PHYS691 Final Exam

Attempt each of the following problems. Attach the resulting file to an email to rhgowdy@vcu.edu.

Due date: Thursday, May 12, 2005.

1 Problem 1: Sound Waves

Use the stress-energy tensor conservation laws to find the speed of sound waves (as a fraction of the speed of light) in a medium that obeys an equation of state of the form

$$p = f(\rho)$$

Do the calculation for an arbitrary curved spacetime.

1.1 Answer: (flat spacetime version 1)

First get the basic idea by doing the problem out in components in flat Minkowski spacetime. The conservation law is then

$$T^{\mu\nu}{}_{,\nu} = 0$$

or, split into time and space parts

$$\begin{aligned} T^{00}{}_{,0} + T^{0m}{}_{,m} &= 0 \\ T^{m0}{}_{,0} + T^{mn}{}_{,n} &= 0 \end{aligned}$$

Assume a fluid with isotropic stress and $p = f(\rho)$

$$T^{00} = \rho$$

$$T^{mn} = f(\rho)\, g^{mn} = f(\rho)\, \delta^{nm}$$

Since we cannot have the coordinates follow the fluid (they are fixed) we have to allow the fluid to move in order to have sound waves. Thus, there must be small non-zero components

$$T^{0m} = T^{m0} = j^m$$

so that the conservation law becomes

$$\begin{aligned} \rho_{,0} + j^m{}_{,m} &= 0 \\ j^m{}_{,0} + \left(\delta^{mn} f(\rho)\right)_{,n} &= 0 \end{aligned}$$

But

$$\left(\delta^{mn} f(\rho)\right)_{,n} = \frac{\partial}{\partial x^m} f(\rho) = \frac{df}{d\rho}\frac{\partial \rho}{\partial x^m} = f'\rho_{,m}$$

and the second equation becomes

$$j^m{}_{,0} + f'\rho_{,m} = 0$$

The essential trick is to eliminate the mass-energy flow variables j^m by taking the spatial divergence of this last result

$$\begin{aligned} j^m{}_{,0m} + f'\rho_{,mm} + f'_{,m}\rho_{,m} &= 0 \\ j^m{}_{,0m} + f'\rho_{,mm} + f''\rho_{,m}\rho_{,m} &= 0 \end{aligned}$$

and comparing that to the time derivative of the first conservation equation

$$\rho_{,00} + j^m{}_{,m0} = 0$$

Subtract the equations and obtain

$$j^m{}_{,0m} + f'\rho_{,mm} + f''\rho_{,m}\rho_{,m} - \rho_{,00} - j^m{}_{,m0} = 0$$

or

$$f'\rho_{,mm} - \rho_{,00} + f''\rho_{,m}\rho_{,m} = 0$$

or

$$-\frac{\partial^2 \rho}{\partial t^2} + f'\nabla^2\rho + f''\left(\vec{\nabla}\rho\right)^2 = 0$$

Compare this equation to the wave equation with propagation velocity v

$$-\frac{\partial^2 \psi}{\partial t^2} + v^2\nabla^2\psi = 0$$

The signal propagation characteristics of the equation are determined by its second derivative terms, so the sound-speed is

$$v = \sqrt{\frac{df}{d\rho}} = \sqrt{\frac{dp}{d\rho}}$$

1.2 Answer: (curved spacetime version)

The straightforward approach is to replace commas by semicolons in the version 1 calculation above, thus introducing a mess of connection coefficients. The coefficients, but not their derivatives, can be made to go away by assuming a local Lorentz Frame. The remaining extra terms do not affect the second derivatives of ρ, so we get the same sound speed result as before.

2 Problem 2: Bosons in Curved Spacetime

In Special Relativity, the wave function for a spin-zero massive particle obeys the Klein Gordon Equation

$$-\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = m^2\psi$$

a) Suppose that such a particle is moving through a curved spacetime and use minimal coupling to find a candidate for its wave equation.

2.1 Answer a)

In a local Lorentz frame, replace ordinary derivatives by covariant derivatives or commas by semicolons. The Special Relativity form of the equation is

$$-\psi_{,00} + \psi_{,mm} = m^2\psi$$

so the curved space form (in a local Lorentz frame) would be

$$-\psi_{;00} + \psi_{;mm} = m^2\psi$$

or, putting in the metric tensor

$$g^{\mu\nu}\psi_{;\mu\nu} = m^2\psi$$

Since this equation is now invariant under coordinate transformations, it will be true in any coordinate system.

b) Write out the candidate equation in detail for the case of a particle moving along the z-axis (so that $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial y}$ are zero).

