] \]
\[ = 2\phi^{-1} \left[ \left( \delta^i_j \hat{D}_{e_r} \phi + \delta^i_r \hat{D}_{e_j} \phi - \hat{h}_{jr} \hat{h}^{is} \hat{D}_{e_s} \phi \right) A^{ij} + \left( \delta^j_i \hat{D}_{e_r} \phi + \delta^j_r \hat{D}_{e_i} \phi - \hat{h}_{ir} \hat{h}^{js} \hat{D}_{e_s} \phi \right) A^{ir} \right] \]
\[ = 2\phi^{-1} \left( A^{ir} \hat{D}_{e_r} \phi + A^{ij} \hat{D}_{e_i} \phi + 3A^{ir} \hat{D}_{e_r} \phi \right) = 10\phi^{-1} A^{ij} \hat{D}_{e_j} \phi \]

The reason for choosing weight 10 for the trace-free part $A^{ij}$ can be seen in the following calculation:

\[ D_{e_j} A^{ij} = \hat{D}_{e_j} A^{ij} + 10\phi^{-1} A^{ij} \hat{D}_{e_j} \phi \]
\[ = \hat{D}_{e_j} \left( \phi^{-10} \hat{A}^{ij} \right) + 10\phi^{-11} \hat{A}^{ij} \hat{D}_{e_j} \phi \]
\[ = \phi^{-10} \hat{D}_{e_j} \hat{A}^{ij} - 10\phi^{-11} \hat{A}^{ij} \hat{D}_{e_j} \phi + 10\phi^{-11} \hat{A}^{ij} \hat{D}_{e_j} \phi \]
\[ = \phi^{-10} \hat{D}_{e_j} \hat{A}^{ij} \]

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D_{e_j} K^{ij} = D_{e_j} A^{ij} + \frac{1}{3} D_{e_j} (K h^{ij})
= \phi^{-10} D_{e_j} \tilde{A}^{ij} + \frac{1}{3} h^{ij} \tilde{D}_{e_j} K

With this choice of weight for \( A^{ij} \), the divergence \( D_{e_j} A^{ij} \) transforms conformally as well. The initial value constraints then become

\(-8\phi^{-5} \tilde{D}^2 \phi + \phi^{-4} R + \frac{2}{3} K^2 - \phi^{-12} \tilde{A}^{ij} \tilde{A}_{ij} = 16\pi G \phi^{-8} \tilde{\rho} \)
\[\phi^{-10} \tilde{D}_{e_j} \tilde{A}^{ij} + \frac{1}{3} h^{ij} \tilde{D}_{e_j} K - h^{ij} \tilde{D}_{e_j} K = 8\pi G \phi^{-10} \tilde{\rho} \]

which take the form of a nonlinear, but elliptic equation for the conformal factor \( \phi \)

\[8 \tilde{D}^2 \phi - R \phi + \phi^{-7} \tilde{A}^{ij} \tilde{A}_{ij} - \frac{2}{3} K^2 \phi^5 + 16\pi G \phi^{-3} \tilde{\rho} = 0\]

and a simple equation that \( \tilde{A}^{ij} \) needs to satisfy

\[\tilde{D}_{e_j} \tilde{A}^{ij} = 8\pi G \tilde{\gamma}^{ij} + \frac{2}{3} \phi^{10} h^{ij} \tilde{D}_{e_j} K \]
\[= 8\pi G \tilde{\gamma}^{ij} + \frac{2}{3} \phi^{10} h^{ij} \tilde{D}_{e_j} K.\]

To solve this last equation, decompose the tensor \( \tilde{A}^{ij} \) into the sum

\[\tilde{A}^{ij} = \tilde{A}^{ij}_{\text{ref}} + \tilde{h}^{ir} \tilde{D}_{e_r} W^j + \tilde{h}^{jr} \tilde{D}_{e_r} W^i - \frac{2}{3} h^{ij} \tilde{D}_r W^r\]

of a “first guess” \( \tilde{A}^{ij}_{\text{ref}} \) and derivatives of a vector field \( W \) so that the initial value equation becomes a system of second order linear differential equations for the vector field \( W \).

\[\tilde{D}_{e_j} \left( \tilde{h}^{ir} \tilde{D}_{e_r} W^j + \tilde{h}^{jr} \tilde{D}_{e_r} W^i - \frac{2}{3} h^{ij} \tilde{D}_r W^r \right) = 8\pi G \tilde{\gamma}^{ij} + \frac{2}{3} \phi^{10} h^{ij} \tilde{D}_{e_j} K - \tilde{D}_{e_j} \tilde{A}^{ij}_{\text{ref}}\]

