Hour Exam No.1

Please attempt all of the following problems before the due date. All problems count the same even though some are more complex than others.

Problem 1

At Minkowski coordinate time, t, an object is located at the Minkowski position coordinates

$$x(t) = \sqrt{t^2+1}; \quad y(t) = z(t) = 0.$$

Using $c = 1$ units, find

a. the object's ordinary Newtonian velocity vector components.

Answer 1a

$v_x = \frac{dx}{dt} = \frac{d}{dt}\left(\sqrt{t^2+1}\right) = \frac{1}{\sqrt{(t^2+1)}}t$
$v_y = v_z = 0.$

b. the components of the object's four-velocity vector.

Answer 1b

Use the formula $u = u^0\left(e_0 + \vec{v}\right)$or

$$\begin{aligned} u^0 &= \frac{1}{\sqrt{1-v^2}} \\ u^i &= u^0 v^i \end{aligned}$$

$v^2 = {v_x}^2 + {v_y}^2 + {v_z}^2 = \left(\frac{1}{\sqrt{(t^2+1)}}t\right)^2 = \frac{1}{t^2+1}t^2$
$1 - v^2 = 1 - \frac{1}{t^2+1}t^2 = \frac{1}{t^2+1}$
$u^0 = \frac{1}{\sqrt{\frac{1}{t^2+1}}} = \sqrt{(t^2+1)}$
$u^1 = u_x = u^0 v_x = \sqrt{(t^2+1)}\frac{1}{\sqrt{(t^2+1)}}t = t$

$$\begin{aligned} u^0 &= \sqrt{(t^2+1)} \\ u_x &= t \\ u_y &= u_z = 0 \end{aligned}$$

Problem 2

Consider a two-dimensional spacetime manifold where we are using the coordinates t, x to locate events and the corresponding holonomic basis vectors

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_x = \frac{\partial}{\partial x}$$

to span each tangent space. A different coordinate system t', x' also locates events in this spacetime where

$$t' = t; \quad x' = x - \sqrt{t^2 + 1},$$

and the corresponding holonomic basis vectors

$$\partial_{t'} = \frac{\partial}{\partial t'}, \quad \partial_{x'} = \frac{\partial}{\partial x'}$$

also span each tangent space.

a. Notice that $\partial_{t'}$ is not equal to ∂_t even though $t' = t$. Express $\partial_{t'}$ in terms of ∂_t and ∂_x.

Answer 2a

From the chain rule for partial derivatives,
$\partial_{t'} = \frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial}{\partial x}$
To evaluate these partials, we need to solve for x, y in terms of x', t'.
$x' = x - \sqrt{t^2 + 1}$
$x = x' + \sqrt{t^2 + 1} = x' + \sqrt{t'^2 + 1}$
$t = t'$
$\frac{\partial t}{\partial t'} = 1, \qquad \frac{\partial x}{\partial t'} = \frac{\partial}{\partial t'}\left(x' + \sqrt{t'^2 + 1}\right) = \frac{t'}{\sqrt{t'^2+1}}$
(Notice that if you use SN–Maple to do this, it gets confused by the primes.)

$$\partial_{t'} = \frac{\partial}{\partial t} + \frac{t'}{\sqrt{t'^2 + 1}}\frac{\partial}{\partial x}$$

b. Express $\partial_{x'}$ in terms of ∂_t and ∂_x.

Answer 2b

$\partial_{x'} = \frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} = \frac{\partial}{\partial x}$

C. Quick answer: What differential forms correspond to the basis dual to the basis vectors ∂_t and ∂_x?

Answer 2c

$$dt, dx$$

Check this (optional):

$$\begin{aligned}
\partial_t \cdot dt &= \frac{\partial t}{\partial t} = 1 \\
\partial_x \cdot dx &= \frac{\partial x}{\partial x} = 1 \\
\partial_t \cdot dx &= \frac{\partial x}{\partial t} = 0 \\
\partial_x \cdot dt &= \frac{\partial t}{\partial x} = 0
\end{aligned}$$

Problem 3

Suppose that the metric tensor on a spacetime has the form

$$g = -dt \otimes dt + dx \otimes dx$$

and you decide to use the "null" coordinates

$$\begin{aligned} u^1 &= t + x \\ u^2 &= t - x \end{aligned}$$

and the corresponding "null basis" vectors

$$e_1 = \frac{\partial}{\partial u^1}, \quad e_2 = \frac{\partial}{\partial u^2}.$$

a. Express the null basis vectors in terms of the basis vectors ∂_t and ∂_x that go with the coordinates x, t.