2.2 Answer b)

$$\psi_{;\mu\nu} = \psi_{,\mu\nu} - \psi_{,\rho}\Gamma^{\rho}{}_{\mu\nu}$$

$$g^{\mu\nu}\psi_{,\mu\nu} - \psi_{,\rho}\Gamma^{\rho}{}_{\mu\nu}g^{\mu\nu} = m^2\psi$$

Notice that the curved space comes in only through the term

$$\Gamma^{\rho} = \Gamma^{\rho}{}_{\mu\nu}g^{\mu\nu}$$

Recall

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\sigma}\left(e_\gamma g_{\sigma\beta} + e_\beta g_{\gamma\sigma} - e_\sigma g_{\beta\gamma}\right)$$

so that

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(e_\nu g_{\sigma\mu} + e_\mu g_{\nu\sigma} - e_\sigma g_{\mu\nu}\right)$$

$$\Gamma^{\rho} = g^{\mu\nu}\frac{1}{2}g^{\rho\sigma}\left(e_\nu g_{\sigma\mu} + e_\mu g_{\nu\sigma} - e_\sigma g_{\mu\nu}\right)$$

Assume that the x, y, z, t axes are orthonormal at a particular point, so that the equation takes the form

$$-\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^2\psi}{\partial z^2} - \Gamma^0\frac{\partial\psi}{\partial t} - \Gamma^3\frac{\partial\psi}{\partial z} = 0$$

with

$$\begin{aligned}
\Gamma^0 &= \frac{1}{2}g^{\mu\nu}\left(e_\nu g_{0\mu} + e_\mu g_{\nu 0} - e_0 g_{\mu\nu}\right) \\
&= -\frac{1}{2}\left(e_0 g_{00} + e_0 g_{00} - e_0 g_{00}\right) + \frac{1}{2}\left(e_m g_{0m} + e_m g_{m0} - e_0 g_{mm}\right) \\
&= -\frac{1}{2}\frac{\partial}{\partial t}g_{00} + \frac{1}{2}\left(2\frac{\partial}{\partial x^m}g_{0m} - \frac{\partial}{\partial t}g_{mm}\right) = \frac{\partial}{\partial x^m}g_{0m} - \frac{1}{2}\frac{\partial}{\partial t}\left(g_{00} + g_{mm}\right)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma^3 &= \frac{1}{2} g^{\mu\nu} \left(e_\nu g_{3\mu} + e_\mu g_{\nu 3} - e_3 g_{\mu\nu} \right) \\
&= -\frac{1}{2} \left(e_0 g_{30} + e_0 g_{03} - e_3 g_{00} \right) + \frac{1}{2} \left(e_m g_{3m} + e_m g_{m3} - e_3 g_{mm} \right) \\
&= -\frac{1}{2} \left(2 \frac{\partial}{\partial t} g_{30} - \frac{\partial}{\partial z} g_{00} \right) + \frac{1}{2} \left(2 e_m g_{3m} - \frac{\partial}{\partial z} g_{mm} \right) \\
&= \frac{\partial}{\partial x^m} g_{3m} - \frac{\partial}{\partial t} g_{30} + \frac{1}{2} \frac{\partial}{\partial z} \left(g_{00} - g_{mm} \right)
\end{aligned}$$

The key point to notice is that the wave equation is modified by terms constructed from the first derivatives of the metric tensor. A plane wave expansion of ψ along with the assumption that Γ^0 and Γ^3 vary slowly can be used to show that these terms cause exponential growth or decay of the wave function in both t and z.

3 Problem 3: Soap Films (Problem of Plateau)

A soap-film suspended on a wire frame with no air trapped anywhere will try to minimize its total surface area because of surface tension.

a) Represent such a film in parametric form in Cartesian coordinates.

3.1 Answer a)

Let x and y be coordinates on the film and use a Cartesian coordinate position vector

$$\vec{X}(x, y) = \left[X(x, y), Y(x, y), Z(x, y) \right]$$

to locate the point (x, y) on the film.

b) Find the differential equations that are obeyed by the functions in this description of a soap film.