This system has unique solutions for the vector \( W \) because the second order differential operator in this case can be derived from the variation of a positive definite integral. A similar argument shows that one can always relate an arbitrary given trace-free tensor \( \tilde{A}^{ij}_{\text{ref}} \) to a unique trace-free tensor \( \tilde{A}^{ij}_{TT} \) that obeys

\[\tilde{D}_{e_j} \tilde{A}^{ij}_{TT} = 0.\]

Such a tensor is called “transverse”. The idea is to find a vector field \( U \) such that

\[\tilde{A}^{ij}_{\text{ref}} = \tilde{A}^{ij}_{TT} + \tilde{h}^{ir} \tilde{D}_{e_r} U^j + \tilde{h}^{jr} \tilde{D}_{e_r} U^i - \frac{2}{3} h^{ij} \tilde{D}_r U^r\]
because the field $U$ then obeys the equation

$$ \left( \tilde{\Delta} U \right)^i = -\tilde{D}_{e_j} \tilde{A}_{ij} $$

where

$$ \left( \tilde{\Delta} U \right)^i = \tilde{D}_{e_j} \left( \tilde{h}^{ir} \tilde{D}_{e_r} U^j + \tilde{h}^{jr} \tilde{D}_{e_r} U^i - \frac{2}{3} \tilde{h}^{ij} \tilde{D}_{e_r} U^r \right) $$

and there is a unique solution for each $\tilde{A}_{ij}^{\text{ref}}$.

### 4.4 Slicing Conditions

The results of the conformal analysis, suggest that the freely specifiable initial data consists of the the conformal equivalence class of the space geometry, which can be specified by giving a particular metric $\tilde{h}_{ij}$ that belongs to the class and the transverse-traceless part $\tilde{A}_{ij}^{\text{TT}}$ of the second fundamental form. A difficulty remains, however, and can be seen as soon as the reduced forms of the initial value constraints are placed together:

$$ 8\tilde{D}^2 \phi - \tilde{R} \phi + \phi^{-7} \tilde{A}_{ij}^{\text{TT}} \tilde{A}_{ij}^{\text{TT}} - \frac{2}{3} K^2 \phi^5 + 16 \pi G \phi^{-3} \phi = 0 $$

$$ \left( \tilde{\Delta} W \right)^i - 8 \pi G \tilde{\phi}^i - \frac{2}{3} \phi^3 \tilde{h}^{jr} \tilde{D}_{e_r} K = 0 $$

where

$$ \tilde{A}_{ij}^{\text{TT}} = \tilde{A}_{ij}^{\text{TT}} + \tilde{h}^{jr} \tilde{D}_{e_r} W^j + \tilde{h}^{jr} \tilde{D}_{e_r} W^i - \frac{2}{3} \tilde{h}^{ij} \tilde{D}_{e_r} W^r. $$

The difficulty is that the equations are, in general, coupled so that the equation for the conformal factor $\phi$ depends on the vector field $W$ (through the $\tilde{A}_{ij}^{\text{TT}} \tilde{A}_{ij}^{\text{TT}}$ term) and the equation for the vector field $W$ depends on the conformal factor.

The coupling can be broken by placing a condition of the trace $K$ of the second fundamental form of the spacelike hypersurfaces. Such a condition is needed in any event because the choice of how to slice spacetime into constant-time hypersurfaces is completely open. In order to obtain a definite time evolution for spacetime, some sort of condition is needed to determine this choice. The condition

$$ K = 0 $$

corresponds to using surfaces of extremal volume and works quite well at producing generalizations of the $t = \text{constant}$ surfaces of Minkowski spacetime in spacetimes that are asymptotically flat. In spacetimes with closed space sections, there is often only one extremal volume surface, but the condition

$$ \tilde{h}^{ir} \tilde{D}_{e_r} K = 0 $$

which can also be written as $K = f(t)$ produces the constant-time surfaces that are usually used and can be used to define a natural time coordinate as well. When these conditions are used, it is possible to solve a linear elliptic system
of equations for the vector field $W$ and then solve a scalar elliptic equation for the conformal factor $\phi$. With the initial value equations solved in this way, one can then turn to the dynamical Einstein equations to evolve the initial data to the next time slice.

