

1 Exercise 6

The following problems all refer to geodesic motion in a static, spherically symmetric metric

\[ ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2d\Omega^2 \]

which has non-zero connection coefficients that were found in the notes to be

\[ \Gamma^0_{01} = \nu' \]

\[ \Gamma^1_{00} = \nu'e^{2(\nu-\lambda)}, \quad \Gamma^1_{11} = \lambda', \quad \Gamma^1_{22} = -2e^{-2\lambda}, \quad \Gamma^1_{33} = -re^{-2\lambda}\sin^2 \theta \]

\[ \Gamma^2_{12} = r^{-1}, \quad \Gamma^2_{33} = -\sin \theta \cos \theta \]

\[ \Gamma^3_{13} = r^{-1}, \quad \Gamma^3_{23} = \cot \theta \]

1.1 Problem 6.1

Specialize the connection coefficients to the case of the Schwartzschild metric

\[ ds^2 = - \left(1 - \frac{2m}{r}\right)dt^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2d\Omega^2 \]

and write out the equations for the functions \( t(\lambda), r(\lambda), \theta(\lambda), \phi(\lambda) \) that describe a geodesic with affine parameter \( \lambda \) in this spacetime.

**Answer 6.1**

Starting from scratch, (It is OK to use the results in the notes, however.)

\[ \left(1 - \frac{2m}{r}\right) = e^{2\nu}, \quad \frac{1}{1 - \frac{2m}{r}} = e^{2\lambda} \]

\[ \frac{2m}{r^2} = 2\nu'e^{2\nu}, \quad -\frac{\frac{2m}{r}}{(1 - \frac{2m}{r})^2} = 2\lambda'e^{2\lambda} \]
\[ \Gamma^0_{01} = \nu' = \frac{1}{2} \frac{2m}{r^2 e^{2\nu}} = \frac{m}{r^2 (1 - \frac{2m}{r})} \]

\[ \Gamma^1_{00} = \nu' e^{2(\nu - \lambda)} = \frac{\nu' e^{2\nu}}{e^{2\lambda}} = \frac{m}{r^2} \frac{1}{1 - \frac{2m}{r}} \]

\[ \Gamma^1_{11} = \chi' = -\frac{1}{2} \frac{2m}{r^2} e^{2\lambda} = -\frac{1}{2} \left( \frac{2m}{r^2} \right) \frac{1}{1 - \frac{2m}{r}} \]

\[ \Gamma^1_{22} = -\frac{2}{e^{2\lambda}} = -\frac{2}{1 - \frac{2m}{r}} = -2 \left( 1 - \frac{2m}{r} \right) \]

\[ \Gamma^1_{33} = -\frac{r \sin^2 \theta}{e^{2\lambda}} = -\frac{r \sin^2 \theta}{1 - \frac{2m}{r}} = -\left( 1 - \frac{2m}{r} \right) r \sin^2 \theta \]

The rest are just as above, so the full list is

\[ \Gamma^0_{01} = \frac{m}{r^2 (1 - \frac{2m}{r})} \]

\[ \Gamma^1_{00} = \frac{m}{r^2} \left( 1 - \frac{2m}{r} \right), \quad \Gamma^1_{11} = -\frac{m}{r^2} \frac{1}{1 - \frac{2m}{r}} \]

\[ \Gamma^1_{22} = -2 \left( 1 - \frac{2m}{r} \right), \quad \Gamma^1_{33} = -\left( 1 - \frac{2m}{r} \right) r \sin^2 \theta \]

\[ \Gamma^2_{12} = r^{-1}, \quad \Gamma^2_{33} = -\sin \theta \cos \theta \]

\[ \Gamma^3_{13} = r^{-1}, \quad \Gamma^3_{23} = \cot \theta \]

Now write out the geodesic equations from the components

\[ \frac{d^2 x^\alpha}{d\lambda^2} + \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} \Gamma^\alpha_{\rho\sigma} = 0 \]

\[ \alpha = 0 \]

\[ \frac{d^2 x^0}{d\lambda^2} + \frac{dx^0}{d\lambda} \frac{dx^0}{d\lambda} \Gamma^0_{00} = 0 \]

\[ \frac{dt}{d\lambda} + 2 \frac{dt}{d\lambda} \frac{dr}{d\lambda} \frac{m}{r^2 (1 - \frac{2m}{r})} = 0 \]
\[ \frac{d^2r}{d\lambda^2} + \left( \frac{dt}{d\lambda} \right) \frac{m}{r^2} \left( 1 - \frac{2m}{r} \right) - \frac{m}{r^2} \left( \frac{dr}{d\lambda} \right)^2 - 2 \left( 1 - \frac{2m}{r} \right) \left( \frac{d\theta}{d\lambda} \right)^2 \frac{dx^2}{d\lambda} + \left( 1 - \frac{2m}{r} \right) r \sin^2 \theta \left( \frac{d\varphi}{d\lambda} \right)^2 = 0 \]