3.2 Answer b)

The area of the film is given by the same sort of expression as the Goto-Nambu string action that we discussed:

$$\begin{aligned}
A &= \int d^2x \sqrt{|g_{HH}|} \\
&= \int d^2x \sqrt{\begin{vmatrix} \frac{\partial \vec{X}}{\partial x} \cdot \frac{\partial \vec{X}}{\partial x} & \frac{\partial \vec{X}}{\partial x} \cdot \frac{\partial \vec{X}}{\partial y} \\ \frac{\partial \vec{X}}{y} \cdot \frac{\partial \vec{X}}{\partial x} & \frac{\partial \vec{X}}{\partial y} \cdot \frac{\partial \vec{X}}{\partial y} \end{vmatrix}} \\
&= \int \int dx dy \left\{ \left(\frac{\partial \vec{X}}{\partial x} \cdot \frac{\partial \vec{X}}{\partial x} \right) \left(\frac{\partial \vec{X}}{\partial y} \cdot \frac{\partial \vec{X}}{\partial y} \right) - \left(\frac{\partial \vec{X}}{\partial y} \cdot \frac{\partial \vec{X}}{\partial x} \right) \left(\frac{\partial \vec{X}}{\partial x} \cdot \frac{\partial \vec{X}}{\partial y} \right) \right\}^{1/2}
\end{aligned}$$

The differential equations satisfied by the functions $\vec{X}$ are obtained from the requirement

$$\delta A = 0$$

which is evaluated just as for the string action but with greek indexes being summed just from 1 to 2.

$$\delta A = \frac{1}{2}\sqrt{|g_{HH}|}g^{HH\alpha\beta}\delta\left(\frac{\partial \vec{X}}{\partial x^\alpha}\cdot\frac{\partial \vec{X}}{\partial x^\beta}\right)$$

$$\begin{aligned}
&= \sqrt{|g_{HH}|}g^{HH\alpha\beta}\left(\frac{\partial \vec{X}}{\partial x^\alpha}\cdot\frac{\partial \delta\vec{X}}{\partial x^\beta}\right) \\
&= \frac{\partial}{\partial x^\beta}\left(\left(\sqrt{|g_{HH}|}g^{HH\alpha\beta}\frac{\partial \vec{X}}{\partial x^\alpha}\right)\cdot\delta\vec{X}\right) - \frac{\partial}{\partial x^\beta}\left(\sqrt{|g_{HH}|}g^{HH\alpha\beta}\frac{\partial \vec{X}}{\partial x^\alpha}\right)\cdot\delta\vec{X}
\end{aligned}$$

so that

$$\delta A = \int\int dxdy\left\{\frac{\partial}{\partial x^\beta}\left(\left(\sqrt{|g_{HH}|}g^{HH\alpha\beta}\frac{\partial \vec{X}}{\partial x^\alpha}\right)\cdot\delta\vec{X}\right) - \frac{\partial}{\partial x^\beta}\left(\sqrt{|g_{HH}|}g^{HH\alpha\beta}\frac{\partial \vec{X}}{\partial x^\alpha}\right)\cdot\delta\vec{X}\right\}$$

Use Green's Theorem on the total divergence term:

$$\int\int dxdy\left\{\frac{\partial}{\partial x^\beta}\left(\left(\sqrt{|g_{HH}|}g^{HH\alpha\beta}\frac{\partial \vec{X}}{\partial x^\alpha}\right)\cdot\delta\vec{X}\right)\right\} = \oint d\ell n_\beta\left(\sqrt{|g_{HH}|}g^{HH\alpha\beta}\frac{\partial \vec{X}}{\partial x^\alpha}\right)\cdot\delta\vec{X}$$

where $d\ell$ is the line element along the wire boundary and n_β is the outward directed normal. So long as the variation is held fixed at the boundary, $\delta\vec{X} = 0$ and this term vanishes. The condition is then

$$\int\int dxdy\left\{\frac{\partial}{\partial x^\beta}\left(\sqrt{|g_{HH}|}g^{HH\alpha\beta}\frac{\partial \vec{X}}{\partial x^\alpha}\right)\cdot\delta\vec{X}\right\} = 0$$

for arbitrary $\delta\vec{X}$ or

$$\frac{\partial}{\partial x^\beta}\left(\sqrt{|g_{HH}|}g^{HH\alpha\beta}\frac{\partial \vec{X}}{\partial x^\alpha}\right) = 0$$

An equivalent form of this expression is just

$${}^{(2)}\nabla^2\vec{X} = 0$$

where ${}^{(2)}\nabla^2$ is the covariant Laplacian on the surface. The famous result of the Plateau problem follows by noticing that this equation is also the condition that X, Y, Z are each real or imaginary parts of analytic functions of $x + iy$. Thus, we can construct soap bubble films from triplets of analytic functions. As a result, the entire problem is solved exactly. For example, choose the analytic functions, $X = x, Y = y, Z = \text{Re}\left((x+iy)^2\right) = x^2 - y^2$ and obtain a "saddle" shaped film of extremal area. Similarly, $X = x, Y = y, Z = \text{Re}\left((x+iy)^3\right)$ describes an extremal area surface that is sometimes called a "monkey saddle".

