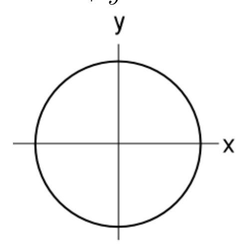
1 The Classical Embedding Problem

1.1 Surfaces vs Manifolds

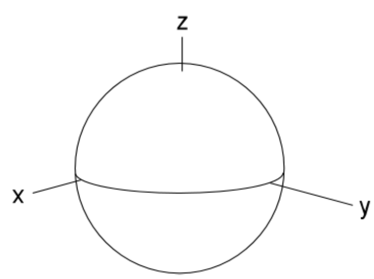
The currently accepted way to represent a non-Euclidean geometry is to start with a *manifold*, defined only in terms of overlapping coordinate charts. That method was not how non-Euclidean geometry began, however. The subject began by considering *surfaces* embedded in Euclidean space. A circle of radius R, for example, was embedded in flat two-dimensional cartesian space according to

$$x^2 + y^2 = R^2$$



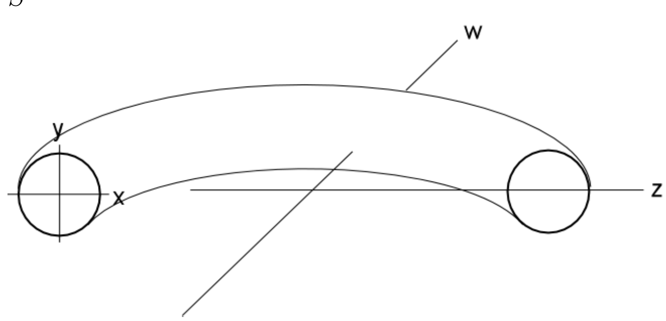
while a two-sphere was the set of number triplets (x, y, z) such that

$$x^2 + y^2 + z^2 = R^2.$$



A smooth, flat, two dimensional torus was the set of number quadruplets (x, y, z, w) such that

$$\begin{aligned} x^2 + y^2 &= R^2 \\ z^2 + w^2 &= S^2 \end{aligned}$$



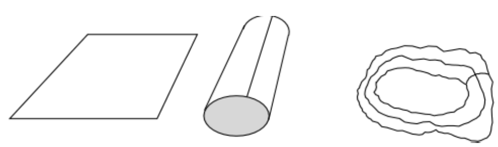
These surfaces inherit their metrics from the embedding space, so they are said to be *isometric embeddings*.

The older embedded surface representation and the modern manifold representation are very different. It was natural to ask if they are really equivalent. Surfaces embedded in Euclidean space inherit a torsion-free, metric-compatible connection. Manifolds with this type of structure are called Riemannian manifolds. It is obvious that embedded surfaces must always be Riemannian manifolds with varying combinations of the Cartesian coordinates forming the needed overlapping coordinate patches. In fact, that is a standard homework exercise in manifold theory. What is not so obvious is that every Riemannian manifold can be represented by a surface isometrically embedded in Euclidean space. That "embedding problem" had to be solved before the manifold picture could become accepted as the primary one.

1.2 The Solutions

The first satisfactory solution of the embedding problem was the famous paper by John Nash [J. Nash, "The Imbedding Problem for Riemannian Manifolds," Annals of Mathematics, **63** (1956) pp 20-63.]. His result was that any compact manifold with a C^k metric (for $k \geq 3$) can be isometrically embedded in N dimensional Euclidean space where $N = \frac{n(3n+11)}{2}$. Later results have cut down the value of N but Nash was the first to show that there is a finite value of N that will always work. A good review of that and related work can be found at http://wwwmaths.anu.edu.au/research.reports/proceedings/040/CMAproc40-andrews.pdf

To understand why large numbers of dimensions might be involved, consider the simplest imaginable case, flat two dimensional space. A sheet of paper models an isometric embedding of such a space in three dimensions. It is somewhat floppy, but one can only bend it in one direction. The sheet can be rolled up to form a tube. However, if one attempts to bend it around to form a doughnut shape or 2-torus, one is forced to crease it somewhere. The embedding then drops from C^∞ to C^1. However, as the example a bit earlier in this section demonstrates, it is possible to have a C^∞ embedding of the 2-torus in four dimensions.



If one now attempts to give the sheet a half-twist before identifying its edges, that works in three dimensions and results in a Mobius strip. However an identification of the remaining edge to make a Klein bottle requires an embedding in 5 dimensions.

For embedding spacetimes, the situation is much worse, but it can still be done. Any spacetime can be embedded isometrically in $\mathbb{R}^N$ with a metric with constant coefficients and $N \leq 90$ and no more than 3 timelike directions. [C.J.S.

Clarke, "On the global isometric embedding of pseudo-Riemannian manifolds," Proceedings of the Royal Society, A314, (1970) pp 417-428.

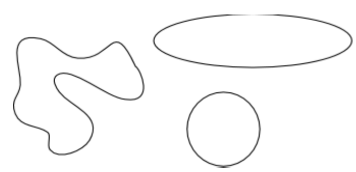
1.3 Why Embed in Flat Spaces?

It is natural to ask about the possibility of embedding a given Riemannian manifold in a higher dimensional Riemannian manifold other than flat (constant metric coefficients) $\mathbb{R}^N$. However, for the classical embedding problem, that is not relevant because embedding that higher dimensional manifold in flat $\mathbb{R}^N$ would yield an embedding of the original manifold in $\mathbb{R}^N$. The key point in the classical embedding problem is just that N should be finite even if it turns out to be very large.

1.4 Is this physics?

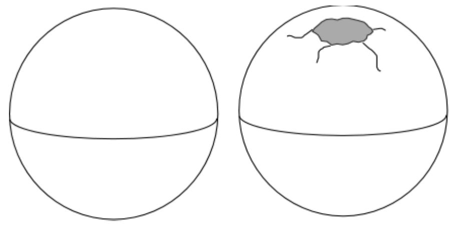
A general point that is usually made about the embedding problem is that it confirms the fact that the surface embedding representation of a Riemannian manifold has nothing at all to do with physics. On this issue (and others), Einstein is said to have listened patiently and then declared, "Interesting, but you had better go back and study some more physics young man".[Sylvia Nasar, "A Beautiful Mind: A Biography of John Forbes Nash, Jr.," Simon & Schuster, (1998)].

