Hour Exam No.2

Please attempt all of the following problems before the due date. All problems count the same even though some are more complex than others.

Several of the following problems require the connection coefficients for the connection compatible with the Schwarzschild metric:

$$\begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} -\left(1-\frac{2m}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2m}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

For the coordinates $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \varphi$ and the corresponding holonomic basis, the non-zero connection coefficients are as follows:

$$\begin{aligned}
\Gamma^0{}_{01} &= \Gamma^0{}_{10} = \frac{m}{r\,(r-2m)} \\
\Gamma^1{}_{00} &= \frac{m}{r^3}\,(r-2m) \\
\Gamma^1{}_{11} &= -\frac{m}{r\,(r-2m)} \\
\Gamma^1{}_{22} &= -\,(r-2m) \\
\Gamma^1{}_{33} &= -\,(r-2m)\sin^2\theta \\
\Gamma^2{}_{12} &= \Gamma^2{}_{21} = \frac{1}{r} \\
\Gamma^2{}_{33} &= -\sin\theta\cos\theta \\
\Gamma^3{}_{13} &= \Gamma^3{}_{31} = \frac{1}{r} \\
\Gamma^3{}_{23} &= \Gamma^3{}_{32} = \cot\theta
\end{aligned}$$

Problem 1

For the connection given above, show that, for any vector fields u, v the following conditions are satisfied:

a. For any function f,

$$D_u D_v f = D_v D_u f$$

Hint: Switch to index notation and see the result immediately.

Answer 1a

Note that $D_u D_v f - D_v D_u f$ is locally linear in u, v. Expand u, v in a holonomic basis ∂_α and find

$$D_u D_v f - D_v D_u f = u^\alpha v^\beta \left(D_{\partial_\alpha} D_{\partial_\beta} f - D_{\partial_\beta} D_{\partial_\alpha} f \right)$$

Now write out the second covariant derivative components in this basis

$$D_{\partial_\alpha} D_{\partial_\beta} f = f_{,\beta\alpha} - f_{,\rho} \Gamma^\rho{}_{\beta\alpha}$$

so that

$$\begin{aligned} D_u D_v f - D_v D_u f &= u^\alpha v^\beta \left(f_{,\beta\alpha} - f_{,\rho} \Gamma^\rho{}_{\beta\alpha} - f_{,\alpha\beta} + f_{,\rho} \Gamma^\rho{}_{\alpha\beta} \right) \\ &= u^\alpha v^\beta \left(f_{,\beta\alpha} - f_{,\alpha\beta} - f_{,\rho} \Gamma^\rho{}_{\beta\alpha} + f_{,\rho} \Gamma^\rho{}_{\alpha\beta} \right) \\ &= \\ &= u^\alpha v^\beta f_{,\rho} \left(-\Gamma^\rho{}_{\beta\alpha} + \Gamma^\rho{}_{\alpha\beta} \right) \end{aligned}$$

Notice that the connection coefficients given above have the symmetry

$$\boxed{\Gamma^\rho{}_{\alpha\beta} = \Gamma^\rho{}_{\beta\alpha}}$$

so that

$$\boxed{D_u D_v f - D_v D_u f = 0.}$$

b. For any vector field w

$$D_w (u \cdot v) = (D_w u) \cdot v + u \cdot D_w v$$

Hint: Switch to index notation and slog it out component by component using Maple to do the algebra.

Answer 1b

This condition is just metric compatibility which, in index notation is

$$g_{\alpha\beta;\delta} = 0$$

or

$$g_{\alpha\beta,\delta} - g_{\alpha\rho}\Gamma^{\rho}{}_{\beta\delta} - g_{\rho\beta}\Gamma^{\rho}{}_{\alpha\delta} = 0$$

Check it component by component, noticing that the expression is symmetric in α and β:

$\alpha = 0$

$g_{0\beta,\delta} - g_{0\rho}\Gamma^{\rho}{}_{\beta\delta} - g_{\rho\beta}\Gamma^{\rho}{}_{0\delta} = 0$

$g_{0\beta,\delta} - g_{00}\Gamma^{0}{}_{\beta\delta} - g_{\rho\beta}\Gamma^{\rho}{}_{0\delta} = 0$

$\beta = 0$

$g_{00,\delta} - g_{00}\Gamma^{0}{}_{0\delta} - g_{\rho 0}\Gamma^{\rho}{}_{0\delta} = 0$

$g_{00,\delta} - g_{00}\Gamma^{0}{}_{0\delta} - g_{00}\Gamma^{0}{}_{0\delta} = 0$

$g_{00,\delta} - 2g_{00}\Gamma^{0}{}_{0\delta} = 0$

The only non-zero $\Gamma^{0}{}_{0\delta}$ is for $\delta = 1$ and similarly the only non-zero $g_{00,\delta}$ is for $\delta = 1$. Thus, we only need to check that case. Use Maple Expand and the Simplify to do the algebra:

$g_{00,1} - 2g_{00}\Gamma^{0}{}_{01} = \frac{d}{dr}\left(-\left(1 - \frac{2m}{r}\right)\right) - 2\left(-\left(1 - \frac{2m}{r}\right)\right)\left(\frac{m}{r(r-2m)}\right) = -2\frac{m}{r^2} - 2\left(-1 + 2\frac{m}{r}\right)\frac{m}{r(r-2m)}$