This expression is already enough for this part, but we will need a bit more for the next part of the problem. Notice that a variation that satisfies the constraint

$$H\delta\vec{X} = 0$$

will not change the surface. It will only change the coordinates x, y on the surface. Thus the area will not change under such a variation. Thus, the following equation is an identity:

$$\int\int dxdy \left\{ \frac{\partial}{\partial x^\beta} \left(\sqrt{|g_{HH}|} g^{HH\alpha\beta} \frac{\partial \vec{X}}{\partial x^\alpha} \right) \cdot H\delta\vec{X} \right\} = 0$$

But that is the same as

$$\int\int dxdy \left\{ H \frac{\partial}{\partial x^\beta} \left(\sqrt{|g_{HH}|} g^{HH\alpha\beta} \frac{\partial \vec{X}}{\partial x^\alpha} \right) \cdot \delta\vec{X} \right\} = 0$$

Since $\delta\vec{X}$ is arbitrary, we have the identity

$$H \frac{\partial}{\partial x^\beta} \left(\sqrt{|g_{HH}|} g^{HH\alpha\beta} \frac{\partial \vec{X}}{\partial x^\alpha} \right) = 0$$

so that the equation that constrains the surface can also be written as

$$V \frac{\partial}{\partial x^\beta} \left(\sqrt{|g_{HH}|} g^{HH\alpha\beta} \frac{\partial \vec{X}}{\partial x^\alpha} \right) = 0$$

Use Leibniz's product rule to obtain

$$V \left\{ \left(\frac{\partial}{\partial x^\beta} \sqrt{|g_{HH}|} g^{HH\alpha\beta} \right) \frac{\partial \vec{X}}{\partial x^\alpha} + \sqrt{|g_{HH}|} g^{HH\alpha\beta} \frac{\partial^2 \vec{X}}{\partial x^\alpha \partial x^\beta} + \right\} = 0$$

But the vectors $\frac{\partial \vec{X}}{\partial x^\alpha}$ lie in the surface, so

$$V \frac{\partial \vec{X}}{\partial x^\alpha} = 0$$

and we get the equation for the soap film surface in the simple form

$$V g^{HH\alpha\beta} \frac{\partial^2 \vec{X}}{\partial x^\alpha \partial x^\beta} = 0.$$

c) Find the condition(s) satisfied by the second fundamental form of such a soap film.

3.3 Answer c)

The straightforward way to do this problem is to take the expression for the projection curvature tensor from the notes

$$h_H{}^c{}_{da} = H_{jd} \frac{\partial X^j}{\partial x^\sigma} g^{\sigma\delta} g^{\alpha\rho} \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} \frac{\partial X^k}{\partial x^\alpha} V_{pa} H^c{}_k$$

This expression has way too many uncontracted indexes to compare with the differential equations that we obtained in part b. Those equations had just one Cartesian index. The only obvious way to get rid of indexes is to contract them and the only way that does not give zero is to contract the first two (the third is projected in the complementary direction).

$$\begin{aligned}
h_{Ha} &= h_H{}^c{}_{ca} = H_{jc} \frac{\partial X^j}{\partial x^\sigma} g^{\sigma\delta} g^{\alpha\rho} \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} \frac{\partial X^k}{\partial x^\alpha} V_{pa} H^c{}_k \\
&= H_{jk} \frac{\partial X^j}{\partial x^\sigma} g^{\sigma\delta} g^{\alpha\rho} \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} \frac{\partial X^k}{\partial x^\alpha} V_{pa} \\
&= \frac{\partial X^j}{\partial x^\sigma} H_{jk} \frac{\partial X^k}{\partial x^\alpha} g^{\sigma\delta} g^{\alpha\rho} \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} V_{pa} \\
&= \frac{\partial \vec{X}}{\partial x^\sigma} \cdot \frac{\partial \vec{X}}{\partial x^\alpha} g^{\sigma\delta} g^{\alpha\rho} \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} V_{pa} \\
&= g_{\sigma\alpha} g^{\sigma\delta} \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} V_{pa} \\
&= \delta^\delta_\alpha g^{\alpha\rho} \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} V_{pa} \\
&= g^{\alpha\beta} \frac{\partial^2 X^p}{\partial x^\alpha \partial x^\beta} V_{pa}
\end{aligned}$$

Now we recognize the result of the previous problem and see that the condition on the second fundamental form is just that it be trace-free:

$$h_{Ha} = 0.$$

4 Problem 4: Gravitational Wave Sources

a) Use the results found in class, but ignore polarization effects and derive an approximate relationship between detector strain, source luminosity, source distance, and source frequency. For this part, just leave everything in Planck units.