5 Numerical Spacetimes

5.1 The Physical Problem

With the initial value problem at least somewhat under control, it has become possible to set up initial data on a spatial grid and evolve it by using computers. The result is a numerical approximation to a solution of Einstein’s field equations. The first few attempts at numerical evolution codes for Einstein’s equations were written pretty much for their own sake. However, there is now a pressing need for these codes to solve a real physical problem and a massive investment of time and effort by a large number of different research groups has already been made.

Black holes will be discussed more thoroughly in the next section of this course. However, we have already seen some clues to just how easily they form. Physically they can be regarded as manifestations of the fact that pressure is a source of gravitation so that, when a star collapses, there comes a point where increasing central pressure accelerates the collapse instead of halting it. Mathematically we have already seen the seeds of black hole solutions in the form of $1/r$ solutions of the elliptic equation for the space geometry conformal factor. Astronomers have become convinced that many massive stars end their lives as objects which have collapsed to the point that light cannot escape from them. These collapsed objects are well represented as black holes. The spacetime around an isolated black hole is fairly easy to find using analytic methods and was, in fact, one of the very first solutions of Einstein’s field equations. The problem that has spurred the development of numerical approaches to solving Einstein’s equations arises when both members of a binary star system collapse and form black holes in orbit around one another. Such orbiting black holes are expected to lose energy by generating gravitational waves, an effect which has already been observed for a slightly less extreme system, the binary pulsar, in which a pair of neutron stars orbit each other. As the orbiting black holes spiral in toward one another, the gravitational radiation rate increases and one expects a rising frequency “chirp” of radiation just before the holes coalesce and form a single black hole.

Understanding this “binary black hole” system has become a matter of urgency because several large-scale laser interferometers have just been completed
and will soon begin to search for the resulting gravitational radiation. Precise predicted waveforms for the signal will be needed to extract it from noise and extensive modeling of what happens will be needed to extract information about the events that these systems detect.

5.2 The Computer Problems

The binary black hole system has no symmetry whatsoever. Thus, it requires a full three-dimensional grid of evaluation points and evolution for a long period of time in order to figure out what is happening. In addition to the usual problems of any numerical simulation such as instabilities and accumulating errors, there are problems peculiar to this system. Because the coordinates evolve along with the spacetime, one cannot be sure that any given mesh of evaluation points will continue to be a good one. Thus, adaptive meshes are required. Because Einstein’s equations generate singularities, one must choose slicing conditions that generate spacelike hypersurfaces that avoid these singularities so that the time evolution will not founder on them. Finally, the goal of the calculation is to predict gravitational radiation and that can be recognized only at a distance of a large number of wavelengths from the coalescing holes.

Thus, one must use a large number of three-dimensional mesh points to describe a far-field region in which very little is actually happening in order to see the waves appear. Similarly, one should not start with initial conditions that correspond to massive amounts of radiation pouring into the system, and that also requires initial conditions over a large far-field region.

The best way to evolve the outgoing gravitational waves is to use a system in which the initial data is given on a lightlike or null hypersurface and is evolved in a lightlike direction. In such a system it is very easy to be sure that there is no radiation coming in to the system and extracting the outgoing wave signal is also very simple. However, the best way to evolve the spacetime near a pair of orbiting black holes is to use initial data on spacelike hypersurfaces because lightlike surfaces tend to fold and develop singularities in this region. Attempts to combine these two approaches usually divide the spacetime into a near-field region where data is given on spacelike surfaces, a far-field region where it is given on lightlike surfaces, and a middle region where the two sets of data are
A hybrid approach that would seem to make sense is to use hypersurfaces that are actually spacelike everywhere, but approach outgoing null surfaces asymptotically. This approach could use the well-developed machinery of the spacelike initial data formalism but would gain the advantage of treating the far-field region efficiently.

The condition $K = \text{constant}$ actually comes close because, in Minkowski spacetime, it produces spacelike hyperbolas that are indeed asymptotic to null cones. The surfaces also turn out to be well-behaved in other spacetimes but have not yet been used for serious numerical calculations.
6 Appendix

6.1 Fundamental Forms of a Hypersurface

Spacetime coordinates

\[ x^i; \quad i = 1, 2, 3 \]
\[ x^0 = t \]

Spacelike hypersurface \( \Sigma (t) \)

\( t = \text{constant} \)

Basis vectors tangent to \( \Sigma (t) \)

\[ \partial_i = \frac{\partial}{\partial x^i} \]

6.2 First fundamental form for \( \Sigma (t) \)

\[ h(u, v) = g(u, v) \]
\( u, v \) both tangent to \( \Sigma (t) \)

\[ h_{ij} = g(\partial_i, \partial_j) = g_{ij} \]