One indication that the embedding representation might not have much to do with physics is that it introduces arbitrariness. For example, the embedding of a circle in two dimensions can be visualized as a loop of string thrown on a table.



The loop can arranged as a circle or an ellipse, or in the shape of a daisy with no effect at all on the one-dimensional manifold that is the circle.

Another indication that the embedding representation might be an unnecessary distraction is the quirky way in which embeddings are sometimes mostly arbitrary and sometimes unique. For example, the embedding of a 2-sphere or any other convex surface in three dimensions is unique up to translations and rotations. It is said to be "rigid." That is the reason that the thin shell of an egg can hold its shape. However, a sphere with a hole in it has many different embeddings, so it is “floppy.”



The results of the classical embedding problem, particularly for spacetimes, provide another indication that this particular problem is not physics. It is difficult to justify representing four dimensional space time by introducing a 90 dimensional embedding spacetime with as many as three different time directions.

There are physical theories that treat four dimensional spacetime as a surface embedded in a larger manifold. However, these theories take the embedding as a physical hypothesis with physically observable consequences for the embedded spacetime. For example, assuming that spacetime is isometrically embedded in a flat Minkowski-like manifold with 89 space dimensions and one time dimension would automatically rule out any spacetime that permits closed timelike lines. In most “brane-world” theories, there are far less than 89 space dimensions, so the embedding hypothesis actually imposes severe restraints on the allowable spacetimes. Understanding these restraints is, essentially, the opposite of the classical embedding problem where one seeks to eliminate the restraints.

2 Specializing the Embedding Structure Equations

2.1 Metricity

For a surface embedded in a flat Riemannian manifold, it is natural to choose a normal projection tensor. The metricity of the embedding space is zero and projecting its various components yields the following equations:

$$\begin{array}{rclcl} Q_H^{HH\mu\nu}{}_\delta &=& 0 & \text{or} & \hat{D}_{H\delta} g_H{}^{\mu\nu} = 0 \\ Q_V^{HH\mu\nu}{}_\delta &=& 0 & \text{or} & \hat{D}_{V\delta} g_H{}^{\mu\nu} = 0 \\ -g^{HH\mu\rho} h^*_{H\rho\delta}{}^\nu + h_H{}^\mu{}_{\delta\rho} g^{VV\rho\nu} &=& 0 & \text{or} & h^*_{H\rho\delta}{}^\nu = h_{H\rho\delta}{}^\nu \\ h_V{}^\nu{}_{\delta\rho} g^{HH\mu\rho} - g^{VV\rho\nu} h^*_{V\rho\delta}{}^\mu &=& 0 & \text{or} & h^*_{V\rho\delta}{}^\nu = h_{V\rho\delta}{}^\nu \end{array}$$

The first equation implies that that the projected derivative within the surface supplies a metric compatible connection within the surface. The second equation indicates that the Fermi derivative perpendicular to the surface preserves the surface projected metric tensor. The third and fourth equations mean that there is no distinction between the projection curvatures and the transpose projection curvatures.

2.2 Torsion

The torsion of the embedding space is zero and projecting its components normal to the embedded surface yields

$$S^{H}_{HH}{}^{\rho}{}_{\mu\nu} = 0$$

so that the connection induced on the surface also has zero torsion,

$$S^{V}_{HH}{}^{\rho}{}_{\mu\nu} = 0$$

so that there is no out-of surface torsion component, and

$$S^{H}_{HV}{}^{\rho}{}_{\mu\nu} = h_{H}{}^{\rho}{}_{\mu\nu}$$

which eliminates the remaining cross-projected torsion component by setting it equal to the projection curvature.

The complementary projections of the torsion yield

$$S^{V}_{VV}{}^{\rho}{}_{\mu\nu} = 0$$
$$S^{H}_{VV}{}^{\rho}{}_{\mu\nu} = 2\omega^{*}_{V\,\mu\nu}{}^{\rho}$$
$$S^{V}_{VH}{}^{\rho}{}_{\mu\nu} = h_{V}{}^{\rho}{}_{\mu\nu}$$

so that all of the cross-projected torsions are eliminated.

2.3 Curvature

2.3.1 Gauss Relations

The curvature of the embedding space is zero. Projecting its components normal to the embedded surface provides the Gauss relation for the intrinsic curvature tensor in terms of the extrinsic curvature

$$R^{H}_{HH\,\rho}{}^{\gamma}{}_{\alpha\beta} = h_{H}{}^{\gamma}{}_{\beta\sigma} h_{H\,\rho\alpha}{}^{\sigma} - h_{H}{}^{\gamma}{}_{\alpha\sigma} h_{H\,\rho\beta}{}^{\sigma}$$

as well as two additional Gauss-like relations for the cross-curvatures:

$$\begin{aligned} R^{H}_{HV\,\rho}{}^{\gamma}{}_{\alpha\beta} &= h_{H}{}^{\gamma}{}_{\alpha\sigma} h_{V}{}^{\sigma}{}_{\beta\rho} - h_{H\,\rho\alpha}{}^{\sigma} h_{V\,\sigma\beta}{}^{\gamma} \\ R^{H}_{VV\,\rho}{}^{\gamma}{}_{\alpha\beta} &= h_{V\,\sigma\beta}{}^{\gamma} h_{V}{}^{\sigma}{}_{\alpha\rho} - h_{V\,\sigma\alpha}{}^{\gamma} h_{V}{}^{\sigma}{}_{\beta\sigma} \end{aligned}$$

In each case, the projected curvature is expressed as a quadratic in the projection curvature. The fully contracted version of the first relation is particularly important:

$$g^{HH\rho\alpha} R^{H}_{HH\,\rho}{}^{\gamma}{}_{\alpha\gamma} = g^{HH\rho\alpha} h_{H}{}^{\gamma}{}_{\gamma\sigma} h_{H\,\rho\alpha}{}^{\sigma} - g^{HH\rho\alpha} h_{H}{}^{\gamma}{}_{\alpha\sigma} h_{H\,\rho\gamma}{}^{\sigma}$$

or

$${}^{H}R = \theta_{H\sigma}\theta_{H}{}^{\sigma} - h_{H}{}^{\gamma\rho}{}_{\sigma} h_{H\,\rho\gamma}{}^{\sigma}$$

where ${}^{H}R$ is the scalar curvature of the surface and is the basis for the Hilbert Action functional for general relativity.

