Exercise 16

Please attempt all of the following problems before the due date. Your grade on this assignment will be calculated from the best three answers.

Problem 16.1

Let $M$ be flat two-dimensional space with Cartesian coordinates $x, y$. Use the definitions of $d$ and $\delta$ to calculate $\Delta f$ for a function $f(x, y)$.

Answer 16.1

$$\Delta f = (\delta d + d\delta) f = \delta df = \delta \left( \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right)$$

The definition of $\delta$ is

$$\delta \beta = (-1)^{n+np+1-(n-t)/2} * d * \beta$$

Here, $n = 2, p = 0, t = 2$ so $n + np + 1 - (n-t)/2 = 2 + 1 - (2-2)/2 = 3$

$\delta \beta = (-1)^3 * d * \beta = - * d * \beta$

$$\Delta f = - * d * \left( \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) = - * d \left( \frac{\partial f}{\partial x} * dx + \frac{\partial f}{\partial y} * dy \right)$$

$*dx = g^{-1}(dx) = *\partial_x = \partial_x dx \wedge dy = dy$

$*dy = g^{-1}(dy) = *\partial_y = \partial_y dx \wedge dy = -dx$

$$\Delta f = - * d \left( \frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx \right) = - * \left( \frac{\partial^2 f}{\partial x^2} dx \wedge dy - \frac{\partial^2 f}{\partial y^2} dy \wedge dx \right)$$

$$= - \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) * (dx \wedge dy)$$

$* (dx \wedge dy) = g^{-1} (dx \wedge dy) = * (\partial_x \wedge \partial_y)$

$*(\partial_x \wedge \partial_y) = \frac{1}{2} (\partial_x \otimes \partial_y - \partial_y \otimes \partial_x).dx \wedge dy = \partial_x \otimes \partial_y, dx \wedge dy = 1$

$$\Delta f = - \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right)$$
Problem 16.2

Let $M$ be flat two-dimensional space with Cartesian coordinates $x, y$. Use the definitions of $d$ and $\delta$ to calculate $\Delta \alpha$ for a one-form $\alpha(x, y) = \alpha_x(x, y) \, dx + \alpha_y(x, y) \, dy$.

Answer 16.2

$\Delta \alpha = d\delta \alpha + \delta d \alpha$

Use the following from the previous problem solution:

$\delta \beta = (-1)^{n+p+1-(n-t)/2} \star d \star \beta$

Here, $n = 2, t = 2$.

For $\beta$ a one-form, the exponent is $2 + 2 + 1 - (2 - 2)/2 = 5$ so that

$\delta \beta = - \star d \star \beta$

For $\beta$ a two-form, the exponent is $2 + 4 + 1 - (2 - 2)/2 = 7$ so that we still get

$\delta \beta = - \star d \star \beta$

$\star(dx \wedge dy) = 1$

$\star dx = dy$

$\star dy = -dx$

From the notes,

$\star \star \beta = (-1)^{\rho(n-p)+(n-t)/2} \beta$

For $p = 2$ the exponent is $2(2 - 2) + (2 - 2)/2 = 0$

$\star \star \beta = \beta$ which gives one more result that we will need:

$\star dx \wedge dy = dx \wedge dy$ or $\star 1 = dx \wedge dy$

Collect all of the results we need together:

<table>
<thead>
<tr>
<th>$\delta \beta$ = $- \star d \star \beta$ for one and two-forms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\star 1 = dx \wedge dy$</td>
</tr>
<tr>
<td>$\star (dx \wedge dy) = 1$</td>
</tr>
<tr>
<td>$\star dx = dy$</td>
</tr>
<tr>
<td>$\star dy = -dx$</td>
</tr>
</tbody>
</table>

$d\alpha = d(\alpha_x dx + \alpha_y dy) = d\alpha_x \wedge dx + d\alpha_y \wedge dy$

$= \frac{\partial \alpha_x}{\partial y} dy \wedge dx + \frac{\partial \alpha_y}{\partial x} dx \wedge dy = \left( \frac{\partial \alpha_y}{\partial x} - \frac{\partial \alpha_x}{\partial y} \right) dx \wedge dy$