$g_{00,1} - 2g_{00}\Gamma^{0}{}_{01} = -2\frac{m}{r^2} - 2\left(-1 + 2\frac{m}{r}\right)\frac{m}{r(r-2m)} = 0$

$\beta = 1$

$g_{01,\delta} - g_{00}\Gamma^{0}{}_{1\delta} - g_{\rho 1}\Gamma^{\rho}{}_{0\delta} = 0$

$-g_{00}\Gamma^{0}{}_{1\delta} - g_{11}\Gamma^{1}{}_{0\delta} = 0$

The only non-zero $\Gamma^{0}{}_{1\delta}$ and also the only non-zero $\Gamma^{1}{}_{0\delta}$ is for $\delta = 0$ so only that case needs to be checked.

$-g_{00}\Gamma^{0}{}_{10} - g_{11}\Gamma^{1}{}_{00} = -\left(-\left(1 - \frac{2m}{r}\right)\right)\left(\frac{m}{r(r-2m)}\right) - \frac{1}{1-\frac{2m}{r}}\frac{m}{r^3}(r - 2m) := 0$

$\beta = 2$

$g_{02,\delta} - g_{00}\Gamma^{0}{}_{2\delta} - g_{\rho 2}\Gamma^{\rho}{}_{0\delta} = 0$

$-g_{00}\Gamma^{0}{}_{2\delta} - g_{22}\Gamma^{2}{}_{0\delta} = 0$

There are no non-zero coefficients with both 2 and 0 indexes, so this is indeed zero.

$\beta = 3$

$g_{03,\delta} - g_{00}\Gamma^{0}{}_{3\delta} - g_{\rho 3}\Gamma^{\rho}{}_{0\delta} = 0$

$-g_{00}\Gamma^{0}{}_{3\delta} - g_{22}\Gamma^{3}{}_{0\delta} = 0$

There are no non-zero coefficients with both 3 and 0 indexes, so these components are also zero.

$\alpha = 1$

$g_{1\beta,\delta} - g_{1\rho}\Gamma^{\rho}{}_{\beta\delta} - g_{\rho\beta}\Gamma^{\rho}{}_{1\delta} = 0$

$g_{1\beta,\delta} - g_{11}\Gamma^{1}{}_{\beta\delta} - g_{\rho\beta}\Gamma^{\rho}{}_{1\delta} = 0$

We already checked $(\alpha, \beta) = (0, 1)$ so $(\alpha, \beta) = (1, 0)$ will also work. No need to check $\beta = 0$.

$\beta = 1$

$g_{11,\delta} - g_{11}\Gamma^{1}{}_{1\delta} - g_{\rho 1}\Gamma^{\rho}{}_{1\delta} = 0$

$g_{11,\delta} - g_{11}\Gamma^{1}{}_{1\delta} - g_{11}\Gamma^{1}{}_{1\delta} = 0$