$$I_{\text{Hilbert}} = \int d^4x \sqrt{|g_{HH}|}\left({}^{H}R\right)$$

2.3.2 Codazzi Relations

The Codazzi relations are somewhat more complex and require the use of the torsion relations. There are two distinct relations that reflect the flatness of the embedding space:

$$\begin{gathered}\hat{D}_{H\beta} h^*_{H\rho\alpha}{}^{\gamma} - \hat{D}_{H\alpha} h^*_{H\rho\beta}{}^{\gamma} = \\ S^H_{HH}{}^{\sigma}{}_{\alpha\beta} h^*_{H\rho\sigma}{}^{\gamma} - S^V_{HH}{}^{\sigma}{}_{\alpha\beta} h_V{}^{\gamma}{}_{\sigma\rho}\end{gathered}$$

$$\begin{gathered}\hat{D}_{V\beta} h^*_{H\rho\alpha}{}^{\gamma} + \hat{D}_{H\alpha} h_V{}^{\gamma}{}_{\beta\rho} = \\ S^H_{HV}{}^{\sigma}{}_{\alpha\beta} h^*_{H\rho\sigma}{}^{\gamma} - S^V_{HV}{}^{\sigma}{}_{\alpha\beta} h_V{}^{\gamma}{}_{\sigma\rho}\end{gathered}$$

and their complements

$$\begin{gathered}\hat{D}_{V\beta} h^*_{V\rho\alpha}{}^{\gamma} - \hat{D}_{V\alpha} h^*_{V\rho\beta}{}^{\gamma} = \\ S^V_{VV}{}^{\sigma}{}_{\alpha\beta} h^*_{V\rho\sigma}{}^{\gamma} - S^H_{VV}{}^{\sigma}{}_{\alpha\beta} h_H{}^{\gamma}{}_{\sigma\rho}\end{gathered}$$

$$\begin{gathered}\hat{D}_{H\beta} h^*_{V\rho\alpha}{}^{\gamma} + \hat{D}_{V\alpha} h_H{}^{\gamma}{}_{\beta\rho} = \\ S^V_{VH}{}^{\sigma}{}_{\alpha\beta} h^*_{V\rho\sigma}{}^{\gamma} - S^H_{VH}{}^{\sigma}{}_{\alpha\beta} h_H{}^{\gamma}{}_{\sigma\rho}\end{gathered}$$

Assume a normal embedding to equate the two types of projection curvature and then replace the cross-projected torsions. In that case, the last equation is the same as the second and just three distinct relations remain:

$$\hat{D}_{H\beta} h_{H\rho\alpha}{}^{\gamma} - \hat{D}_{H\alpha} h_{H\rho\beta}{}^{\gamma} = 0$$

$$\hat{D}_{V\beta} h_{H\rho\alpha}{}^{\gamma} + \hat{D}_{H\alpha} h_V{}^{\gamma}{}_{\beta\rho} = h_H{}^{\sigma}{}_{\alpha\beta} h_{H\rho\sigma}{}^{\gamma} + h_V{}^{\sigma}{}_{\beta\alpha} h_V{}^{\gamma}{}_{\sigma\rho}$$

$$-\hat{D}_{V\beta} h_V{}^{\gamma}{}_{\alpha\rho} + \hat{D}_{V\alpha} h_V{}^{\gamma}{}_{\beta\rho} = 2\omega_{V\alpha\beta}{}^{\sigma} h_{H\rho\sigma}{}^{\gamma}$$

Notice that these relations contain only the projection curvatures and their projected derivatives.

3 Describing Embeddings

3.1 Intrinsic Description

3.1.1 Minkowski Bulk Coordinates as Functions on the Manifold

The intrinsic description of an embedded manifold uses the coordinate patches on the manifold as parameters. Denote the coordinates on the flat embedding space B by X^a where a ranges from 0 to N. The basis vectors on the tangent spaces to B are then

$$\vec{\partial}_a = \frac{\partial}{\partial X^a}$$

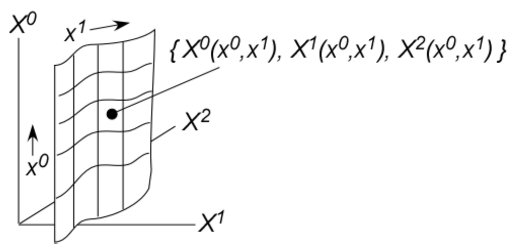
The metric on B is required to be diagonal with components equal to ± 1.

$$\vec{\partial}_a \cdot \vec{\partial}_b = \eta_{ab} = \begin{cases} 0 & \text{for} \quad a \neq b \\ s(a) & \text{for} \quad a = b \end{cases}$$

Denote coordinates on the embedded manifold M by x^α where α ranges from 0 to n. The embedding is then described by specifying the $N+1$ functions X^aas functions on the manifold M by giving the values

$$X^a\left(x^0, x^1, ..., x^n\right)$$

for coordinates x^a in each chart on M.



3.1.2 Manifold Metric and Projection Tensor from Minkowski Bulk Coordinates

In this description, one constructs the basis vectors tangent to the embedded manifold

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial X^a}{\partial x^\mu}\vec{\partial}_a$$

and the induced manifold metric components

$$g_{\mu\nu} = \partial_\mu \cdot \partial_\nu = \frac{\partial X^a}{\partial x^\mu}\frac{\partial X^b}{\partial x^\nu}\eta_{ab} = \frac{\partial \vec{X}}{\partial x^\mu} \cdot \frac{\partial \vec{X}}{\partial x^\nu}$$

The projection into the embedded surface tangent space can then be defined by

$$H\left(v\right) = \left(v \cdot \partial_\mu\right) g^{\mu\nu}\partial_\nu$$

Check this expression by assuming that v is tangent to the surface. It can then be expanded in terms of the tangent basis vectors

$$v = v^\alpha \partial_\alpha$$

and

$$\begin{aligned} H\left(v\right) &= \left(v^\alpha \partial_\alpha \cdot \partial_\mu\right) g^{\mu\nu}\partial_\nu = v^\alpha g_{\alpha\mu} g^{\mu\nu}\partial_\nu \\ &= v^\alpha \delta^\nu_\alpha \partial_\nu = v^\alpha \partial_\alpha = v. \end{aligned}$$