4.1 Answer a)

The detector strain is given by the expression

$$h^{jk}\left(x^0, x^i\right) = \frac{2}{r} P^j{}_r P^k{}_s \ddot{I}^{rs}\left(x^0 - r\right)$$

The various projection operators are combinations of trig functions that are of order one, so if we ignore polarization the result is

$$\text{strain} = \frac{2}{r}\ddot{I}$$

where I is the dominant component of the quadrupole moment tensor. For a source oscillating with angular frequency ω, differentiating by time just produces a factor of ω. Thus,

$$\text{strain} = \frac{2\omega^2}{r} I$$

The source luminosity is given by the quadrupole formula

$$L = \frac{1}{5} \dddot{I}^{jk} \dddot{I}^{jk}$$

which, for an oscillating source, with one dominant quadrupole moment, gives

$$L = \frac{\omega^6}{5} I^2$$

Use this expression to elliminate the quadrupole moment

$$I = \sqrt{\frac{5L}{\omega^6}} = \omega^{-3}\sqrt{5L}$$

and obtain the desired relation

$$\text{strain} = 2\frac{\omega^2}{r}\omega^{-3}\sqrt{5L} = \frac{2\sqrt{5L}}{\omega r}$$

b) For this part, you will need to look up some constants and conversions. Find the greatest distance (in light years) that a gravitational wave detector with a strain sensitivity of 10^{-18} could respond to an event that dumps one full solar mass of energy into a one second pulse of gravitational waves at an angular frequency of a kiloHertz.

4.2 Answer b)

The mass of the sun is about 2×10^{30} kg. Converting that much mass into energy would yield

$$\begin{aligned} E &= mc^2 = 2 \times 10^{30}\,\text{kg} \times \left(3 \times 10^8\,\text{m/s}\right)^2 \\ &= 18 \times 10^{46}\,\text{J} \end{aligned}$$

and doing it in one second would yield a luminosity of

$$\begin{aligned} L &= 18 \times 10^{46}\,\text{W} \\ &= 18 \times 10^{46} \times 10^{-52.560}\ \text{Planck Power units} \\ &= 5.0 \times 10^{-6}\ \text{Planck Power units} \end{aligned}$$

The other input that we need in dimensionless form is the angular frequency

$$\begin{aligned}\omega &= 10^3\,\mathrm{Hz} \\ &= 10^3 \times 10^{-43.268} = 10^{-40.268} \\ &= 5.4 \times 10^{-41}\end{aligned}$$

Now solve the relation from the last part

$$\text{strain} = \frac{2\sqrt{5L}}{\omega r}$$

for the distance r

$$\begin{aligned}r &= \frac{2\sqrt{5L}}{\omega \text{strain}} \\ &= \frac{2\sqrt{5 \times 5 \times 10^{-6}}}{5.4 \times 10^{-41} \times 10^{-18}} \\ &= 1.8519 \times 10^{56} \\ &= 2 \times 10^{56}\text{Planck distance units} \\ &= 2 \times 10^{56} \times \frac{1\,\mathrm{m}}{10^{34.791}} \\ &= 3 \times 10^{21}\,\mathrm{m}\end{aligned}$$

A light-year is

$$\begin{aligned}1\text{ly} &= 3 \times 10^8\,\mathrm{m/s} \times 3.15 \times 10^7\,\mathrm{s} \\ &= 10^{16}\,\mathrm{m}\end{aligned}$$

so the detectability distance is

$$\begin{aligned}r &= 3 \times 10^{21}\,\mathrm{m} = 3 \times 10^5 \text{ light years} \\ &= 300,000 \text{ light years}\end{aligned}$$

Since our galaxy is about 100,000 light years across, the event would have to be somewhere within our galaxy or possibly in the Large or Small Magellanic clouds that orbit our galaxy.

From this calculation, you can also see that each factor of ten improvement in strain sensitivity multiplies the range by a factor of ten. You can also see that lower frequency signals can be detected at much longer range. That is one reason for the LISA proposal to use orbiting spacecraft to detect frequencies well below one Hertz.

5 Problem 5: Lapse and Shift

Find the lapse and shift functions that correspond to the spacetime metric tensor

$$ds^2 = -(1 - 2m/r)\,dt^2 + 2vP^{-1/2}dtdr + r^4P^{-1}dr^2 + r^2d\Omega^2$$

where v, P are polynomials

$$P = v^2 + (1 - 2m/r)\, r^4$$

$$v = Kr^3/3 - H$$

By the way, this is the metric of a black hole of mass m in peculiar coordinates.