$g_{11,\delta} - 2g_{11}\Gamma^1{}_{1\delta} = 0$
Both terms are zero except for $\delta = 1$ so check that case.
$g_{11,1} - 2g_{11}\Gamma^1{}_{11} = \frac{d}{dr}\left(\frac{1}{1-\frac{2m}{r}}\right) - 2\left(\frac{1}{1-\frac{2m}{r}}\right)\left(-\frac{m}{r(r-2m)}\right) = -2\frac{m}{(-r+2m)^2} + \frac{2}{1-2\frac{m}{r}}\frac{m}{r(r-2m)} = 0.$
$\beta = 2$
$g_{12,\delta} - g_{11}\Gamma^1{}_{2\delta} - g_{\rho 2}\Gamma^\rho{}_{1\delta} = 0$
$-g_{11}\Gamma^1{}_{2\delta} - g_{22}\Gamma^2{}_{1\delta} = 0$
The coefficients are zero except for $\delta = 2$ so check that case.
$-g_{11}\Gamma^1{}_{22} - g_{22}\Gamma^2{}_{12} = -\left(\frac{1}{1-\frac{2m}{r}}\right)(-(r-2m)) - r^2\left(\frac{1}{r}\right) = 0.$
$\beta = 3$
$g_{13,\delta} - g_{11}\Gamma^1{}_{3\delta} - g_{\rho 3}\Gamma^\rho{}_{1\delta} = 0$
$-g_{11}\Gamma^1{}_{3\delta} - g_{33}\Gamma^3{}_{1\delta} = 0$
The coefficients are zero except for $\delta = 3$ so check that case.
$-g_{11}\Gamma^1{}_{33} - g_{33}\Gamma^3{}_{13} = -\left(\frac{1}{1-\frac{2m}{r}}\right)\left(-(r-2m)\sin^2\theta\right) - \left(r^2\sin^2\theta\right)\frac{1}{r} = 0.$
$\alpha = 2$
$g_{2\beta,\delta} - g_{2\rho}\Gamma^\rho{}_{\beta\delta} - g_{\rho\beta}\Gamma^\rho{}_{2\delta} = 0$
$g_{2\beta,\delta} - g_{22}\Gamma^2{}_{\beta\delta} - g_{\rho\beta}\Gamma^\rho{}_{2\delta} = 0$
No need to check $\beta = 0$ or $\beta = 1$ because of symmetry.
$\beta = 2$
$g_{22,\delta} - g_{22}\Gamma^2{}_{2\delta} - g_{\rho 2}\Gamma^\rho{}_{2\delta} = 0$
$g_{22,\delta} - g_{22}\Gamma^2{}_{2\delta} - g_{22}\Gamma^2{}_{2\delta} = 0$
$g_{22,\delta} - 2g_{22}\Gamma^2{}_{2\delta} = 0$
Both terms are zero except for $\delta = 1$ so check that case.
$g_{22,1} - 2g_{22}\Gamma^2{}_{21} = \frac{d}{dr}\left(r^2\right) - 2r^2\frac{1}{r} = 0.$
$\beta = 3$
$g_{23,\delta} - g_{22}\Gamma^2{}_{3\delta} - g_{\rho 3}\Gamma^\rho{}_{2\delta} = 0$
$-g_{22}\Gamma^2{}_{3\delta} - g_{33}\Gamma^3{}_{2\delta} = 0$
Both terms are zero except for $\delta = 3$ so check that case.
$-g_{22}\Gamma^2{}_{33} - g_{33}\Gamma^3{}_{23} = -r^2(-\sin\theta\cos\theta) - \left(r^2\sin^2\theta\right)\cot\theta = 0.$
$\alpha = 3$
$g_{3\beta,\delta} - g_{3\rho}\Gamma^\rho{}_{\beta\delta} - g_{\rho\beta}\Gamma^\rho{}_{3\delta} = 0$
$g_{3\beta,\delta} - g_{33}\Gamma^3{}_{\beta\delta} - g_{\rho\beta}\Gamma^\rho{}_{3\delta} = 0$
No need to check $\beta = 0$ or $\beta = 1$ or $\beta = 2$ because of symmetry.
$\beta = 3$
$g_{33,\delta} - g_{33}\Gamma^3{}_{3\delta} - g_{\rho 3}\Gamma^\rho{}_{3\delta} = 0$
$g_{33,\delta} - g_{33}\Gamma^3{}_{3\delta} - g_{33}\Gamma^3{}_{3\delta} = 0$
$g_{33,\delta} - 2g_{33}\Gamma^3{}_{3\delta} = 0$
Both terms are zero except for $\delta = 1$ so check that case.
$g_{33,1} - 2g_{33}\Gamma^3{}_{31} = \frac{d}{dr}\left(r^2\sin^2\theta\right) - 2\left(r^2\sin^2\theta\right)\frac{1}{r} = 0.$
All components vanish and the expression is verified.

Problem 2

The wave operator on a scalar field can be written in the form

$$\nabla^2\Phi = g^{\alpha\beta}\Phi_{;\alpha\beta}$$

Use the connection coefficients given at the beginning of this exam to find the wave equation near a black hole of mass m.