Answer 3a

From the chain rule for partial derivatives,

$e_1 = \frac{\partial}{\partial u^1} = \frac{\partial t}{\partial u^1}\frac{\partial}{\partial t} + \frac{\partial x}{\partial u^1}\frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial u^2} = \frac{\partial t}{\partial u^2}\frac{\partial}{\partial t} + \frac{\partial x}{\partial u^2}\frac{\partial}{\partial x}$

Invert the relation to get x, t in terms of u^1, u^2

$t = \frac{1}{2}\left(u^1 + u^2\right), \qquad x = \frac{1}{2}\left(u^1 - u^2\right)$

$\frac{\partial t}{\partial u^1} = \frac{1}{2}, \qquad \frac{\partial x}{\partial u^1} = +\frac{1}{2}, \qquad \frac{\partial t}{\partial u^2} = \frac{1}{2}, \qquad \frac{\partial x}{\partial u^2} - \frac{1}{2}$

$$\begin{aligned} e_1 &= \frac{1}{2}\left(\partial_t + \partial_x\right) \\ e_2 &= \frac{1}{2}\left(\partial_t - \partial_x\right) \end{aligned}$$

b. Find the metric components g_{11}, g_{12}, g_{22} in the null basis.

Answer 3b

$g_{11} = \frac{1}{2}\left(\partial_t + \partial_x\right) \cdot \frac{1}{2}\left(\partial_t + \partial_x\right) = \frac{1}{4}\left(\partial_t \cdot \partial_t + \partial_x \cdot \partial_x\right) = \frac{1}{4}\left(-1 + 1\right) = 0$

$g_{12} = \frac{1}{2}\left(\partial_t + \partial_x\right) \cdot \frac{1}{2}\left(\partial_t - \partial_x\right) = \frac{1}{4}\left(\partial_t \cdot \partial_t - \partial_x \cdot \partial_x\right) = \frac{1}{4}\left(-1 - 1\right) = -\frac{1}{2}$

$g_{22} = \frac{1}{2}\left(\partial_t - \partial_x\right) \cdot \frac{1}{2}\left(\partial_t - \partial_x\right) = \frac{1}{4}\left(\partial_t \cdot \partial_t + \partial_x \cdot \partial_x\right) = \frac{1}{4}\left(-1 + 1\right) = 0$

Problem 4

An electromagnetic field two-form is given by

$$f = r^{-2} dt \wedge dr$$

where t is the usual Minkowski time function and $r = \sqrt{x^2 + y^2 + z^2}$ is a radius coordinate. In the following, use the basis vectors

$$\partial_0 = \frac{\partial}{\partial t}, \quad \partial_1 = \frac{\partial}{\partial r}$$

and their dual basis forms and assume $c = 1$ units.

a. Find the components $F_{00}, F_{01}, F_{10}, F_{11}$, of f.

Answer 4a

$f = r^{-2} dt \wedge dr = r^{-2} (dt \otimes dr - dr \otimes dt)$
The components are
$F_{00} = f(\partial_0, \partial_0) = r^{-2} (dt \otimes dr - dr \otimes dt)(\partial_0, \partial_0)$
$= r^{-2} (dt(\partial_0) dr(\partial_0) - dr(\partial_0) dt(\partial_0))$
$= r^{-2} (1 \times 0 - 0 \times 1) = 0$

Copy the above three lines several times and just change the subscripts to get the rest.
$F_{01} = f(\partial_0, \partial_1) = r^{-2} (dt \otimes dr - dr \otimes dt)(\partial_0, \partial_1)$
$= r^{-2} (dt(\partial_0) dr(\partial_1) - dr(\partial_0) dt(\partial_1))$
$= r^{-2} (1 \times 1 - 0 \times 0) = r^{-2}$
$F_{10} = f(\partial_1, \partial_0) = r^{-2} (dt \otimes dr - dr \otimes dt)(\partial_1, \partial_0)$
$= r^{-2} (dt(\partial_1) dr(\partial_0) - dr(\partial_1) dt(\partial_0))$
$= r^{-2} (0 \times 0 - 1 \times 1) = -r^{-2}$
$F_{11} = f(\partial_1, \partial_1) = r^{-2} (dt \otimes dr - dr \otimes dt)(\partial_1, \partial_1)$
$= r^{-2} (dt(\partial_1) dr(\partial_1) - dr(\partial_1) dt(\partial_1))$
$= r^{-2} (1 \times 1 - 1 \times 1) = 0$

$$\begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix} = \begin{pmatrix} 0 & r^{-2} \\ -r^{-2} & 0 \end{pmatrix}$$

b. Find the components $F^1{}_0$ and $F^0{}_1$ of the related tensor.

Answer 4b

$F^1{}_0 = F_{10} = -r^{-2}$
$F^0{}_1 = -F_{01} = -r^{-2}$

c. Identify the sort of object that would make this electromagnetic field.