In terms of the embedding space basis, the projection tensor is defined by

$$\begin{aligned} H\left(v\right) &= \left(v \cdot \frac{\partial X^a}{\partial x^\mu}\vec{\partial}_a\right) g^{\mu\nu}\frac{\partial X^b}{\partial x^\nu}\vec{\partial}_b \\ &= g^{\mu\nu}\frac{\partial X^a}{\partial x^\mu}\frac{\partial X^b}{\partial x^\nu}\left(v \cdot \vec{\partial}_a\right)\vec{\partial}_b \\ &= g^{\mu\nu}\frac{\partial X^a}{\partial x^\mu}\frac{\partial X^b}{\partial x^\nu}\eta_{ar}\left(v \cdot dX^r\right)\vec{\partial}_b \end{aligned}$$

so that

$$H = g^{\mu\nu}\frac{\partial X^a}{\partial x^\mu}\frac{\partial X^b}{\partial x^\nu}\eta_{ar}dX^r \otimes \vec{\partial}_b$$

or, in components

$$H^b{}_r = g^{\mu\nu}\frac{\partial X^a}{\partial x^\mu}\frac{\partial X^b}{\partial x^\nu}\eta_{ar}$$

The component expressions are simplified by using commas to denote derivatives and raising and lowering indexes with the appropriate metrics.

$$H^b{}_r = X^{b,\nu}X_{r,\nu} = \vec{\nabla}X^b \cdot \vec{\nabla}X_r$$

3.1.3 Proection Curvature from Minkowski Bulk Coordinates

Covariant derivatives in the embedding space are just ordinary derivatives with respect to the coordinates X^a. Thus, we can evaluate the projection curvature directly

$$h_H\left(v\right)^c{}_a = h_H{}^c{}_{ba}v^b = H^c{}_r H^r{}_{a,d}H^d{}_b v^b$$

The derivative is projected tangent to the embedded manifold, so it is enough to evaluate the components

$$h_H\left(\partial_\delta\right)^c{}_a = H^c{}_r H^r{}_{a,\delta}$$

$$H^r{}_a = g^{\alpha\beta}\frac{\partial X^k}{\partial x^\alpha}\frac{\partial X^r}{\partial x^\beta}\eta_{ka} = \vec{\nabla}X^r \cdot \vec{\nabla}X_a$$

$$\begin{aligned} H^r{}_{a,\delta} &= \frac{\partial}{\partial x^\delta}\left(g^{\alpha\beta}\frac{\partial X^k}{\partial x^\alpha}\frac{\partial X^r}{\partial x^\beta}\eta_{ka}\right) \\ &= g^{\alpha\beta}{}_{,\delta}\frac{\partial X^k}{\partial x^\alpha}\frac{\partial X^r}{\partial x^\beta}\eta_{ka} + g^{\alpha\beta}\frac{\partial^2 X^k}{\partial x^\alpha \partial x^\delta}\frac{\partial X^r}{\partial x^\beta}\eta_{ka} + g^{\alpha\beta}\frac{\partial X^k}{\partial x^\alpha}\frac{\partial^2 X^r}{\partial x^\beta \partial x^\delta}\eta_{ka} \end{aligned}$$

note that

$$g^{\alpha\beta}{}_{,\delta} = -g^{\alpha\rho}g_{\rho\sigma,\delta}g^{\sigma\beta}$$

so that

$$\begin{aligned} g^{\alpha\beta}{}_{,\delta}\frac{\partial X^k}{\partial x^\alpha}\frac{\partial X^r}{\partial x^\beta}\eta_{ka} &= -g^{\alpha\rho}g_{\rho\sigma,\delta}g^{\sigma\beta}\frac{\partial X^k}{\partial x^\alpha}\frac{\partial X^r}{\partial x^\beta}\eta_{ka} \\ g_{\rho\sigma} &= \frac{\partial X^p}{\partial x^\rho}\frac{\partial X^s}{\partial x^\sigma}\eta_{ps} \\ g_{\rho\sigma,\delta} &= \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta}\frac{\partial X^s}{\partial x^\sigma}\eta_{ps} + \frac{\partial X^p}{\partial x^\rho}\frac{\partial^2 X^s}{\partial x^\sigma \partial x^\delta}\eta_{ps} \end{aligned}$$

Notice that the expression for $g_{\rho\sigma}$ uses the dummy index p instead of r because the expression that we will be substituting into is already using r as a dummy index. Now do the substitution:

$$
\begin{aligned}
& {g^{\alpha\beta}}_{,\delta} \frac{\partial X^k}{\partial x^\alpha} \frac{\partial X^r}{\partial x^\beta} \eta_{ka} \\
= \quad & -g^{\alpha\rho} g^{\sigma\beta} \left(\frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} \frac{\partial X^s}{\partial x^\sigma} \eta_{ps} + \frac{\partial X^p}{\partial x^\rho} \frac{\partial^2 X^s}{\partial x^\sigma \partial x^\delta} \eta_{ps} \right) \frac{\partial X^k}{\partial x^\alpha} \frac{\partial X^r}{\partial x^\beta} \eta_{ka} \\
= \quad & -g^{\alpha\rho} g^{\sigma\beta} \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} \frac{\partial X^s}{\partial x^\sigma} \eta_{ps} \frac{\partial X^k}{\partial x^\alpha} \frac{\partial X^r}{\partial x^\beta} \eta_{ka} \\
& -g^{\alpha\rho} g^{\sigma\beta} \frac{\partial X^p}{\partial x^\rho} \frac{\partial^2 X^s}{\partial x^\sigma \partial x^\delta} \eta_{ps} \frac{\partial X^k}{\partial x^\alpha} \frac{\partial X^r}{\partial x^\beta} \eta_{ka}
\end{aligned}
$$

It is useful to switch as much of this expression to index-free notation as possible.