Answer 2

$\Phi_{;\alpha\beta} = \Phi_{,\alpha\beta} - \Phi_{,\rho}\Gamma^{\rho}{}_{\alpha\beta}$
$\nabla^2\Phi = g^{\alpha\beta}\Phi_{;\alpha\beta} = g^{\alpha\beta}\left(\Phi_{,\alpha\beta} - \Phi_{,\rho}\Gamma^{\rho}{}_{\alpha\beta}\right)$
$\nabla^2\Phi = g^{\alpha\beta}\Phi_{,\alpha\beta} - \Phi_{,\rho}g^{\alpha\beta}\Gamma^{\rho}{}_{\alpha\beta}$
Now work out what $g^{\alpha\beta}\Gamma^{\rho}{}_{\alpha\beta}$ is, component by component.
Note that the inverse metric tensor components are:
$g^{00} = (g_{00})^{-1} = \left(-\left(1-\frac{2m}{r}\right)\right)^{-1} = \frac{1}{-1+2\frac{m}{r}} = \frac{r}{-r+2m} = -\frac{r}{r-2m}$ is true
$g^{11} = (g_{11})^{-1} = \left(\frac{1}{1-\frac{2m}{r}}\right)^{-1} = -\frac{-r+2m}{r} = \frac{r-2m}{r}$ is true
$g^{22} = (g_{22})^{-1} = \left(r^2\right)^{-1} = \frac{1}{r^2}$
$g^{33} = (g_{33})^{-1} = \left(r^2\sin^2\theta\right)^{-1} = \frac{1}{r^2\sin^2\theta}$
so
$g^{\alpha\beta}\Gamma^{0}{}_{\alpha\beta} = g^{00}\Gamma^{0}{}_{00} + g^{11}\Gamma^{0}{}_{11} + g^{22}\Gamma^{0}{}_{22} + g^{33}\Gamma^{0}{}_{33} = 0$
$g^{\alpha\beta}\Gamma^{1}{}_{\alpha\beta} = g^{00}\Gamma^{1}{}_{00} + g^{11}\Gamma^{1}{}_{11} + g^{22}\Gamma^{1}{}_{22} + g^{33}\Gamma^{1}{}_{33}$
$g^{\alpha\beta}\Gamma^{1}{}_{\alpha\beta} = \left(-\frac{r}{r-2m}\right)\frac{m}{r^3}(r-2m) + \left(\frac{r-2m}{r}\right)\left(-\frac{m}{r(r-2m)}\right) + \frac{1}{r^2}\left(-(r-2m)\right) + \frac{1}{r^2\sin^2\theta}\left(-(r-2m)\sin^2\theta\right) = 2\frac{m-r}{r^2}$ is true
$g^{\alpha\beta}\Gamma^{2}{}_{\alpha\beta} = g^{00}\Gamma^{2}{}_{00} + g^{11}\Gamma^{2}{}_{11} + g^{22}\Gamma^{2}{}_{22} + g^{33}\Gamma^{2}{}_{33}$
$= \left(-\frac{r}{r-2m}\right)\Gamma^{2}{}_{00} + \left(\frac{r-2m}{r}\right)\Gamma^{2}{}_{11} + \frac{1}{r^2}\Gamma^{2}{}_{22} + \frac{1}{r^2\sin^2\theta}\Gamma^{2}{}_{33}$
$= \left(-\frac{r}{r-2m}\right)\Gamma^{2}{}_{00} + \left(\frac{r-2m}{r}\right)\Gamma^{2}{}_{11} + \frac{1}{r^2}\Gamma^{2}{}_{22} + \frac{1}{r^2\sin^2\theta}\Gamma^{2}{}_{33} = \frac{1}{r^2\sin^2\theta}\Gamma^{2}{}_{33} = -\frac{\sin\theta\cos\theta}{r^2\sin^2\theta} = -\frac{\cot\theta}{r^2}$ is true
$g^{\alpha\beta}\Gamma^{3}{}_{\alpha\beta} = g^{00}\Gamma^{3}{}_{00} + g^{11}\Gamma^{3}{}_{11} + g^{22}\Gamma^{3}{}_{22} + g^{33}\Gamma^{3}{}_{33} = 0$
and thus
$\Phi_{,\rho}g^{\alpha\beta}\Gamma^{\rho}{}_{\alpha\beta} = \Phi_{,0}g^{\alpha\beta}\Gamma^{0}{}_{\alpha\beta} + \Phi_{,1}g^{\alpha\beta}\Gamma^{1}{}_{\alpha\beta} + \Phi_{,2}g^{\alpha\beta}\Gamma^{2}{}_{\alpha\beta} + \Phi_{,3}g^{\alpha\beta}\Gamma^{3}{}_{\alpha\beta}$

$$\begin{aligned}\Phi_{,\rho}g^{\alpha\beta}\Gamma^{\rho}{}_{\alpha\beta} &= 2\frac{m-r}{r^2}\Phi_{,1} - \frac{\cot\theta}{r^2}\Phi_{,2} \\ &= 2\frac{m-r}{r^2}\frac{\partial\Phi}{\partial r} - \frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta}\end{aligned}$$

$g^{\alpha\beta}\Phi_{,\alpha\beta} = g^{00}\Phi_{,00} + g^{11}\Phi_{,11} + g^{22}\Phi_{,22} + g^{33}\Phi_{,33}$
$g^{\alpha\beta}\Phi_{,\alpha\beta} = \left(-\frac{r}{r-2m}\right)\Phi_{,00} + \left(\frac{r-2m}{r}\right)\Phi_{,11} + \frac{1}{r^2}\Phi_{,22} + \frac{1}{r^2\sin^2\theta}\Phi_{,33}$
$=\left(-\frac{r}{r-2m}\right)\frac{\partial^2\Phi}{\partial t^2} + \left(\frac{r-2m}{r}\right)\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$

$$\nabla^2\Phi = g^{\alpha\beta}\Phi_{,\alpha\beta} - \Phi_{,\rho}g^{\alpha\beta}\Gamma^{\rho}{}_{\alpha\beta} = \left(-\frac{r}{r-2m}\right)\frac{\partial^2\Phi}{\partial t^2} + \left(\frac{r-2m}{r}\right)\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} - 2\frac{m-r}{r^2}\frac{\partial\Phi}{\partial r} + -\frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta}$$

$$\boxed{\nabla^2\Phi = \left(-\frac{r}{r-2m}\right)\frac{\partial^2\Phi}{\partial t^2} + \left(\frac{r-2m}{r}\right)\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} - 2\frac{m-r}{r^2}\frac{\partial\Phi}{\partial r} + -\frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta}}$$

Problem 3

Suppose that an object is in free fall near a black hole (or any other spherically symmetric object) of mass m. The world-line of the object can be described by its Schwarzschild coordinate functions $t(\tau), r(\tau), \theta(\tau), \varphi(\tau)$ where τ represents proper time along that world-line.

a. Obtain the system of ordinary differential equations that determine these functions. These should be fully explicit relations among the derivatives of the functions $t(\tau), r(\tau), \theta(\tau), \varphi(\tau)$.

Answer 3a

Start with the geodesic equation

$$D_u u = 0$$

and convert it to index notation in these coordinates. $u^\alpha = \frac{dx^\alpha}{d\tau} = \dot{x}^\alpha$ where τ is the proper time.