\[ \alpha = 1 \]

\[ \frac{d^2x^1}{d\lambda^2} + \frac{dx^0}{d\lambda} \frac{dx^0}{d\lambda} \Gamma^1_{00} + \frac{dx^1}{d\lambda} \frac{dx^1}{d\lambda} \Gamma^1_{11} + \frac{dx^2}{d\lambda} \frac{dx^2}{d\lambda} \Gamma^1_{22} + \frac{dx^3}{d\lambda} \frac{dx^3}{d\lambda} \Gamma^1_{33} = 0 \]

\[ \frac{d^2r}{d\lambda^2} + \left( \frac{dt}{d\lambda} \right) \frac{m}{r^2} \left( 1 - \frac{2m}{r} \right) - \frac{m}{r^2} \left( \frac{dr}{d\lambda} \right)^2 - 2 \left( 1 - \frac{2m}{r} \right) \left( \frac{d\theta}{d\lambda} \right)^2 \frac{dx^2}{d\lambda} + \left( 1 - \frac{2m}{r} \right) r \sin^2 \theta \left( \frac{d\varphi}{d\lambda} \right)^2 = 0 \]

\[ \alpha = 2 \]

\[ \frac{d^2x^2}{d\lambda^2} + \frac{dx^0}{d\lambda} \frac{dx^0}{d\lambda} \Gamma^2_{00} = 0 \]

\[ \frac{d^2x^2}{d\lambda^2} + 2 \frac{dx^1}{d\lambda} \frac{dx^2}{d\lambda} \Gamma^2_{12} + \frac{dx^3}{d\lambda} \frac{dx^3}{d\lambda} \Gamma^2_{33} = 0 \]

\[ \frac{d^2r}{d\lambda^2} + 2 \frac{dr}{d\lambda} \frac{d\theta}{d\lambda} \Gamma^2_{12} + \left( \frac{d\varphi}{d\lambda} \right)^2 \Gamma^2_{33} = 0 \]

\[ \frac{d^2\theta}{d\lambda^2} + \frac{2 d\theta}{d\lambda} \frac{dr}{d\lambda} - \sin \theta \cos \theta \left( \frac{d\varphi}{d\lambda} \right)^2 = 0 \]

\[ \alpha = 3 \]

\[ \frac{d^2x^3}{d\lambda^2} + \frac{dx^0}{d\lambda} \frac{dx^0}{d\lambda} \Gamma^3_{00} = 0 \]

\[ \frac{d^2x^3}{d\lambda^2} + 2 \frac{dx^1}{d\lambda} \frac{dx^3}{d\lambda} \Gamma^3_{13} + \frac{dx^2}{d\lambda} \frac{dx^3}{d\lambda} \Gamma^3_{23} = 0 \]

\[ \frac{d^2\varphi}{d\lambda^2} + \frac{2 dr}{d\lambda} \frac{d\varphi}{d\lambda} + \frac{d\theta}{d\lambda} \frac{d\varphi}{d\lambda} \cot \theta = 0 \]
1.2 Problem 6.2

Assume that a geodesic starts off in (and tangent to) the \( \theta = \frac{\pi}{2} \) plane with a four-velocity \( u \) that satisfies

\[
\frac{\partial}{\partial t} \cdot u = E \quad \frac{\partial}{\partial \varphi} \cdot u = \ell \quad u \cdot u = -k = \begin{cases} 0 & \text{for a lightlike geodesic} \\ -1 & \text{for a timelike geodesic} \end{cases}
\]

and reduce the system of equations to a single differential equation for \( r(t) \).

Answer 6.2

The \( \alpha = 2 \) equation

\[
\frac{d^2 \theta}{d\lambda^2} + \frac{2}{r} \frac{dr}{d\lambda} \frac{d\theta}{d\lambda} - \sin \theta \cos \theta \left( \frac{d\varphi}{d\lambda} \right)^2 = 0
\]

is solved by \( \theta = \frac{\pi}{2}, \frac{d\theta}{d\lambda} = 0 \) so the geodesic will stay in the \( \theta = \frac{\pi}{2} \) plane.

Now write out the conserved quantities

\[
\frac{\partial}{\partial t} \cdot u = E \\
e_0 \cdot (u^0 e_0 + u^1 e_1 + ...) = E \\
g_{00} u^0 = E \\
- \left(1 - \frac{2m}{r}\right) \frac{dt}{d\lambda} = E
\]

\[
\frac{d^2 \theta}{d\lambda^2} + \frac{g_{00}(u^0)^2 + g_{11}(u^1)^2 + g_{22}(u^2)^2 + g_{33}(u^3)^2}{1 - \frac{2m}{r}} = -k \\
- \left(1 - \frac{2m}{r}\right) \left( \frac{dt}{d\lambda} \right)^2 + \frac{1}{1 - \frac{2m}{r}} \left( \frac{dr}{d\lambda} \right)^2 + r^2 \left( \frac{d\theta}{d\lambda} \right)^2 + r^2 \sin^2 \theta \left( \frac{d\varphi}{d\lambda} \right)^2 = -k
\]

\[
e_3 \cdot (u^0 e_0) = \ell \\
g_{33} u^3 = \ell \\
r^2 \sin^2 \theta \frac{d\varphi}{d\lambda} = \ell
\]