$\delta d\alpha = - \star d \star \left( \frac{\partial \alpha_x}{\partial x} - \frac{\partial \alpha_x}{\partial y} \right) dx \wedge dy$

$= - \star \left( d \left( \frac{\partial \alpha_x}{\partial x} - \frac{\partial \alpha_x}{\partial y} \right) \wedge (dx \wedge dy) \right) = - \star \left( d \left( \frac{\partial \alpha_y}{\partial x} - \frac{\partial \alpha_x}{\partial y} \right) \wedge 1 \right)$

$= - \star \left( d \left( \frac{\partial^2 \alpha_y}{\partial x^2} - \frac{\partial^2 \alpha_y}{\partial x \partial y} \right) dx + \left( \frac{\partial^2 \alpha_x}{\partial x \partial y} - \frac{\partial^2 \alpha_x}{\partial y^2} \right) dy \right)$

$= - \left( \frac{\partial^2 \alpha_y}{\partial x^2} - \frac{\partial^2 \alpha_y}{\partial x \partial y} \right) \star dx - \left( \frac{\partial^2 \alpha_x}{\partial x \partial y} - \frac{\partial^2 \alpha_x}{\partial y^2} \right) \star dy$
\[
\delta \alpha = - \left( \frac{\partial^2 \alpha y}{\partial x^2} - \frac{\partial^2 \alpha x}{\partial x \partial y} \right) dy + \left( \frac{\partial^2 \alpha y}{\partial x \partial y} - \frac{\partial^2 \alpha x}{\partial y^2} \right) dx
\]

where

\[
\delta \alpha = -\star d \star \alpha = -\star d (\alpha_x \star dx + \alpha_y \star dy) = -\star d (\alpha_x dy - \alpha_y dx)
\]

\[
= -\star \left( \frac{\partial \alpha_x}{\partial x} dx \wedge dy - \frac{\partial \alpha_x}{\partial y} dy \wedge dx \right) = -\star \left( \frac{\partial \alpha_x}{\partial x} dx \wedge dy + \frac{\partial \alpha_x}{\partial y} dx \wedge dy \right)
\]

\[
\delta \alpha = \left( \frac{\partial \alpha_x}{\partial x} + \frac{\partial \alpha_y}{\partial y} \right) (dx \wedge dy) = - \left( \frac{\partial \alpha_x}{\partial x} + \frac{\partial \alpha_y}{\partial y} \right)
\]

d\delta \alpha = -d \left( \frac{\partial \alpha_x}{\partial x} + \frac{\partial \alpha_y}{\partial y} \right) = - \left( \frac{\partial^2 \alpha_x}{\partial x^2} + \frac{\partial^2 \alpha_y}{\partial x \partial y} \right) dx + \left( \frac{\partial^2 \alpha_x}{\partial x \partial y} + \frac{\partial^2 \alpha_y}{\partial y^2} \right) dy
\]

\[
\Delta \alpha = d \delta \alpha + \delta \alpha
\]

\[
= - \left( \frac{\partial^2 \alpha_x}{\partial x^2} + \frac{\partial^2 \alpha_y}{\partial y^2} \right) dx - \left( \frac{\partial^2 \alpha_x}{\partial x \partial y} + \frac{\partial^2 \alpha_y}{\partial y^2} \right) dy - \left( \frac{\partial^2 \alpha_x}{\partial x \partial y} - \frac{\partial^2 \alpha_y}{\partial y^2} \right) dy + \left( \frac{\partial^2 \alpha_x}{\partial x \partial y} - \frac{\partial^2 \alpha_y}{\partial y^2} \right) dx
\]

\[
= - \left( \frac{\partial^2 \alpha_x}{\partial x^2} + \frac{\partial^2 \alpha_y}{\partial y^2} - \frac{\partial^2 \alpha_x}{\partial x \partial y} + \frac{\partial^2 \alpha_y}{\partial y^2} \right) dx - \left( \frac{\partial^2 \alpha_x}{\partial x \partial y} + \frac{\partial^2 \alpha_y}{\partial y^2} \right) dy + \left( \frac{\partial^2 \alpha_x}{\partial x \partial y} - \frac{\partial^2 \alpha_y}{\partial y^2} \right) dx
\]

\[
\Delta \alpha = - \left( \frac{\partial^2 \alpha_x}{\partial x^2} + \frac{\partial^2 \alpha_y}{\partial y^2} \right) dx - \left( \frac{\partial^2 \alpha_x}{\partial x^2} + \frac{\partial^2 \alpha_y}{\partial y^2} \right) dy
\]
Problem 16.3

Take $M$ to be the unit two dimensional sphere and use the orthonormal basis forms

$$
\omega^1 = d\theta \\
\omega^2 = \sin \theta d\phi
$$

to calculate $\Delta f$ for a function $f(\theta, \phi)$.