Answer 4c

You can probably guess that this thing is a point charge. However, to get the sign and amount of the charge right, you should recall the form of the force law that we are using: For a test particle of charge e and mass m

$$\frac{dp}{d\tau} = \frac{e}{m} g^{-1} \left(f \left(p \right) \right)$$

or, in terms of components

$$\frac{dp^\alpha}{d\tau} = \frac{e}{m} g^{\alpha\sigma} f_{\sigma\rho} p^\rho = \frac{e}{m} F^\alpha{}_\rho p^\rho$$

For a test charge at rest, $p^r = 0$, $p^0 = m$, and $\tau = t$

$$\frac{dp^1}{dt} = eF^1{}_0 = -er^{-2}$$

Because e_1 points radially, we find a radial force of $-\frac{e}{r^2}$. Compare this result to the usual form of Coulomb's law:

$$k\frac{Qe}{r^2} = -\frac{e}{r^2}$$

The charge that generates this field must be

$$Q = -\frac{1}{k}$$

or about -1.1×10^{-10} Coulombs.

If you want the units to come out right, you need to attach the correct units (N/C) to the field two-form.

Problem 5 was the same as problem 3.

Problem 6

A set of observers are sitting on a flat turntable that is rotating with angular velocity ω. Each observer is located at a fixed pair of Cartesian coordinates x', y' that rotate with the disk. In terms of the non-rotating Minkowski coordinates t, x, y the position of an given observer at x', y' is

$$\begin{aligned} x &= x' \cos \omega t - y' \sin \omega t \\ y &= x' \sin \omega t + y' \cos \omega t \end{aligned}$$

Using $c = 1$ units,

a. find the four-velocity vector u of the observer at

$$\begin{aligned} x' &= r \\ y' &= 0 \end{aligned}$$

in terms of the non-rotating Minkowski basis vectors $\partial_t, \partial_x, \partial_y$

Answer 6a

First find the Newtonian velocity components of an object at constant x', y'.
$v_x = \frac{dx}{dt} = \frac{d}{dt}\left(x' \cos \omega t - y' \sin \omega t\right) = -x' \left(\sin \omega t\right) \omega - y' \left(\cos \omega t\right) \omega$
$v_y = \frac{dy}{dt} = \frac{d}{dt}\left(x' \sin \omega t + y' \cos \omega t\right) = x' \left(\cos \omega t\right) \omega - y' \left(\sin \omega t\right) \omega$
The value of v^2 we know should be $\left(x'^2 + y'^2\right) \omega^2 = r^2 \omega^2$.
Now use the formulas for the four-velocity components
$u^0 = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-r^2\omega^2}}$
$u_x = u^0 v_x = \frac{-x'(\sin \omega t)\omega - y'(\cos \omega t)\omega}{\sqrt{1-r^2\omega^2}} = -\frac{\omega r}{\sqrt{1-r^2\omega^2}} \sin \omega t$
$u_y = u^0 v_y = \frac{x'(\cos \omega t)\omega - y'(\sin \omega t)\omega}{\sqrt{1-r^2\omega^2}} = \frac{\omega r}{\sqrt{1-r^2\omega^2}} \cos \omega t$
and put these together with the basis vectors to form the four-velocity

$$u = \frac{1}{\sqrt{1 - r^2\omega^2}} \left(\partial_t - \omega r \sin \omega t \partial_x + \omega r \cos \omega t \partial_y\right)$$

b. The co-rotating coordinate system consists of $t' = t, x', y'$. Note that it is still the same t. Express the co-rotating holonomic basis vectors $\partial_{x'}$ and $\partial_{y'}$ at

$$\begin{aligned} x' &= r \\ y' &= 0 \end{aligned}$$

in terms of the non-rotating Minkowski basis vectors $\partial_t, \partial_x, \partial_y$

Answer 6b

First notice that the partials with respect to x', y' are holding t' and therefore t constant and the same is true of the partials with respect to x, y. Thus t and t' are just constants at this point.

$\partial_{x'} = \frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial}{\partial y} = \cos\omega t \partial_x + \sin\omega t \partial_y$

$\partial_{y'} = \frac{\partial}{\partial y'} = \frac{\partial x}{\partial y'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial y'}\frac{\partial}{\partial y} = -\sin\omega t \partial_x + \cos\omega t \partial_y$

Notice that the values of x' and y' do not actually matter.

C. Suppose that the observer at

$$\begin{aligned} x' &= r \\ y' &= 0 \end{aligned}$$

uses the basis vectors

$$\begin{aligned} e_0 &= u \\ e_1 &= \partial_{x'} \\ e_2 &= \partial_{y'} \end{aligned}$$

Find the components of the metric tensor in this basis.

Answer 6c

Collect the results of parts a,b to express the e_i in terms of the Minkowkski basis vectors.