$$
\begin{aligned}
& {g^{\alpha\beta}}_{,\delta} \frac{\partial X^k}{\partial x^\alpha} \frac{\partial X^r}{\partial x^\beta} \eta_{ka} \\
= \quad & -g^{\alpha\rho} \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} \left(\frac{\partial X^r}{\partial x^\beta} g^{\sigma\beta} \frac{\partial X^s}{\partial x^\sigma} \right) \eta_{ps} \frac{\partial X^k}{\partial x^\alpha} \eta_{ka} \\
& -g^{\sigma\beta} \left(\frac{\partial X^k}{\partial x^\alpha} g^{\alpha\rho} \frac{\partial X^p}{\partial x^\rho} \right) \eta_{ps} \frac{\partial^2 X^s}{\partial x^\sigma \partial x^\delta} \frac{\partial X^r}{\partial x^\beta} \eta_{ka} \\
= \quad & -g^{\alpha\rho} \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} \left(\vec{\nabla} X^r \cdot \vec{\nabla} X^s \right) \eta_{ps} \frac{\partial X^k}{\partial x^\alpha} \eta_{ka} \\
& -g^{\sigma\beta} \left(\vec{\nabla} X^k \cdot \vec{\nabla} X^p \right) \eta_{ps} \frac{\partial^2 X^s}{\partial x^\sigma \partial x^\delta} \frac{\partial X^r}{\partial x^\beta} \eta_{ka}
\end{aligned}
$$

Use the bulk metric tensor η to raise and lower bulk indexes to eliminate still more index clutter.

$$
\begin{aligned}
& {g^{\alpha\beta}}_{,\delta} \frac{\partial X^k}{\partial x^\alpha} \frac{\partial X^r}{\partial x^\beta} \eta_{ka} \\
= \quad & -g^{\alpha\rho} \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} \left(\vec{\nabla} X^r \cdot \vec{\nabla} X_p \right) \frac{\partial X_a}{\partial x^\alpha} \\
& -g^{\sigma\beta} \left(\vec{\nabla} X_s \cdot \vec{\nabla} X_a \right) \frac{\partial^2 X^s}{\partial x^\sigma \partial x^\delta} \frac{\partial X^r}{\partial x^\beta}
\end{aligned}
$$

Rename some dummy indexes to get these terms to look alike.

$$
\begin{aligned}
& {g^{\alpha\beta}}_{,\delta} \frac{\partial X^k}{\partial x^\alpha} \frac{\partial X^r}{\partial x^\beta} \eta_{ka} \\
= \quad & -g^{\alpha\rho} \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} \left(\vec{\nabla} X^r \cdot \vec{\nabla} X_p \right) \frac{\partial X_a}{\partial x^\alpha} \\
& -g^{\rho\alpha} \left(\vec{\nabla} X_p \cdot \vec{\nabla} X_a \right) \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} \frac{\partial X^r}{\partial x^\alpha} \\
= \quad & -g^{\alpha\rho} \frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta} \left[\left(\vec{\nabla} X^r \cdot \vec{\nabla} X_p \right) \frac{\partial X_a}{\partial x^\alpha} + \left(\vec{\nabla} X_p \cdot \vec{\nabla} X_a \right) \frac{\partial X^r}{\partial x^\alpha} \right]
\end{aligned}
$$

Next, put this first term into the expression for the projection gradient

$$\begin{aligned}
H^r{}_{a,\delta} &= -g^{\alpha\rho}\frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta}\left[\left(\vec{\nabla}X^r\cdot\vec{\nabla}X_p\right)\frac{\partial X_a}{\partial x^\alpha}+\left(\vec{\nabla}X_p\cdot\vec{\nabla}X_a\right)\frac{\partial X^r}{\partial x^\alpha}\right] \\
&\quad +g^{\alpha\beta}\frac{\partial^2 X^k}{\partial x^\alpha \partial x^\delta}\frac{\partial X^r}{\partial x^\beta}\eta_{ka}+g^{\alpha\beta}\frac{\partial X^k}{\partial x^\alpha}\frac{\partial^2 X^r}{\partial x^\beta \partial x^\delta}\eta_{ka}
\end{aligned}$$

Rename dummy indexes to get the second derivative terms to look alike.

$$\begin{aligned}
H^r{}_{a,\delta} &= -g^{\alpha\rho}\frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta}\left[\left(\vec{\nabla}X^r\cdot\vec{\nabla}X_p\right)\frac{\partial X_a}{\partial x^\alpha}-\left(\vec{\nabla}X_p\cdot\vec{\nabla}X_a\right)\frac{\partial X^r}{\partial x^\alpha}\right] \\
&\quad +g^{\rho\alpha}\frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta}\frac{\partial X^r}{\partial x^\alpha}\eta_{pa}+g^{\alpha\rho}\frac{\partial^2 X^r}{\partial x^\rho \partial x^\delta}\frac{\partial X^k}{\partial x^\alpha}\eta_{ka}
\end{aligned}$$

$$\begin{aligned}
H^r{}_{a,\delta} &= g^{\alpha\rho}\frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta}\left[-\left(\vec{\nabla}X_p\cdot\vec{\nabla}X_a\right)\frac{\partial X^r}{\partial x^\alpha}-\left(\vec{\nabla}X^r\cdot\vec{\nabla}X_p\right)\frac{\partial X_a}{\partial x^\alpha}+\frac{\partial X^r}{\partial x^\alpha}\eta_{pa}+\delta^r_p\frac{\partial X^k}{\partial x^\alpha}\eta_{ka}\right] \\
&= g^{\alpha\rho}\frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta}\frac{\partial X^k}{\partial x^\alpha}\left(-H_{pa}\delta^r_k-H^r{}_p\eta_{ka}+\delta^r_k\eta_{pa}+\delta^r_p\eta_{ka}\right) \\
&= g^{\alpha\rho}\frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta}\frac{\partial X^k}{\partial x^\alpha}\left(V_{pa}\delta^r_k+V^r{}_p\eta_{ka}\right)
\end{aligned}$$