6.3 Unit vector normal to \( \Sigma (t) \)

Start with the fact that the form \( dt \) is already normal to the surface since

\[ dt \cdot u = 0 \]

for any vector \( u \) tangent to \( \Sigma (t) \). The form just needs a normalization factor to make it a unit normal

\[ d\tau = N dt \]

where we need

\[ d\tau \cdot d\tau = -1 \]

But

\[ N dt \cdot N dt = N^2 dx^0 \cdot dx^0 = N^2 g^{00} \]

so

\[ N^2 g^{00} = -1 \]

\[ N = (-g^{00})^{-1/2} \]

or

\[ g^{00} = -N^{-2} \]

To get the unit vector normal to the surface, use the metric

\[ n = \pm g^{-1}(d\tau) = \pm g^{-1}(N dx^0) \]
Either sign will work. The minus sign leads to the future-pointing normal that is usually used.

\[ n = -g^{-1} (d\tau) = n^0 \frac{\partial}{\partial t} + n^i \partial_i \]

where

\[ n^0 = -dx^0 \cdot n = -dx^0 \cdot g^{-1} (N dx^0) = -Ng^{-1} (dx^0, dx^0) = -Ng^{00} \]

\[ N^2 g^{00} = -1 \]

\[ Ng^{00} = N^{-1} \]

\[ n^0 = N^{-1} \]

\[ n^i = -dx^i \cdot n = -dx^i \cdot g^{-1} (N dx^0) = -Ng^{0i} \]

In ADM notation,

\[ N^i = N^2 g^{0i} \]

so

\[ n^i = -N^{-1} N^i \]

and

\[ n = N^{-1} \left( \frac{\partial}{\partial t} - N^i \partial_i \right) \]

### 6.4 Second fundamental form for \( \Sigma (t) \)

\[ k (u, v) = -u \cdot D_v n \]

\( u, v \) both tangent to \( \Sigma (t) \)

Side note: This definition is equivalent to

\[ k (u, v) = u \cdot D_v g^{-1} (d\tau) = u \cdot D_v (d\tau) \]

and this is the form that should be used when the metric signature is in doubt. For example, it gives the correct result for surfaces of constant \( x^2 + y^2 + z^2 \) in Cartesian space. The minus sign associated with the normal vector \( n \) is peculiar to spacelike hypersurfaces in spacetime.

Start by assuming hypersurface orthogonal coordinates with \( x^0 = \tau = \) proper time. These can always be chosen near a particular hypersurface.

\[ n = \partial_0 \]

\[ [g] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & g_{13} \\ 0 & g_{21} & g_{22} & g_{23} \\ 0 & g_{31} & g_{32} & g_{33} \end{bmatrix} \]
\[ D_{\partial} n = D_{\partial} \partial_0 = \Gamma^\rho_{\partial 0} \partial_\rho = \Gamma^0_{\partial 0} \partial_0 + \Gamma^\nu_{\partial 0} \partial_\nu \]

\[ \Gamma^\alpha_{\partial \gamma} = \frac{1}{2} g^{\alpha \sigma} (e_\gamma g_{\sigma \beta} + e_\beta g_{\gamma \sigma} - e_\tau g_{\nu \nu}) \]

\[ \Gamma^0_{0i} = \frac{1}{2} g^{0 \sigma} (\partial_i g_{0\sigma} + \partial_0 g_{i\sigma} - \partial_\sigma g_{0i}) \]
\[ = \frac{1}{2} g^{00} (\partial_i g_{00} + \partial_0 g_{i0} - \partial_0 g_{0i}) \]
\[ = 0 \]

\[ \Gamma^{r}_{0i} = \frac{1}{2} g^{r \sigma} (\partial_i g_{r\sigma} + \partial_0 g_{i\sigma} - \partial_\sigma g_{r0}) \]
\[ = \frac{1}{2} g^{rs} \partial_0 g_{is} \]
\[ D_{\partial} n = \frac{1}{2} g^{rs} \partial_0 g_{is} \partial_r \]

Define the time derivative taken in this hypersurface orthogonal system as

\[ \partial_0 g_{is} = \frac{\partial g_{is}}{\partial \tau} \]
\[ D_v n = \frac{1}{2} v^s g^{rs} \frac{\partial g_{is}}{\partial \tau} \partial_r \]