$D_u u = u^\delta u^\alpha{}_{;\delta} e_\alpha = \frac{dx^\delta}{d\tau}\left(u^\alpha{}_{,\delta} + u^\rho \Gamma^\alpha{}_{\rho\delta}\right) e_\alpha$

Note that $\frac{dx^\delta}{d\tau} u^\alpha{}_{,\delta} = \frac{dx^\delta}{d\tau}\frac{\partial u^\alpha}{\partial x^\delta} = \frac{du^\alpha}{d\tau} = \dot{u}^\alpha$

$D_u u = \left(\dot{u}^\alpha + u^\delta u^\rho \Gamma^\alpha{}_{\rho\delta}\right) e_\alpha$

so the system of equations turns out to be

$$\dot{u}^\alpha + u^\delta u^\rho \Gamma^\alpha{}_{\rho\delta} = 0$$

or, using $u^\alpha = \frac{dx^\alpha}{d\tau}$,

$$\frac{d^2 x^\alpha}{d\tau^2} + \frac{dx^\delta}{d\tau}\frac{dx^\rho}{d\tau}\Gamma^\alpha{}_{\rho\delta} = 0$$

Write the equations out in detail:

$\alpha = 0$

$\frac{d^2 t}{d\tau^2} + \frac{dx^\delta}{d\tau}\frac{dx^\rho}{d\tau}\Gamma^0{}_{\rho\delta} = 0$

There are only two non-zero connection coefficients with the first index equal to zero.

$\frac{d^2 t}{d\tau^2} + 2\frac{dx^0}{d\tau}\frac{dx^1}{d\tau}\frac{m}{r(r-2m)} = 0$

$$\begin{aligned}\frac{d^2 t}{d\tau^2} + 2\frac{dt}{d\tau}\frac{dr}{d\tau}\frac{m}{r(r-2m)} &= 0\\ \ddot{t} + 2\dot{t}\dot{r}\frac{m}{r(r-2m)} &= 0\end{aligned}$$

$\alpha = 1$

$\frac{d^2 x^1}{d\tau^2} + \frac{dx^\delta}{d\tau}\frac{dx^\rho}{d\tau}\Gamma^1{}_{\rho\delta} = 0$

Note that all of the nonzero $\Gamma^1{}_{\rho\delta}$ have $\rho = \delta$.

$\ddot{r} + \frac{dx^0}{d\tau}\frac{dx^0}{d\tau}\Gamma^1{}_{00} + \frac{dx^1}{d\tau}\frac{dx^1}{d\tau}\Gamma^1{}_{11} + \frac{dx^2}{d\tau}\frac{dx^2}{d\tau}\Gamma^1{}_{22} + \frac{dx^3}{d\tau}\frac{dx^3}{d\tau}\Gamma^1{}_{33} = 0$

$\ddot{r}+\dot{t}^2\frac{m}{r^3}(r-2m)+\dot{r}^2\left(-\frac{m}{r(r-2m)}\right)+\dot{\theta}^2(-(r-2m))+\dot{\varphi}^2\left(-(r-2m)\sin^2\theta\right) = 0$

$$\ddot{r} + \dot{t}^2\frac{m}{r^3}(r-2m) - \dot{r}^2\frac{m}{r(r-2m)} - \dot{\theta}^2(r-2m) - \dot{\varphi}^2(r-2m)\sin^2\theta = 0$$

$\alpha = 2$

$\frac{d^2x^2}{d\tau^2} + \frac{dx^\delta}{d\tau}\frac{dx^\rho}{d\tau}\Gamma^2{}_{\rho\delta} = 0$

$\frac{d^2\theta}{d\tau^2} + \frac{dx^2}{d\tau}\frac{dx^1}{d\tau}\Gamma^2{}_{12} + \frac{dx^1}{d\tau}\frac{dx^2}{d\tau}\Gamma^2{}_{21} + \frac{dx^3}{d\tau}\frac{dx^3}{d\tau}\Gamma^2{}_{33} = 0$

$\ddot{\theta} + 2\dot{\theta}\dot{r}\Gamma^2{}_{12} + \dot{\varphi}^2\Gamma^2{}_{33} = 0$

$\ddot{\theta} + 2\dot{\theta}\dot{r}\frac{1}{r} + \dot{\varphi}^2(-\sin\theta\cos\theta) = 0$

$$\ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} - \dot{\varphi}^2\sin\theta\cos\theta = 0$$

$\alpha = 3$

$\frac{d^2x^3}{d\tau^2} + \frac{dx^\delta}{d\tau}\frac{dx^\rho}{d\tau}\Gamma^3{}_{\rho\delta} = 0$

$\ddot{\varphi} + 2\frac{dx^1}{d\tau}\frac{dx^3}{d\tau}\Gamma^3{}_{31} + 2\frac{dx^2}{d\tau}\frac{dx^3}{d\tau}\Gamma^3{}_{32} = 0$

$\ddot{\varphi} + 2\dot{r}\dot{\varphi}\Gamma^3{}_{31} + 2\dot{\theta}\dot{\varphi}\Gamma^3{}_{32} = 0$

$\ddot{\varphi} + 2\dot{r}\dot{\varphi}\frac{1}{r} + 2\dot{\theta}\dot{\varphi}\cot\theta = 0$

$$\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} + 2\dot{\theta}\dot{\varphi}\cot\theta = 0$$