Now project this result with H to get the final result:

$$\begin{aligned}
h_H\left(\partial_\delta\right)^c{}_a &= H^c{}_r H^r{}_{a,\delta} \\
&= H^c{}_r g^{\alpha\rho}\frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta}\frac{\partial X^k}{\partial x^\alpha}\left(V_{pa}\delta^r_k+V^r{}_p\eta_{ka}\right) \\
&= g^{\alpha\rho}\frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta}\frac{\partial X^k}{\partial x^\alpha}\left(V_{pa}\delta^r_k H^c{}_r+H^c{}_r V^r{}_p\eta_{ka}\right) \\
&= g^{\alpha\rho}\frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta}\frac{\partial X^k}{\partial x^\alpha}V_{pa}H^c{}_k
\end{aligned}$$

The full expression for the projection curvature is then

$$h_H\left(\partial_\delta\right)^c{}_a = g^{\alpha\rho}\frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta}\frac{\partial X^k}{\partial x^\alpha}V_{pa}H^c{}_k$$

where the components $g^{\alpha\rho}$ are defined by

$$g^{\alpha\rho}\left(\frac{\partial\vec{X}}{\partial x^\rho}\cdot\frac{\partial\vec{X}}{\partial x^\beta}\right)=\delta^\alpha_\beta$$

and

$$\begin{aligned}
V_{pa} &= \eta_{pa}-\vec{\nabla}X_p\cdot\vec{\nabla}X_a \\
H^c{}_k &= \vec{\nabla}X^c\cdot\vec{\nabla}X_k
\end{aligned}$$

One last task remains. The manifold tangent basis vectors ∂_δ need to be replaced by bulk tangent vectors in HT_P

$$H\vec{\partial}_d = M_d^\delta \partial_\delta$$

Find the coefficients M_d^δ by using the chain rule

$$\partial_\sigma = \frac{\partial X^c}{\partial x^\sigma}\vec{\partial}_c$$

to obtain

$$\frac{\partial X^j}{\partial x^\sigma}\vec{\partial}_j \cdot H\vec{\partial}_d = M_d^\delta \partial_\delta \cdot \partial_\sigma$$

or

$$\frac{\partial X^j}{\partial x^\sigma}H_{jd} = M_d^\delta g_{\delta\sigma}$$

so that

$$M_d^\delta = g^{\sigma\delta}\frac{\partial X^j}{\partial x^\sigma}H_{jd}$$

and thus

$$\begin{aligned} H\vec{\partial}_d &= \frac{\partial X^j}{\partial x^\sigma}H_{jd}g^{\sigma\delta}\partial_\delta \\ &= \left(\vec{\nabla}X_j \cdot \vec{\nabla}X_d\right)\frac{\partial X^j}{\partial x^\sigma}g^{\sigma\delta}\partial_\delta \end{aligned}$$

The full expansion of the projection curvature tensor is then

$$h_H{}^c{}_{da} = \left(\vec{\nabla}X_j \cdot \vec{\nabla}X_d\right)\frac{\partial X^j}{\partial x^\sigma}g^{\sigma\delta}h_H\left(\partial_\delta\right)^c{}_a$$

or

$$h_H{}^c{}_{da} = H_{jd}\frac{\partial X^j}{\partial x^\sigma}g^{\sigma\delta}g^{\alpha\rho}\frac{\partial^2 X^p}{\partial x^\rho \partial x^\delta}\frac{\partial X^k}{\partial x^\alpha}V_{pa}H^c{}_k$$

where we recall the definitions

$$g^{\alpha\rho}\left(\frac{\partial\vec{X}}{\partial x^\rho}\cdot\frac{\partial\vec{X}}{\partial x^\beta}\right) = \delta_\beta^\alpha$$

and

$$\begin{aligned} V_{pa} &= \eta_{pa} - \vec{\nabla}X_p \cdot \vec{\nabla}X_a \\ H^c{}_k &= \vec{\nabla}X^c \cdot \vec{\nabla}X_k \end{aligned}$$

Notice that the other projection curvature, h_V is not defined when H projects onto a single embedded manifold. The definition of h_V requires differentiation in directions perpendicular to the manifold and that is only possible when the projection tensor fields are defined away from the manifold. In other words, a local foliation of the embedding space by manifolds is required for h_V to be defined.

3.2 Extrinsic Description

The extrinsic description of a surface provides $N-n$ functions $y^A : B \to \mathbb{R}$ and defines the surface by

$$y^A = 0$$

for $A = 1, 2, ..., N-n$. The forms dy^A and the corresponding gradient vectors

$$\vec{\nabla} y^A = g^{-1}\left(dy^A\right)$$

can then be used to construct the projection perpendicular to the surface:

$$V = g_{AB} \vec{\nabla} y^A \otimes dy^B$$

where g_{AB} is defined by the condition

$$g_{AB}\left(dy^B \cdot dy^C\right) = \delta^C_A$$

Check that this is a projection tensor.

$$\begin{aligned}
V\left(V\left(v\right)\right) &= V\left(g_{AB}\vec{\nabla} y^A\left(dy^B \cdot v\right)\right) = g_{AB} V\left(\vec{\nabla} y^A\right)\left(dy^B \cdot v\right) \\
&= g_{AB} g_{CD} \vec{\nabla} y^C \left(dy^D \cdot \vec{\nabla} y^A\right)\left(dy^B \cdot v\right) \\
&= g_{AB} g_{CD} \vec{\nabla} y^C \left(dy^D \cdot g^{-1}\left(dy^A\right)\right)\left(dy^B \cdot v\right) \\
&= g_{AB} g_{CD} \vec{\nabla} y^C g^{-1}\left(dy^D, dy^A\right)\left(dy^B \cdot v\right) \\
&= g_{AB} g_{CD} \vec{\nabla} y^C \left(dy^D \cdot dy^A\right)\left(dy^B \cdot v\right) \\
&= g_{AB} \delta^A_C \vec{\nabla} y^C \left(dy^B \cdot v\right) \\
&= g_{CB} \vec{\nabla} y^C \left(dy^B \cdot v\right) \\
&= V\left(v\right)
\end{aligned}$$

so that

$$V^2 = V.$$

Check that it annihilates any vector tangent to the surface. For such a vector, v

$$v\left(y^A\right) = v \cdot dy^A = 0$$

for all A. Thus

$$V\left(v\right) = \left(g_{AB}\vec{\nabla} y^A \otimes dy^B\right) \cdot v = g_{AB}\vec{\nabla} y^A\left(dy^B \cdot v\right) = 0.$$

The complementary projection

$$H = 1 - V = 1 - g_{AB}\vec{\nabla} y^A \otimes dy^B$$

then projects into the surface.