5.1 Answer

The metric components have the form

$$[g] = \begin{bmatrix} 1\,(1-2m/r) & vP^{-1/2} & 0 & 0 \\ vP^{-1/2} & r^4P^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

while the inverse metric components are

$$\left[g^{-1}\right] = \begin{bmatrix} 1\,(1-2m/r) & vP^{-1/2} & 0 & 0 \\ vP^{-1/2} & r^4P^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}^{-1}$$

or :

$$\left[g^{-1}\right] = \begin{bmatrix} -\frac{r^4}{-r^4+v^2+2mr^3} & P\frac{v}{-\sqrt{P}r^4+\sqrt{P}v^2+2\sqrt{P}mr^3} & 0 & 0 \\ P\frac{v}{-\sqrt{P}r^4+\sqrt{P}v^2+2\sqrt{P}mr^3} & \frac{2Pm-Pr}{-r^5+2mr^4+rv^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}$$

Compare these expressions to the ones given in the notes:

$$\begin{aligned} [g] &= \begin{pmatrix} \left|\vec{N}\right|^2 - N^2 & h\left(\vec{N}\right)^T \\ h\left(\vec{N}\right) & [h] \end{pmatrix} \\ \left[g^{-1}\right] &= \begin{pmatrix} -1/N^2 & \vec{N}^T/N^2 \\ \vec{N}/N^2 & [h]^{-1} - \vec{N}\vec{N}^T/N^2 \end{pmatrix} \end{aligned}$$

The spacelike metric is evidently

$$[h] = \begin{bmatrix} r^4P^{-1} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

and we can read off the shift vector components with their indexes lowered by this metric:

$$N_i = g_{0i}$$

or

$$\begin{aligned} N_1 &= vP^{-1/2} \\ N_2 &= N_3 = 0 \end{aligned}$$

Raise the index using h^{-1}

$$\begin{aligned} N^1 &= r^{-4}PvP^{-1/2} = \frac{v}{r^4}\sqrt{P} \\ N^2 &= N^3 = 0 \end{aligned}$$

The lapse function is obtained by comparing the inverse metric expressions

$$1/N^2 = \frac{r^4}{-r^4 + v^2 + 2mr^3} = \frac{1}{v^2 r^{-4} - (1 - 2m/r)}$$

$$N = \sqrt{v^2 r^{-4} - (1 - 2m/r)} = r^{-2}\sqrt{v^2 - (1 - 2m/r)\, r^4}$$

Collect the final non-zero results in the form

$$\begin{aligned} N^1 &= \frac{v}{r^4}\sqrt{v^2 + (1 - 2m/r)\, r^4} \\ N &= r^{-2}\sqrt{v^2 - (1 - 2m/r)\, r^4} \end{aligned}$$

The important thing to notice about these results is that they are well-behaved at $r = 2m$. Thus, these t =constant surfaces are regular across the black hole event horizon and t is a well-behaved time coordinate there. One might not have guessed that from the original form of the metric tensor.

The example is a static, regular slicing of a black hole by a set of hyperbolic constant-time surfaces that are asymptotically lightlike.

6 Problem 6: Initial Data

Suppose that you wish to set up time-symmetric initial data for two black holes of identical mass separated by about ten Schwarzschild radii. The data is to be set up on a Cartesian coordinate grid (x, y, z) with the holes on the z-axis.

For a single black hole the horizon corresponds to the minimal area $r =$ constant surface at the instant of time symmetry. Assume that this relationship is approximately true for these interacting black holes so that their minimal area surfaces (now somewhat distorted) correspond to their horizons and give the spacetime Cartesian metric tensor components as functions of the coordinates (x, y, z).

6.1 Answer

The spatial metric tensor is taken to be

$$g_{ij} = \phi^4 \delta_{ij}$$

where a single black hole with mass m would be represented by

$$\phi = 1 + \frac{2m}{\tilde{r}}$$

with $\tilde{r}$ the radius in terms of the flat space metric

$$\tilde{r} = \sqrt{x^2 + y^2 + z^2}$$

The area at constant $\tilde{r}$ is just

$$A(\tilde{r}) = 4\pi \left(1 + \frac{2m}{\tilde{r}}\right)^4 \tilde{r}^2$$

which goes through a minimum at $\tilde{r} = 2m$ so that is the location of the "throat" in these coordinates.