Collecting the equations together,

$$\begin{array}{|l|}\hline
\ddot{t} + 2\dot{t}\dot{r}\frac{m}{r(r-2m)} = 0 \\
\ddot{r} + \dot{t}^2\frac{m}{r^3}(r-2m) - \dot{r}^2\frac{m}{r(r-2m)} - \dot{\theta}^2(r-2m) - \dot{\varphi}^2(r-2m)\sin^2\theta = 0 \\
\ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} - \dot{\varphi}^2\sin\theta\cos\theta = 0 \\
\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} + 2\dot{\theta}\dot{\varphi}\cot\theta = 0 \\
\hline\end{array}$$

b. Specialize the equations to the zero-velocity case and compare the predicted initial acceleration to what Newton's theory would predict.

Answer 3b

For zero velocity, we have $\dot{r} = \dot{\theta} = \dot{\varphi} = 0$ and $\dot{t} = 1$. The equations then become

$$\begin{aligned}
\ddot{t} &= 0 \\
\ddot{r} + \frac{m}{r^3}(r-2m) &= 0 \\
\ddot{\theta} &= 0 \\
\ddot{\varphi} &= 0
\end{aligned}$$

So there is a radial acceleration with

$$\boxed{\ddot{r} = -\tfrac{m}{r^3}(r-2m) = -\tfrac{m}{r^2} + \tfrac{2m^2}{r^3}.}$$

The first term is exactly what Newton's theory predicts. The second term is a correction that becomes important at distances comparable to the Schwarzschild radius $2m$.

C. Find the conditions that must be satisfied by an object in an equatorial circular orbit around a black hole. Do not forget the constraint $u \cdot u = -1$ because you will need it.

Answer 3c

For an equatorial orbit, we have $\theta = \frac{\pi}{2}$ so that $\dot{\theta} = 0$. Since the orbit is circular, we also have $\dot{r} = 0$. The equations then yield

$$\begin{aligned} \ddot{t} &= 0 \\ \dot{t}^2 \frac{m}{r^3}(r-2m) - \dot{\varphi}^2 (r-2m) &= 0 \\ 0 &= 0 \\ \ddot{\varphi} &= 0 \end{aligned}$$

So we see that both $\dot{t}$ (which is the red-shift factor, by the way) and $\dot{\varphi}$ are constant and related by $\dot{t}^2 \frac{m}{r^3} - \dot{\varphi}^2 = 0$ or

$$\dot{\varphi}^2 = \dot{t}^2 \frac{m}{r^3}$$

These derivatives are also related by the constraint

$$u \cdot u = -1$$

which takes the form
$g_{\alpha\beta} u^\alpha u^\beta = g_{00} \dot{t}^2 + g_{33} \dot{\varphi}^2 = -1$
$-\left(1 - \frac{2m}{r}\right) \dot{t}^2 + r^2 \dot{\varphi}^2 = -1$
Plug the circular orbit condition into this constraint:
$-\left(1 - \frac{2m}{r}\right) \dot{t}^2 + r^2 \dot{t}^2 \frac{m}{r^3} = -1$
$-\left(1 - \frac{2m}{r}\right) \dot{t}^2 + \dot{t}^2 \frac{m}{r} = \dot{t}^2 \frac{-r+3m}{r} = -1$

$$\begin{aligned} \dot{t}^2 &= \frac{r}{r-3m} \\ \dot{\varphi}^2 &= \frac{m}{r^3} \frac{r}{r-3m} \end{aligned}$$

Notice that the solution exists only for $\boxed{r > 3m}$. Thus, that is the radius of the smallest stable circular orbit. Objects closer than that spiral into the hole even though they are still outside the event horizon.

Problem 4

Show that the difference between two tangent-space connections D and D' on a given manifold

$$K = D' - D$$

can be regarded as a tensor field.

1. Answer 4

Insert the appropriate arguments. D and D' each need a derivative vector v and a vector to act on, so we have

$$K_v u =$$

which yields a vector field. To get a scalar field, we need to use a 1-form field α and obtain the scalar function

$$\begin{aligned} K(\alpha, u, v) &= \alpha(D'_v u - D_v u) \\ &= \alpha(D'_v u) - \alpha(D_v u). \end{aligned}$$

Now check that this function in locally linear by multiplying each argument by a scalar field f.

$K(f\alpha, u, v) = f\alpha(D'_v u - D_v u) = fK(\alpha, u, v)$

so the first argument works.

Now check the second argument.

$K(\alpha, fu, v) = \alpha(D'_v(fu)) - \alpha(D_v(fu))$

$= \alpha((D'_v f)u + fD'_v u) - \alpha((D_v f)u + fD_v u)$ Leibniz's Rule

$= \alpha((D'_v f)u) + \alpha(fD'_v u) - \alpha((D_v f)u) - \alpha(fD_v u)$

$= (D'_v f)\alpha(u) + f\alpha(D'_v u) - (D_v f)\alpha(u) - f\alpha(D_v u)$ Linearity of α.