3.2.1 In Terms of Bulk Coordinates

Each of the functions y^A is a function of the $N+1$ bulk coordinates X^a so that

$$\vec{\nabla} y^A = g^{-1}\left(dy^A\right) = g^{-1}\left(\frac{\partial y^A}{\partial X^r} dX^r\right)$$

or

$$\vec{\nabla} y^A = \frac{\partial y^A}{\partial X^r}\eta^{rs}\vec{\partial}_s$$

and the definition of g_{AB} becomes just the inverse

$$g_{AB}g^{BC} = \delta^C_A$$

where

$$g^{BC} = \frac{\partial y^B}{\partial X^r}\frac{\partial y^C}{\partial X^s}\eta^{rs}$$

and the components of the projection tensor V are

$$V^a{}_b = \eta^{ak} g_{AB}\frac{\partial y^A}{\partial X^k}\frac{\partial y^B}{\partial X^b}$$

3.2.2 Arbitrariness

Although there is no need for arbitrary coordinate patches on the surface, this description does harbor its own kind of arbitrariness. The functions y^A can be replaced by combinations $\sum_B f^A_B y^B$ without changing the surface that they are describing. The array of coefficients f^A_B just needs to have a non-zero determinant everywhere.

3.2.3 Projection Curvature in Extrinsic Form

The curvature of the surface has components

$$h_H{}^c{}_{ba} = H^c{}_r H^r{}_{a,d} H^d{}_b$$

$$\begin{aligned} H^r{}_{a,d} &= -\frac{\partial}{\partial X^d}\eta^{rk} g_{AB}\frac{\partial y^A}{\partial X^k}\frac{\partial y^B}{\partial X^a} \\ &= \eta^{rk} g_{AR} g^{RS}{}_{,d} g_{SB}\frac{\partial y^A}{\partial X^k}\frac{\partial y^B}{\partial X^a} - \eta^{rk} g_{AB}\frac{\partial^2 y^A}{\partial X^k \partial X^d}\frac{\partial y^B}{\partial X^a} - \eta^{rk} g_{AB}\frac{\partial y^A}{\partial X^k}\frac{\partial^2 y^B}{\partial X^a \partial X^d} \end{aligned}$$

where

$$\begin{aligned} g^{RS}{}_{,d} &= \frac{\partial}{\partial X^d}\frac{\partial y^R}{\partial X^j}\frac{\partial y^S}{\partial X^s}\eta^{js} \\ &= \frac{\partial^2 y^R}{\partial X^j \partial X^d}\frac{\partial y^S}{\partial X^s}\eta^{js} + \frac{\partial y^R}{\partial X^j}\frac{\partial^2 y^S}{\partial X^s \partial X^d}\eta^{js} \end{aligned}$$

so the projection gradient is

$$
\begin{aligned}
H^r{}_{a,d} &= \eta^{rk} g_{AR} g_{SB} \frac{\partial y^A}{\partial X^k} \frac{\partial y^B}{\partial X^a} \frac{\partial^2 y^R}{\partial X^j \partial X^d} \frac{\partial y^S}{\partial X^s} \eta^{js} + \eta^{rk} g_{AR} g_{SB} \frac{\partial y^A}{\partial X^k} \frac{\partial y^B}{\partial X^a} \frac{\partial y^R}{\partial X^j} \frac{\partial^2 y^S}{\partial X^s \partial X^d} \eta^{js} \\
& \quad - \eta^{rk} g_{AB} \frac{\partial^2 y^A}{\partial X^k \partial X^d} \frac{\partial y^B}{\partial X^a} - \eta^{rk} g_{AB} \frac{\partial y^A}{\partial X^k} \frac{\partial^2 y^B}{\partial X^a \partial X^d}
\end{aligned}
$$

Now notice that many terms of the expression $H^c{}_r H^r{}_{a,d}$ involve the combination

$$
H^c{}_r \eta^{rk} \frac{\partial y^A}{\partial X^k}
$$

which is zero. The surviving term is just

$$
H^c{}_r H^r{}_{a,d} = -H^c{}_r \eta^{rk} g_{AB} \frac{\partial^2 y^A}{\partial X^k \partial X^d} \frac{\partial y^B}{\partial X^a}
$$

and the projection curvature components are

$$
h_H{}^c{}_{ba} = -H^c{}_r \eta^{rk} g_{AB} \frac{\partial^2 y^A}{\partial X^k \partial X^d} \frac{\partial y^B}{\partial X^a} H^d{}_b
$$

where

$$
H^a{}_b = \delta^a_b - \eta^{ak} g_{AB} \frac{\partial y^A}{\partial X^k} \frac{\partial y^B}{\partial X^b}
$$

and g_{AB} is the matrix inverse of

$$
g^{BC} = \frac{\partial y^B}{\partial X^r} \frac{\partial y^C}{\partial X^s} \eta^{rs}
$$

4 Dynamical Actions

4.1 Action Principles

The most efficient way to obtain consistent (and thus solvable) equations of motion for a system is to define an action functional I that assigns a number $I(\mathcal{K})$ to each possible history $\mathcal{K}$ of the system. For a given one-parameter family of histories $\mathcal{K}(\varepsilon)$ one can define the variational derivative of any function f that is associated with these histories by

$$
\delta f = \left. \frac{d}{d\varepsilon} f(\mathcal{K}(\varepsilon)) \right|_{\varepsilon=0}
$$

The simplest example would be a free particle in one dimension with position $x(t)$ at time t. The action functional is just

$$
I = \int \left(\frac{1}{2} m \dot{x}^2 \right) dt
$$

so that the variational derivative is

$$\delta I = \int (m\dot{x}\delta\dot{x})\, dt.$$

The variational principle would require

$$\delta I = 0$$

for arbitrary distance variations δx that vanish outside of a finite time interval. One then used integration by parts to obtain

$$\int \frac{d\,(m\dot{x})}{dt} \delta x dt. = 0$$

or, since δx is arbitrary,

$$\frac{d\,(m\dot{x})}{dt} = 0.$$

Notice, however, that we had to specify that it is δx that is arbitrary. We could just as easily have specified that $\delta\dot{x}$ is the quantity that is supposed to be arbitrary. In that case, the variational principle would have led to

$$m\dot{x} = 0.$$

Thus, one can have different theories from the same action functional by specifying different things to be arbitrary.