$e_0 = \frac{1}{\sqrt{1-r^2\omega^2}}\left(\partial_t - \omega r \sin\omega t \partial_x + \omega r \cos\omega t \partial_y\right)$

$e_1 = \cos\omega t \partial_x + \sin\omega t \partial_y$

$e_2 = -\sin\omega t \partial_x + \cos\omega t \partial_y$

Take dot products using the usual Minkowski metric tensor.

$g_{00} = e_0 \cdot e_0 = u \cdot u = -1$

$g_{01} = e_0 \cdot e_1$

$= \frac{1}{\sqrt{1-r^2\omega^2}}\left(\partial_t - \omega r \sin\omega t \partial_x + \omega r \cos\omega t \partial_y\right) \cdot \left(\cos\omega t \partial_x + \sin\omega t \partial_y\right)$

$= \frac{1}{\sqrt{1-r^2\omega^2}}\left(-\omega r \sin\omega t \partial_x \cdot \cos\omega t \partial_x + \omega r \cos\omega t \partial_y \cdot \sin\omega t \partial_y\right)$

$= \frac{1}{\sqrt{1-r^2\omega^2}}\left(-\omega r \sin\omega t \cos\omega t + \omega r \cos\omega t \sin\omega t\right) = 0$

$g_{02} = e_0 \cdot e_2$

$= \frac{1}{\sqrt{1-r^2\omega^2}}\left(\partial_t - \omega r \sin\omega t \partial_x + \omega r \cos\omega t \partial_y\right) \cdot \left(-\sin\omega t \partial_x + \cos\omega t \partial_y\right)$

$= \frac{1}{\sqrt{1-r^2\omega^2}}\left(-\omega r \sin\omega t \partial_x \cdot (-\sin\omega t \partial_x) + \omega r \cos\omega t \partial_y \cdot \cos\omega t \partial_y\right)$

$= \frac{1}{\sqrt{1-r^2\omega^2}}\left(\omega r \sin^2\omega t + \omega r \cos^2\omega t\right) = \frac{\omega r}{\sqrt{1-r^2\omega^2}}$

$g_{11} = e_1 \cdot e_1 = \left(\cos\omega t \partial_x + \sin\omega t \partial_y\right) \cdot \left(\cos\omega t \partial_x + \sin\omega t \partial_y\right)$

$= \left(\cos^2\omega t + \sin^2\omega t\right) = 1$

$g_{12} = e_1 \cdot e_2 = \left(\cos\omega t \partial_x + \sin\omega t \partial_y\right) \cdot \left(-\sin\omega t \partial_x + \cos\omega t \partial_y\right)$

$= \cos\omega t \partial_x \cdot (-\sin\omega t \partial_x) + \sin\omega t \partial_y \cdot \cos\omega t \partial_y$

$= -\sin\omega t\cos\omega t + \sin\omega t\cos\omega t = 0$
$g_{22} = e_2 \cdot e_2 = (-\sin\omega t\partial_x + \cos\omega t\partial_y)\cdot(-\sin\omega t\partial_x + \cos\omega t\partial_y)$
$= (-\sin\omega t\partial_x)\cdot(-\sin\omega t\partial_x) + \cos\omega t\partial_y \cdot \cos\omega t\partial_y$
$= \sin^2\omega t + \cos^2\omega t = 1$

$$\begin{pmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} -1 & 0 & \frac{\omega r}{\sqrt{1-r^2\omega^2}} \\ 0 & 1 & 0 \\ \frac{\omega r}{\sqrt{1-r^2\omega^2}} & 0 & 1 \end{pmatrix}$$

Added Note:

At the point $x' = r, y' = 0$ the basis vectors $e_1 = \partial_{x'}, e_2 = \partial_{y'}$ can be expressed in terms of polar coordinates on the turntable as
$e_1 = \partial_r$
$e_2 = r^{-1}\partial_\varphi$
$\partial_\varphi = re_2$
so that the corresponding metric components are

$$\begin{pmatrix} g_{00} & g_{0r} & g_{0\varphi} \\ g_{r0} & g_{rr} & g_{r\varphi} \\ g_{\varphi 0} & g_{\varphi r} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} -1 & 0 & \frac{\omega r^2}{\sqrt{1-r^2\omega^2}} \\ 0 & 1 & 0 \\ \frac{\omega r^2}{\sqrt{1-r^2\omega^2}} & 0 & r^2 \end{pmatrix}$$

and the standard form of the metric is

$$ds^2 = -dt^2 + 2\frac{\omega r^2}{\sqrt{1-r^2\omega^2}}dtd\varphi + r^2d\varphi^2$$

The cross-term in $dtd\varphi$ is the signal that this is the metric of a rotating system.