For two identical black holes, we would have

$$\phi = 1 + \frac{2m}{\tilde{r}_1} + \frac{2m}{\tilde{r}_2}$$

where $\tilde{r}_1$ and $\tilde{r}_2$ are distances from different points calculated using the flat metric. The simplest *starting assumption* to make is that these distance are calculated from points a distance $20m$ apart in the flat metric. Put one at $z = -10m$ and the other at $z = 10m$ along the z-axis so that

$$\begin{aligned} \tilde{r}_1 &= \sqrt{x^2 + y^2 + (z + 10m)^2} \\ \tilde{r}_2 &= \sqrt{x^2 + y^2 + (z - 10m)^2} \end{aligned}$$

and thus the proposed initial metric is

$$g_{ij} = \left(1 + \frac{2m}{\sqrt{x^2 + y^2 + (z + 10m)^2}} + \frac{2m}{\sqrt{x^2 + y^2 + (z - 10m)^2}}\right)^4 \delta_{ij}$$

Of course, the flat metric does not measure physical distances, so we still need to check what the actual distance between these black holes is. Along the z-axis,

$$ds^2 = \left(1 + \frac{2m}{\sqrt{(z + 10m)^2}} + \frac{2m}{\sqrt{(z - 10m)^2}}\right)^4 dz^2$$

so that the distance element is

$$ds = \left(1 + \frac{2m}{|z + 10m|} + \frac{2m}{|z - 10m|}\right)^2 dz$$

Integrate this from $z = 0$ to the approximate surface of one of the holes at $z = 8m$ to get half the physical separation

$$\begin{aligned}
\frac{1}{2}D &= \int_0^{8m} \left(1 + \frac{2m}{z+10m} - \frac{2m}{z-10m}\right)^2 dz \\
&= \int_0^{8m} \left(1 + 2m\left(\frac{1}{z+10m} - \frac{1}{z-10m}\right)\right)^2 dz \\
&= \int_0^{8m} \left(1 - 2m\left(\frac{20m}{z^2-100m^2}\right)\right)^2 dz \\
&= \int_0^{8m} \left(1 - \frac{40m^2}{z^2-100m^2}\right)^2 dz \\
&= m\int_0^{8} \left(1 - \frac{40}{x^2-100}\right)^2 dx \\
&= \left(\frac{22}{5}\ln 18 - \frac{22}{5}\ln 2 + \frac{88}{9}\right) m \\
&= 19.45m
\end{aligned}$$

Evidently we have the holes separated by about twenty Shwarzschild radii, so try putting them closer at $z = \pm 5m$ and integrate half the distance from zero to $3m$.

$$\begin{aligned}
\frac{1}{2}D &= \int_0^{3m} \left(1 + \frac{2m}{z+5m} - \frac{2m}{z-5m}\right)^2 dz \\
&= \int_0^{3m} \left(1 + 2m\left(\frac{1}{z+5m} - \frac{1}{z-5m}\right)\right)^2 dz \\
&= \int_0^{3m} \left(1 - 2m\left(\frac{10m}{z^2-25m^2}\right)\right)^2 dz \\
&= \int_0^{3m} \left(1 - \frac{20m^2}{z^2-25m^2}\right)^2 dz \\
&= m\int_0^{3} \left(1 - \frac{20}{z^2-25}\right)^2 dz \\
&= m\left(\frac{24}{5}\ln 8 - \frac{24}{5}\ln 2 + \frac{9}{2}\right) \\
&= 11.154m
\end{aligned}$$

Now we are closer, with the holes separated by about 11 Schwarzschild radii.

7 Problem 7: Isometries

Use the procedures that we applied to the case of static spherical symmetry and construct a simple form for the metric of a static, cylindrically symmetric spacetime. Take the coordinates to be (t, r, z, θ). In this case, the Killing vectors are $\frac{\partial}{\partial t}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \theta}$. Be sure to justify each specialization.

7.1 Answer

Because the spacetime is static, there is a timelike Killing vector field that we can take to be $\frac{\partial}{\partial t}$. There is also a reflection symmetry under time reversal, so the spactime metric cannot have cross-terms between time and space coordinates. The spacetime metric then takes the form

$$ds^2 = -f dt^2 + d\ell^2$$

where $d\ell^2$ is a three-dimensional space metric and, along with f is independent of the time t.

For cylindrical symmetry, there are two more Killing vector fields. These generate group orbits that are cylinders. Choose one of these group orbits and put the usual coordinates z, θ on it, with θ an angle so that its metric is

$${}^{(2)}dc^2 = Bdz^2 + Cd\theta^2$$

with

$$\begin{aligned} -\infty &< z < \infty \\ -\pi &< \theta \leq \pi \end{aligned}$$