4.2 String Theory

The action functional of a Goto-Nambu string has the form

$$\begin{aligned}
I_{\text{G-N}} &= \int d^2x \sqrt{|g_{HH}|} \\
&= \int d^2x \sqrt{\left| \begin{array}{cc} \frac{\partial\vec{X}}{\partial x^0} \cdot \frac{\partial\vec{X}}{\partial x^0} & \frac{\partial\vec{X}}{\partial x^0} \cdot \frac{\partial\vec{X}}{\partial x^1} \\ \frac{\partial\vec{X}}{\partial x^1} \cdot \frac{\partial\vec{X}}{\partial x^0} & \frac{\partial\vec{X}}{\partial x^1} \cdot \frac{\partial\vec{X}}{\partial x^1} \end{array} \right|} \\
&= \int\int dx^0 dx^1 \left\{ \left(\frac{\partial\vec{X}}{\partial x^0} \cdot \frac{\partial\vec{X}}{\partial x^0} \right) \left(\frac{\partial\vec{X}}{\partial x^1} \cdot \frac{\partial\vec{X}}{\partial x^1} \right) - \left(\frac{\partial\vec{X}}{\partial x^1} \cdot \frac{\partial\vec{X}}{\partial x^0} \right) \left(\frac{\partial\vec{X}}{\partial x^0} \cdot \frac{\partial\vec{X}}{\partial x^1} \right) \right\}^{1/2}
\end{aligned}$$

Now evaluate a variation with respect to an arbitrary deformation of the string metric $g_{HH\alpha\beta}$.

$$\delta\sqrt{|g_{HH}|} = \frac{1}{2}\sqrt{|g_{HH}|} g^{HH\alpha\beta} \delta g_{HH\alpha\beta}$$

In this case, the requirement

$$\delta I_{\text{G-N}} = 0$$

leads to conditions

$$\sqrt{|g_{HH}|}g^{HH\alpha\beta} = 0$$

that can only be met by a null or "lightlike" string.

Next, evaluate a variation with respect to an arbitrary deformation (over a compact region) of the embedding functions $\vec{X}$.

$$\begin{aligned}
\delta\sqrt{|g_{HH}|} &= \frac{1}{2}\sqrt{|g_{HH}|}g^{HH\alpha\beta}\delta\left(\frac{\partial\vec{X}}{\partial x^\alpha}\cdot\frac{\partial\vec{X}}{\partial x^\beta}\right) \\
&= \sqrt{|g_{HH}|}g^{HH\alpha\beta}\left(\frac{\partial\vec{X}}{\partial x^\alpha}\cdot\frac{\partial\delta\vec{X}}{\partial x^\beta}\right) \\
&= \frac{\partial}{\partial x^\beta}\left(\left(\sqrt{|g_{HH}|}g^{HH\alpha\beta}\frac{\partial\vec{X}}{\partial x^\alpha}\right)\cdot\delta\vec{X}\right) - \frac{\partial}{\partial x^\beta}\left(\sqrt{|g_{HH}|}g^{HH\alpha\beta}\frac{\partial\vec{X}}{\partial x^\alpha}\right)\cdot\delta\vec{X}
\end{aligned}$$

The variation of the action is then

$$\delta I_{\text{G-N}} = -\int d^2x\left\{\frac{\partial}{\partial x^\beta}\left(\sqrt{|g_{HH}|}g^{HH\alpha\beta}\frac{\partial\vec{X}}{\partial x^\alpha}\right)\cdot\delta\vec{X}\right\}$$

where the total divergence just leads to a suface integral that is zero if $\delta\vec{X}$ vanishes outside of a compact region. The equations of motion of the string are then

$$\frac{\partial}{\partial x^\beta}\left(\sqrt{|g_{HH}|}g^{HH\alpha\beta}\frac{\partial\vec{X}}{\partial x^\alpha}\right) = 0$$

and the string need no longer be lightlike.

4.3 Stringlike Actions for General Relativity

Einstein's field equations can be obtained from the Hilbert Action functional

$$I_{\text{Hilbert}} = \int d^4x\sqrt{|g_{HH}|}\left({}^H R\right)$$

in a surprising variety of ways. Here, we are thinking of spacetime as embedded in a larger manifold with H to projection onto the spacetime surface. To complete the action principal, one must say what quantities will be varied arbitrarily. Letting $\delta g_{HH\alpha\beta}$ be arbitrary yields the Einstein Vacuum field equations in the form

$${}^H G^{\alpha\beta} = 0.$$

One can decide that the connection coefficients $\Gamma^\alpha{}_{\beta\delta}$ will also be arbitrary and get this same result, together with the expression for the metric-compatible connection coefficients in terms of derivatives of the metric tensor. That version is called the Pallatini Action Principle.

Now suppose we use the Gauss equation to write the scalar curvature in terms of the projection curvature. The action functional then becomes

$$I_{\text{Hilbert}} = \int d^4x \sqrt{|g_{HH}|} \left(\theta_{Hs} \theta_H{}^s - h_H{}^{kr}{}_s h_{Hrk}{}^s \right)$$

and we can express it entirely in terms of derivatives of the embedding functions X^a. Now do the variation with respect to these embedding functions. Work this through and you will find that the resulting field equations are just

$$\frac{\partial^2 \vec{X}}{\partial x^\alpha \partial x^\beta} \left({}^H G^{\alpha\beta} \right) + \frac{\partial \vec{X}}{\partial x^\alpha} \left({}^H G^{\alpha\beta}{}_{;\beta} \right) = 0$$

or, using the contracted Bianchi Identities,

$$\frac{\partial^2 \vec{X}}{\partial x^\alpha \partial x^\beta} \left({}^H G^{\alpha\beta} \right) = 0.$$

Thus, Einstein spacetimes with

$${}^H G^{\alpha\beta} = 0$$

are certainly solutions but there can be other solutions as well.