$= f\alpha(D'_v u) - f\alpha(D_v u) + (D'_v f - D_v f)\alpha(u)$

However, our definition of a connection includes the condition

$$D'_v f = vf, \qquad D_v f = vf$$

so

$K(\alpha, fu, v) = f\alpha(D'_v u) - f\alpha(D_v u) + (vf - vf)\alpha(u) = fK(\alpha, u, v)$

and K is locally linear in the second argument.

Finally, check the third argument, using the assumed property

$$D'_{fv} u = fD'_v u, \qquad D_{fv} u = fD_v u$$

$K(\alpha, u, fv) = D'_{fv} u - D_{fv} u = fD'_v u - fD_v u = fK(\alpha, u, v)$

and K is locally linear in the third argument and is therefore a tensor field.

Problem 5

Suppose that V_P is the space of Weyl spinors at the point P and $\dot{V}_P$ is the complex conjugate space. Let E_A be basis vectors for V_P and let $E_{\dot{A}}$ be basis vectors for $\dot{V}_P$. The self-conjugate (i.e. real) spin-tensors

$$e_{A\dot{A}} = E_A \otimes E_{\dot{A}} + E_{\dot{A}} \otimes E_A$$

can be identified as basis vectors for the spacetime tangent space T_P. An orthonormal set of tangent space basis vectors e_α can be expanded in terms of these as

$$e_\alpha = \gamma^{A\dot{A}}{}_\alpha e_{A\dot{A}}$$

where the coefficients $\gamma^{A\dot{A}}{}_\alpha$ are constants and there will be an inverse expansion

$$e_{A\dot{A}} = \gamma_{A\dot{A}}{}^\alpha e_\alpha.$$

a. Define the connection coefficients for the spaces V_P and $\dot{V}_P$. These complex functions are called the 'spin connection coefficients'. How many such coefficients are there?

Answer 5a

The basis vectors for the space V_P are E_A so the connection coefficients for that space are just the components of the derivatives of those basis vectors in each tangent-space basis vector direction.

$$D_{e_a} E_A = \Gamma^B{}_{Aa} E_B$$

or

$$\boxed{\Gamma^B{}_{Aa} = \Omega^B \cdot D_{e_a} E_A.}$$

Similarly, the connection coefficients for $\dot{V}_P$ are

$$\boxed{\Gamma^{\dot{B}}{}_{\dot{A}a} = \Omega^{\dot{B}} \cdot D_{e_a} E_{\dot{A}}}$$

With no conditions placed on these connections, the independent complex components are

$$\Gamma^1{}_{1a}, \Gamma^1{}_{2a}, \Gamma^2{}_{1a}, \Gamma^2{}_{2a}$$

and a similar list for the conjugate space. Thus, there are eight complex four-vectors for a total of 32 independent complex components or 64 real numbers.

b. Express the spacetime connection coefficients $\Gamma^{\alpha}{}_{\beta\delta}$ for an orthonormal basis e_α in terms of the spin connection coefficients, the coefficients $\gamma^{A\dot{A}}{}_{\alpha}$ and their inverses $\gamma_{A\dot{A}}{}^{\alpha}$.

Answer 5b

The defining equation for the $\Gamma^{\alpha}{}_{\beta\delta}$ is

$$D_{e_\delta} e_\beta = \Gamma^{\alpha}{}_{\beta\delta} e_\alpha$$

Now put in the expansion of e_β in terms of spinor basis vectors and do the derivative.

$e_\beta = \gamma^{B\dot{B}}{}_{\beta} e_{B\dot{B}} = \gamma^{B\dot{B}}{}_{\beta} \left(E_B \otimes E_{\dot{B}} + E_{\dot{B}} \otimes E_B\right)$

$D_{e_\delta} e_\beta = \gamma^{B\dot{B}}{}_{\beta} D_{e_\delta} \left(E_B \otimes E_{\dot{B}} + E_{\dot{B}} \otimes E_B\right)$ Use the fact that the $\gamma^{B\dot{B}}{}_{\beta}$ are constants.

$D_{e_\delta} e_\beta$
$= \gamma^{B\dot{B}}{}_{\beta} \left(D_{e_\delta} E_B \otimes E_{\dot{B}} + E_B \otimes D_{e_\delta} E_{\dot{B}} + D_{e_\delta} E_{\dot{B}} \otimes E_B + E_{\dot{B}} \otimes D_{e_\delta} E_B\right)$
$= \gamma^{B\dot{B}}{}_{\beta} \left(\Gamma^{A}{}_{B\delta} E_A \otimes E_{\dot{B}} + E_B \otimes \Gamma^{\dot{A}}{}_{\dot{B}\delta} E_{\dot{A}} + \Gamma^{\dot{A}}{}_{\dot{B}\delta} E_{\dot{A}} \otimes E_B + E_{\dot{B}} \otimes \Gamma^{A}{}_{B\delta} E_A\right)$
$= \gamma^{B\dot{B}}{}_{\beta} \Gamma^{A}{}_{B\delta} E_A \otimes E_{\dot{B}} + \gamma^{B\dot{B}}{}_{\beta} \Gamma^{\dot{A}}{}_{\dot{B}\delta} E_B \otimes E_{\dot{A}}$
$+ \gamma^{B\dot{B}}{}_{\beta} \Gamma^{\dot{A}}{}_{\dot{B}\delta} E_{\dot{A}} \otimes E_B + \gamma^{B\dot{B}}{}_{\beta} \Gamma^{A}{}_{B\delta} E_{\dot{B}} \otimes E_A$

Now rename the dummy spinor indexes so that the basis tensors look the same. Just switch all the A and B indexes in half the terms.