Now find the components of the second fundamental form

\[ k_{ab} = k (\partial_a, \partial_b) = -\partial_a \cdot D_{\partial} n \]
\[ = -\partial_a \cdot \frac{1}{2} g^{rs} \frac{\partial g_{bs}}{\partial \tau} \partial_r \]
\[ = -\frac{1}{2} g^{rs} \frac{\partial g_{bs}}{\partial \tau} \]
\[ = -\frac{1}{2} g^{rs} \frac{\partial g_{bs}}{\partial \tau} g_{ar} = -\frac{1}{2} g^{rs} \frac{\partial g_{bs}}{\partial \tau} \]
\[ k_{ab} = -\frac{1}{2} \frac{\partial g_{ab}}{\partial \tau} = -\frac{1}{2} \frac{\partial g_{ab}}{\partial \tau} \]

Now state this expression in coordinate-independent language. The partial derivative says to drag the space components of the metric tensor along the vector field

\[ n = \frac{\partial}{\partial \tau} \]

and calculate their rate of change. That operation is the Lie derivative

\[ \frac{\partial g_{ab}}{\partial \tau} = (\mathcal{L}_n g)(\partial_a, \partial_b) \]
\[ g_{ab} = g(\partial_a, \partial_b) \]
Now use
\[
n = N^{-1} \left( \frac{\partial}{\partial t} - N^i \partial_i \right)
\]
to obtain
\[
\frac{\partial g_{ab}}{\partial \tau} = \left( \mathcal{L}_{N^{-1}(\frac{\partial}{\partial t} - N^i \partial_i)} g \right) (\partial_a, \partial_b)
\]
The Lie derivative works just like other derivatives that act on tensors, so we subtract counter-terms for each argument.
\[
(\mathcal{L}_u g) (\partial_a, \partial_b) = u (g_{ab}) - g (\mathcal{L}_u \partial_a, \partial_b) - g (\partial_a, \mathcal{L}_u \partial_b)
\]
\[
\left( \mathcal{L}_{\frac{\partial}{\partial t}} g \right) (\partial_a, \partial_b) = \frac{\partial g_{ab}}{\partial t} - g \left( \mathcal{L}_{\frac{\partial}{\partial t}} \partial_a, \partial_b \right) - g \left( \partial_a, \mathcal{L}_{\frac{\partial}{\partial t}} \partial_b \right)
\]
\[
\frac{\partial g_{ab}}{\partial \tau} = \left( \mathcal{L}_{N^{-1}(\frac{\partial}{\partial t} - N^i \partial_i)} g \right) (\partial_a, \partial_b)
\]
\[
= \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} - N^i \partial_i \right) g (\partial_a, \partial_b) - \frac{\partial}{\partial t} \left( \mathcal{L}_{N^{-1}(\frac{\partial}{\partial t} - N^i \partial_i)} \partial_a \right) \partial_b - \frac{\partial}{\partial t} \mathcal{L}_{\frac{\partial}{\partial t}} (\partial_a, \partial_b)
\]
\[
= \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} - N^i \partial_i \right) g (\partial_a, \partial_b) - \frac{\partial}{\partial t} \left( \mathcal{L}_{N^{-1}(\frac{\partial}{\partial t} - N^i \partial_i)} \partial_a \right) \partial_b - \frac{\partial}{\partial t} \mathcal{L}_{\frac{\partial}{\partial t}} (\partial_a, \partial_b)
\]
Recall that the Lie derivative of a vector field by another one is just the commutator so that
\[
\mathcal{L}_{N^{-1}(\frac{\partial}{\partial t} - N^i \partial_i)} \partial_a = \left[ N^{-1} \left( \frac{\partial}{\partial t} - N^i \partial_i \right) , \partial_a \right] = N^{-1} \left( \frac{\partial}{\partial t} - N^i \partial_i \right) \partial_a - \partial_a \left( N^{-1} \left( \frac{\partial}{\partial t} - N^i \partial_i \right) \right)
\]
\[
= N^{-1} \left( \frac{\partial}{\partial t} - N^i \partial_i \right) \partial_a + N^{-2} (\partial_a N) \left( \frac{\partial}{\partial t} - N^i \partial_i \right) - \left( \frac{\partial}{\partial t} \partial_a \right) \left( \frac{\partial}{\partial t} - N^i \partial_i \right)
\]
\[
= N^{-1} \left( \frac{\partial}{\partial t} - N^i \partial_i \right) \partial_a + N^{-2} (\partial_a N) \left( \frac{\partial}{\partial t} - N^i \partial_i \right) - N^{-1} \left( \frac{\partial}{\partial t} - N^i \partial_i \right) \partial_a + N^{-1} (\partial_a N) \partial_i
\]
\[
= + N^{-2} (\partial_a N) \left( \frac{\partial}{\partial t} - N^i \partial_i \right) + N^{-1} (\partial_a N^i) \partial_i
\]
\[
\mathcal{L}_{N^{-1}(\frac{\partial}{\partial t} - N^i \partial_i)} \partial_b = + N^{-2} (\partial_b N) \left( \frac{\partial}{\partial t} - N^i \partial_i \right) + N^{-1} (\partial_b N^i) \partial_i
\]
The derivative of $N$ looks like bad news, but persevere
\[
g \left( \mathcal{L}_{N^{-1}(\frac{\partial}{\partial t} - N^i \partial_i)} \partial_a \right) \partial_b = \left( g \left( N^{-2} (\partial_a N) \left( \frac{\partial}{\partial t} - N^i \partial_i \right) + N^{-1} (\partial_a N^i) \partial_i \right) \partial_b \right)
\]
\[
= g \left( N^{-2} (\partial_a N) \left( \frac{\partial}{\partial t} - N^i \partial_i \right) \partial_b \right) - N^{-2} (\partial_a N) N^i \partial_b + N^{-1} (\partial_a N^i) \partial_b \partial_b
\]
\[
= N^{-2} (\partial_a N) g_{ab} - N^{-2} (\partial_a N) N^i g_{ib} + N^{-1} (\partial_a N^i) g_{ib}
\]
Note, from the ADM decomposition of the spacetime metric, that

