Hour Exam No.2

Please attempt all of the following problems before the due date. All problems count the same even though some are more complex than others.

Several of the following problems require the connection coefficients for the connection compatible with the Schwarzschild metric:

\[ \begin{bmatrix}
  g_{00} & g_{01} & g_{02} & g_{03} \\
  g_{10} & g_{11} & g_{12} & g_{13} \\
  g_{20} & g_{21} & g_{22} & g_{23} \\
  g_{30} & g_{31} & g_{32} & g_{33} \\
\end{bmatrix} = \begin{bmatrix}
  -(1 - \frac{2m}{r}) & 0 & 0 & 0 \\
  0 & \frac{1}{r} & 0 & 0 \\
  0 & 0 & r^2 & 0 \\
  0 & 0 & 0 & r^2 \sin^2 \theta \\
\end{bmatrix} \]

For the coordinates \( x^0 = t, x^1 = r, x^2 = \theta, x^3 = \varphi \) and the corresponding holonomic basis, the non-zero connection coefficients are as follows:

\[
\begin{align*}
\Gamma^0_{01} &= \Gamma^0_{10} = \frac{m}{r (r - 2m)} \\
\Gamma^1_{00} &= \frac{m}{r^3} (r - 2m) \\
\Gamma^1_{11} &= -\frac{m}{r (r - 2m)} \\
\Gamma^1_{22} &= -(r - 2m) \\
\Gamma^1_{33} &= -(r - 2m) \sin^2 \theta \\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{1}{r} \\
\Gamma^2_{33} &= -\sin \theta \cos \theta \\
\Gamma^3_{13} &= \Gamma^3_{31} = \frac{1}{r} \\
\Gamma^3_{23} &= \Gamma^3_{32} = \cot \theta \\
\end{align*}
\]
Problem 1

For the connection given above, show that, for any vector fields \( u, v \) the following conditions are satisfied:

a. For any function \( f \),
\[ D_u D_v f = D_v D_u f \]

Hint: Switch to index notation and see the result immediately.

Answer 1a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

b. For any vector field \( w \)
\[ D_w (u \cdot v) = (D_w u) \cdot v + u \cdot D_w v \]

Hint: Switch to index notation and slog it out component by component using Maple to do the algebra.

Answer 1b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.
Problem 2

The wave operator on a scalar field can be written in the form

\[ \nabla^2 \Phi = g^{\alpha \beta} \Phi_{;\alpha ;\beta} \]

Use the connection coefficients given at the beginning of this exam to find the wave equation near a black hole of mass \( m \).

1. Answer 2

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.
Problem 3

Suppose that an object is in free fall near a black hole (or any other spherically symmetric object) of mass $m$. The world-line of the object can be described by its Schwarzschild coordinate functions $t(\tau), r(\tau), \theta(\tau), \varphi(\tau)$ where $\tau$ represents proper time along that world-line.

**a.** Obtain the system of ordinary differential equations that determine these functions. These should be fully explicit relations among the derivatives of the functions $t(\tau), r(\tau), \theta(\tau), \varphi(\tau)$.

**Answer 3a**

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

**b.** Specialize the equations to the zero-velocity case and compare the predicted initial acceleration to what Newton’s theory would predict.

**Answer 3b**

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

**c.** Find the conditions that must be satisfied by an object in an equatorial circular orbit around a black hole. Calculate the value of $r$ for the smallest such orbit. Do not forget the constraint $u \cdot u = -1$ because you will need it.

**Answer 3c**

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.
Problem 4

Show that the difference between two tangent-space connections $D$ and $D'$ on a given manifold

$$K = D' - D$$

can be regarded as a tensor field.

1. Answer 4
Problem 5

Suppose that \( V_P \) is the space of two-component Weyl spinors at the point \( P \) and \( \bar{V}_P \) is the complex conjugate space. Let \( E_A \) be basis vectors for \( V_P \) and let \( \bar{E}_{\bar{A}} \) be basis vectors for \( \bar{V}_P \). The self-conjugate (i.e. real) spin-tensors

\[
e_{A\bar{A}} = E_A \otimes \bar{E}_{\bar{A}} + \bar{E}_{\bar{A}} \otimes E_A
\]

can be identified as basis vectors for the spacetime tangent space \( T_P \). An orthonormal set of tangent space basis vectors \( e_\alpha \) can be expanded in terms of these as

\[
e_\alpha = \gamma^{A\bar{A}}_\alpha e_{A\bar{A}}
\]

where the coefficients \( \gamma^{A\bar{A}}_\alpha \) are constants and there will be an inverse expansion

\[
e_{A\bar{A}} = \gamma_{A\bar{A}}^\alpha e_\alpha.
\]

a. Define the connection coefficients for the spaces \( V_P \) and \( \bar{V}_P \). These complex functions are called the ‘spin connection coefficients’. How many such coefficients are there?

Answer 5a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

b. Express the spacetime connection coefficients \( \Gamma^\alpha_{\beta\delta} \) for an orthonormal basis \( e_\alpha \) in terms of the spin connection coefficients, the coefficients \( \gamma^{A\bar{A}}_\alpha \) and their inverses \( \gamma_{A\bar{A}}^\alpha \).

Answer 5b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.