$D_{e_\delta} e_\beta$
$= \gamma^{B\dot{B}}{}_{\beta} \Gamma^{A}{}_{B\delta} E_A \otimes E_{\dot{B}} + \gamma^{A\dot{A}}{}_{\beta} \Gamma^{\dot{B}}{}_{\dot{A}\delta} E_A \otimes E_{\dot{B}}$
$+ \gamma^{B\dot{B}}{}_{\beta} \Gamma^{\dot{A}}{}_{\dot{B}\delta} E_{\dot{A}} \otimes E_B + \gamma^{A\dot{A}}{}_{\beta} \Gamma^{B}{}_{A\delta} E_{\dot{A}} \otimes E_B$
$= \left(\gamma^{B\dot{B}}{}_{\beta} \Gamma^{A}{}_{B\delta} + \gamma^{A\dot{A}}{}_{\beta} \Gamma^{\dot{B}}{}_{\dot{A}\delta}\right) E_A \otimes E_{\dot{B}}$
$+ \left(\gamma^{B\dot{B}}{}_{\beta} \Gamma^{\dot{A}}{}_{\dot{B}\delta} + \gamma^{A\dot{A}}{}_{\beta} \Gamma^{B}{}_{A\delta}\right) E_{\dot{A}} \otimes E_B$

Rename these again to get basis tensors of the form $E_C \otimes E_{\dot{C}}$ and transpose.
$A \to C, \dot{B} \to \dot{C}$ in the first term
$\dot{A} \to \dot{C}, B \to C$ in the second term.

$D_{e_\delta} e_\beta$
$= \left(\gamma^{B\dot{C}}{}_{\beta} \Gamma^{C}{}_{B\delta} + \gamma^{C\dot{A}}{}_{\beta} \Gamma^{\dot{C}}{}_{\dot{A}\delta}\right) E_C \otimes E_{\dot{C}}$
$+ \left(\gamma^{C\dot{B}}{}_{\beta} \Gamma^{\dot{C}}{}_{\dot{B}\delta} + \gamma^{A\dot{C}}{}_{\beta} \Gamma^{C}{}_{A\delta}\right) E_{\dot{C}} \otimes E_C$

Rename the dummy indexes inside the parentheses:

$D_{e_\delta} e_\beta$
$= \left(\gamma^{K\dot{C}}{}_{\beta} \Gamma^{C}{}_{K\delta} + \gamma^{C\dot{K}}{}_{\beta} \Gamma^{\dot{C}}{}_{\dot{K}\delta}\right) E_C \otimes E_{\dot{C}}$
$+ \left(\gamma^{C\dot{K}}{}_{\beta} \Gamma^{\dot{C}}{}_{\dot{K}\delta} + \gamma^{K\dot{C}}{}_{\beta} \Gamma^{C}{}_{K\delta}\right) E_{\dot{C}} \otimes E_C$

and notice that the terms inside the parentheses are now identical so we get

$$D_{e_\delta} e_\beta = \left(\gamma^{K\dot{C}}{}_{\beta} \Gamma^{C}{}_{K\delta} + \gamma^{C\dot{K}}{}_{\beta} \Gamma^{\dot{C}}{}_{\dot{K}\delta}\right) \left(E_C \otimes E_{\dot{C}} + E_{\dot{C}} \otimes E_C\right)$$

or

$$D_{e_\delta} e_\beta = \left(\gamma^{K\dot{C}}{}_\beta \Gamma^C{}_{K\delta} + \gamma^{C\dot{K}}{}_\beta \Gamma^{\dot{C}}{}_{\dot{K}\delta} \right) e_{C\dot{C}}$$

and, using the inverse expansion,

$$e_{C\dot{C}} = \gamma_{C\dot{C}}{}^\alpha e_\alpha.$$

$$D_{e_\delta} e_\beta = \left(\gamma^{K\dot{C}}{}_\beta \Gamma^C{}_{K\delta} + \gamma^{C\dot{K}}{}_\beta \Gamma^{\dot{C}}{}_{\dot{K}\delta} \right) \gamma_{C\dot{C}}{}^\alpha e_\alpha$$

so that

$$\Gamma^\alpha{}_{\beta\delta} e_\alpha = \left(\gamma^{K\dot{C}}{}_\beta \Gamma^C{}_{K\delta} + \gamma^{C\dot{K}}{}_\beta \Gamma^{\dot{C}}{}_{\dot{K}\delta} \right) \gamma_{C\dot{C}}{}^\alpha e_\alpha$$

and we finally can identify

$$\boxed{\Gamma^\alpha{}_{\beta\delta} = \left(\gamma^{K\dot{C}}{}_\beta \Gamma^C{}_{K\delta} + \gamma^{C\dot{K}}{}_\beta \Gamma^{\dot{C}}{}_{\dot{K}\delta} \right) \gamma_{C\dot{C}}{}^\alpha}$$