Now specialize to \( \theta = \frac{\pi}{2} \) and write these equations as

\[
\frac{d\varphi}{d\lambda} = \frac{\ell}{r^2}
\]
\[
\frac{dt}{d\lambda} = -\frac{E}{1 - \frac{2m}{r}} - \left(1 - \frac{2m}{r}\right) \left(\frac{dt}{d\lambda}\right)^2 + \frac{1}{1 - \frac{2m}{r}} \left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\varphi}{d\lambda}\right)^2 = -k
\]

Eliminate \( \frac{d\varphi}{d\lambda} \) and \( \frac{dt}{d\lambda} \) from the last equation.

\[- \left(1 - \frac{2m}{r}\right) \left(\frac{E}{1 - \frac{2m}{r}}\right)^2 + \frac{1}{1 - \frac{2m}{r}} \left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{\ell}{r^2}\right)^2 = -k \]

\[- \frac{E^2}{1 - \frac{2m}{r}} + \frac{1}{1 - \frac{2m}{r}} \left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2} = -k \]

Now we are down to just one differential equation for the function \( r(\lambda) \).

\[\left(\frac{dr}{d\lambda}\right)^2 + \left(\frac{\ell^2}{r^2} + k\right) \left(1 - \frac{2m}{r}\right) = E^2\]

If \( t \) is made the independent variable, then it gets a bit more complex and nothing much is gained.
1.3 Problem 6.3

The radial equation that you found in problem 2 looks like the equation of motion for a particle in a one dimensional potential. Find that potential function for $k = 0$ and discuss the conditions to have a stable photon orbit around a black hole. Such an orbit would stay between upper and lower bounds for the coordinate $r$.

Answer 6.3

The equation in this case is

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2} \left(1 - \frac{2m}{r}\right) = E^2$$

For comparison, the equation of motion of a particle in one dimension would be of the form

$$\frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + V = E.$$

or

$$\left(\frac{dr}{dt}\right)^2 = \frac{2}{m} (E - V)$$

Recall that the particle is only allowed to be where $E - V$ has positive or zero values and has turning points where $E - V = 0$. For an oscillating solution (Keplerian orbit), we need $E - V$ to have two zeros with positive values in between them. That means that $V$ needs to have a local minimum.

Here we have

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \tilde{V}^2$$

where the function playing the role of $V$ is

$$\tilde{V}^2 = \frac{\ell^2}{r^2} \left(1 - \frac{2m}{r}\right) = \frac{\ell^2}{r^3} (r - 2m)$$

First check to see if the photon can possibly be in a Keplerian orbit. For that, $\tilde{V}^2$ needs to have a local minimum. For an extremum,

$$\frac{d}{dr} \tilde{V}^2 = 0.$$  

But

$$\frac{d}{dr} \tilde{V}^2 = \frac{d}{dr} \left(\frac{\ell^2 (r - 2m)}{r^2}\right) = \frac{1}{r^3} \ell^2 + \frac{1}{r^4} (6ml^2 - 3r \ell^2) = \left(\frac{1}{r^3} + \frac{1}{r^4} (6m - 3r)\right) \ell^2$$

Notice that $\ell^2$ factors out and the extremum condition is

$$\frac{1}{r^3} + \frac{1}{r^4} (6m - 3r) = 0.$$
or, since \( r > 0 \), it is OK to multiply by \( r^4 \)

\[
    r + 6m - 3r = 0
\]

so that there should be some sort of equilibrium possible at

\[
    r = 3m.
\]

Now check the value of the second derivative \( \frac{d^2}{dr^2} \tilde{V}^2 \) at that point.

\[
\frac{d}{dr} \tilde{V}^2 = \left( \frac{1}{r^3} + \frac{1}{r^4} (6m - 3r) \right)
\]

\[
\frac{d^2}{dr^2} \tilde{V}^2 = \ell^2 \frac{d}{dr} \left( \frac{1}{r^3} + \frac{1}{r^4} (6m - 3r) \right)
\]

\[
= \ell^2 \left( \frac{1}{r^3} (12r - 24m) - \frac{6}{r^4} \right)
\]

\[
= \ell^2 \left( \frac{1}{(3m)^3} (12(3m) - 24m) - \frac{6}{(3m)^4} \right)
\]

\[
= -\frac{2}{81m^4} \ell^2
\]

The value is negative, so the orbit is only metastable. Photons that do not approach as closely as \( r = 3m \) return back to infinity. Photons approaching closer than \( r = 3m \) fall into the hole.