Answer 16.3

The $\star$ operator is strictly local and will work in the same way that it did for $dx$ and $dy$, so we can use the results of the previous two problems:

$$
\begin{align*}
\star 1 &= \omega^1 \wedge \omega^2 \\
\star (\omega^1 \wedge \omega^2) &= 1 \\
\star \omega^1 &= \omega^2 \\
\star \omega^2 &= -\omega^1
\end{align*}
$$

Since $\delta$ reduces the order of a form by one, it must give zero when acting on a zero-form. To confirm this, try it:

$$
\begin{align*}
\delta f &= (-1)^{n+p+1-(n-t)/2} \star d \star f = (-1)^{2+1-0/2} \star d \star f \\
&= - \star d \star f = - \star d (f \omega^1 \wedge \omega^2) \\
\delta f &= - \star (df \wedge \omega^1 \wedge \omega^2 + fd \omega^1 \wedge \omega^2 - f \omega^1 \wedge d\omega^2) \\
&= - \star (df \wedge \omega^1 \wedge \omega^2 + fd \omega^1 \wedge \omega^2 - f \omega^1 \wedge d\omega^2)
\end{align*}
$$

Each term inside the parentheses contains something that is a sum of triple wedge products and all of those are zero.

$$
\begin{align*}
\Delta f &= \delta \delta f + \delta df = \delta df = - \star d \star df \\
df &= \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi = \frac{\partial f}{\partial \theta} \omega^1 + \frac{\partial f}{\partial \phi} \omega^2 \\
\star df &= - \star \left( \frac{\partial f}{\partial \theta} \omega^1 + \frac{\partial f}{\partial \phi} \omega^2 \right) \\
\Delta f &= - \star \left( \left( \frac{\partial f}{\partial \theta} \right) \wedge \omega^2 + \frac{\partial f}{\partial \phi} d\omega^2 - d \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \phi} \right) \wedge \omega^1 \right)
\end{align*}
$$

From the definitions of the basis forms,

$$
\begin{align*}
d\omega^1 &= 0 \\
d\omega^2 &= d (\sin \theta d\phi) = \cos \theta d\theta \wedge d\phi = \cot \theta \omega^1 \wedge \omega^2
\end{align*}
$$

$$
\begin{align*}
\Delta f &= - \star \left( \left( \frac{\partial f}{\partial \theta} \right) \wedge \omega^2 + \frac{\partial f}{\partial \phi} \cot \theta \omega^1 \wedge \omega^2 - d \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \phi} \right) \wedge \omega^1 \right) \\
&= - \star \left( \frac{\partial f}{\partial \theta} \omega^1 \wedge \omega^2 + \frac{\partial f}{\partial \phi} \cot \theta \omega^1 \wedge \omega^2 + \frac{1}{\sin \theta} \frac{\partial^2 f}{\partial \phi^2} \omega^1 \wedge \omega^2 \right) \\
\Delta f &= \left( \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial f}{\partial \phi} \cot \theta + \frac{1}{\sin \theta} \frac{\partial^2 f}{\partial \phi^2} \right) \star (\omega^1 \wedge \omega^2)
\end{align*}
$$
Problem 16.4

Take $M$ to be the unit circle with orthonormal basis form $\omega = d\varphi$. Show that the basis form field $\omega$ is itself a non-zero harmonic form. Is there a continuous function $f$ on the circle such that $\omega = df$?

Answer 16.4

Calculate $*\omega = *g^{-1}\omega = *\partial_\varphi = \partial_\varphi | \omega = 1$

$*d* \omega = * (d1) = 0$

so $\delta \omega = 0$.