The family of curves perpendicular to the group orbits can then be used to map these z, θ coordinates onto all of the other orbits. The function f in the spacetime metric will then be independent of z, θ as well as t. With orbits labeled by a coordinate r the space metric is then

$$d\ell^2 = A(r)\, dr^2 + B(r)\, dz^2 + C(r)\, d\theta^2$$

and the spacetime metric is

$$ds^2 = -f(r)\, dt^2 + A(r)\, dr^2 + B(r)\, dz^2 + C(r)\, d\theta^2$$

Just as for the Schwarzschild metric, the radial coordinate that labels the orbits can be defined, thus eliminating one function. If r is defined to be $\frac{1}{2\pi}$ times the orbit circumference, then the metric becomes

$$ds^2 = -f(r)\, dt^2 + A(r)\, dr^2 + B(r)\, dz^2 + r^2 d\theta^2$$

8 Problem 8:

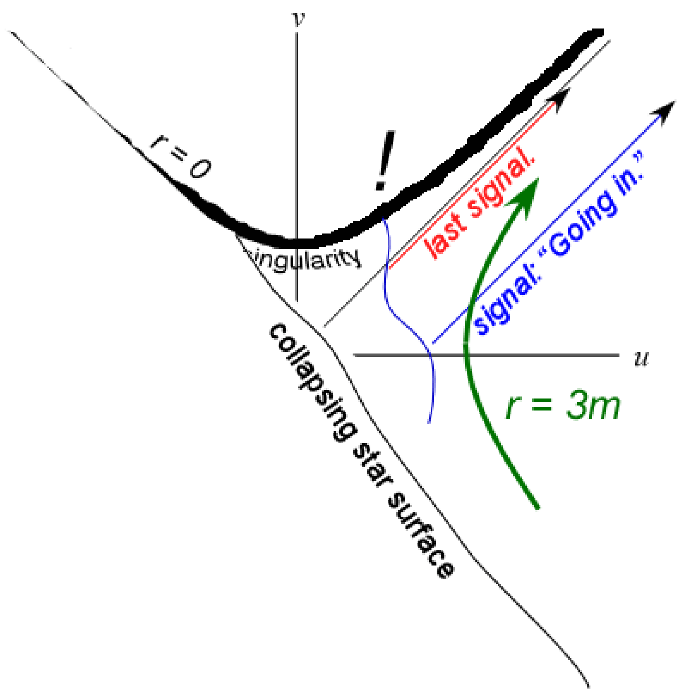
For this problem, you will have to draw some pictures.

Use a Kruskal Diagram to show the geometry near the surface of a star that is collapsing to a black hole. An observer is standing off from the collapse at a constant luminosity distance of $r = 3m$.

a) What happens to the initial $r = 0$ singularity of the Kruskal metric in this picture?

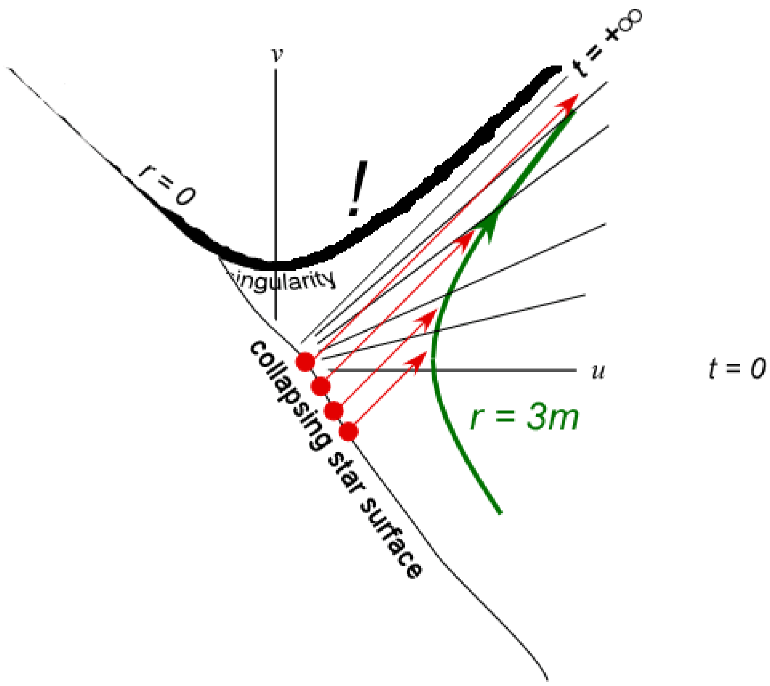
8.1 Answer a)

The picture is actually in the notes.



The initial singularity is the bottom branch of the hyperbola. It is replaced by the spacetime geometry inside the star. Thus, the initial singularity is not actually present in this spacetime.

b) Suppose that a clock is on the surface of the star and is sending out light signals are regular intervals. Use the Kruskal Diagram to explain what the $r = 3m$ observer will see in terms of the time, t for which the external geometry is static.



The Schwarzschild time coordinate goes to infinity near the horizon, so the signals from regular events reach the $r = 3m$ hyperbola at increasing time intervals as the surface nears the horizon. Light from the star would be red-shifted until it becomes undetectable.