\[ g_{ab} = N_b = N^i g_{ib} \]

so

\[
g \left( L_{N^{-1}} \left( \frac{\partial}{\partial t} - N^i \partial_i \right) \partial_a, \partial_b \right) = N^{-2} (\partial_a N) N^i g_{ib} - N^{-2} (\partial_a N) N^i g_{ib} + N^{-1} (\partial_a N^i) g_{ib}
\]

\[ = N^{-1} (\partial_a N^i) g_{ib} \]

The bad news has cancelled out!

The other term we have to calculate is just the same, with the indexes \( a, b \) reversed:

\[
g \left( \partial_a, L_{N^{-1}} \left( \frac{\partial}{\partial t} - N^i \partial_i \right) \partial_b \right) = N^{-1} (\partial_b N^i) g_{ia}
\]

Putting the various terms together,

\[
\frac{\partial g_{ab}}{\partial t} = N^{-1} \left( \frac{\partial g_{ab}}{\partial t} - N^i \partial_i g_{ab} \right) - g \left( L_{N^{-1}} \left( \frac{\partial}{\partial t} - N^i \partial_i \right) \partial_a, \partial_b \right) - g \left( \partial_a, L_{N^{-1}} \left( \frac{\partial}{\partial t} - N^i \partial_i \right) \partial_b \right)
\]

\[ = N^{-1} \left( \frac{\partial g_{ab}}{\partial t} - N^i \partial_i g_{ab} \right) - N^{-1} (\partial_a N^i) g_{ib} - N^{-1} (\partial_b N^i) g_{ia} \]

We can work this out in detail, but the easiest thing to do at this point is to note that we can always choose Riemann normal coordinates within the hypersurface at a particular point. In those coordinates, the expression becomes

\[
\frac{\partial g_{ab}}{\partial t} = N^{-1} \left( \frac{\partial g_{ab}}{\partial t} - g_{ib} \partial_i N^i - g_{ia} \partial_b N^i \right)
\]

and, since the space metric tensor has vanishing derivatives and the connection coefficients all vanish at our chosen point, the ordinary derivatives can become covariant derivatives and the metric can be used to lower indexes so that the expression can also be written as

\[
\frac{\partial g_{ab}}{\partial t} = N^{-1} \left( \frac{\partial g_{ab}}{\partial t} - g_{ib} \partial_i N^i - g_{ia} \partial_b N^i \right)
\]

Since that form of the expression is coordinate invariant, it must hold in any other coordinates. The final expression for the second fundamental form is then

\[
k_{ab} = -\frac{1}{2} \frac{\partial g_{ab}}{\partial t} = -\frac{1}{2N} \left( \frac{\partial g_{ab}}{\partial t} - g_{ib} \partial_i N^i - g_{ia} \partial_b N^i \right)
\]

\[ = -\frac{1}{2N} \left( \frac{\partial g_{ab}}{\partial t} - N_{a|b} - N_{b|a} \right).
\]