$d\omega = d^2 \varphi = 0$.

So, wherever the coordinate $\varphi$ is defined,

$$\Delta \omega = d\delta \omega + \delta d\omega = 0.$$  

But we can define $\varphi$ everywhere except for one point (where $\varphi$ jumps from $2\pi$ to 0, for example). The choice of the point where $\varphi$ is discontinuous does not affect the form $d\varphi$ so we find that $\Delta \omega = 0$ everywhere and the form $\omega$ is harmonic.

Now assume that there is a function $f$ such that $\omega = df$ everywhere and consider the integral

$$\int_M \omega = \int_0^{2\pi} d\varphi = 2\pi.$$  

By using the coordinate $\varphi$ to evaluate the integral, only one point is left out and we can assume that point to have measure zero. But we also have the integral identity

$$\int_M \omega = \int_M df = \int_{\partial M} f$$

and the boundary $\partial M$ of a circle is empty, so

$$\int_M \omega = 0$$

and the assumption yields a contradiction.
Problem 16.5

Consider plane waves in a flat spacetime. Take the orthonormal frame to be
\[ \omega^\mu = dx^\mu \]
and the transverse and longitudinal parts of the vector potential to be
\[ a_T = A_T e^{ik \cdot x}, \quad a_L = A_L e^{ik \cdot x} \]
where \( A_T, A_L, \) and \( k \) are all one-forms. Use the equations \( da_L = 0 \) and \( \delta a_T = 0 \) and \( \Delta a_T = 0 \) to find the resulting contraints on these one-forms. Express the field two-form \( f \) in terms of these one-forms.

Answer 16.5

There are two things we can do here.

First, the abstract relations:
\[ f = da = d(a_T + a_L) = da_T + da_L \]

which solves the \( df = 0 \) Maxwell equations. The remaining Maxwell equations
\[ \delta f = 4\pi j \]
then become
\[ \delta da_T = 4\pi j \]
or, since \( \delta a_T = 0 \),
\[ (\delta d + d\delta) a_T = \Delta a_T = 4\pi j. \]
so that, in a vacuum, \( \Delta a_T = 0 \).

Second we work out what these relations look like for the proposed expressions:
\[ da_L = d(A_L e^{ik \cdot x}) = A_L \wedge de^{ik \cdot x} = A_L \wedge (ik \cdot dx) e^{ik \cdot x} = 0 \]
\[ A_L \wedge (k_0 dx^0 + k_1 dx^1 + k_2 dx^2 + k_3 dx^3) = 0. \]
Notice that we can regard \( k \) as the one-form
\[ k = k_0 dx^0 + k_1 dx^1 + k_2 dx^2 + k_3 dx^3 \]
in which case, the constraint is just
\[ A_L \wedge k = 0 \]

A solution to this constraint is
\[ A_L = Ck \]
where \( C \) can be any constant. It is easy to see that any components that are linearly independent of \( k \) will give a non-zero wedge product with \( k \) so this solution for \( A_L \) is the only one.
Recall that
\[
\delta \beta = (-1)^{n+p+1-(n-t)/2} * d * \beta
\]
For spacetime, \( n + np + 1 - (n - t) / 2 = 4 + 4p + 1 - 0 = 3 + 4p \) so
\[
\delta \beta = - * d * \beta
\]
Now look at the condition \( \delta a_T = 0 \) or \( * d * a_T = 0 \)
\[
* d * (A_T e^{ik \cdot x}) = * d (* (A_T) e^{ik \cdot x}) = * ((* A_T) \wedge d e^{ik \cdot x}) = * ((* A_T) \wedge i k e^{ik \cdot x})
\]
so the condition becomes
\[
* ((* A_T) \wedge k) = 0
\]
which is equivalent to
\[
A_T \cdot k = 0
\]
and requires \( A_T \) to be orthogonal to the propagation form \( k \).
Finally, the harmonic condition becomes
\[
\Delta a_T = \Delta (A_T e^{ik \cdot x}) = \delta d (A_T e^{ik \cdot x}) + d \delta (A_T e^{ik \cdot x}) = 0
\]
or
\[
\delta (A_T \wedge d e^{ik \cdot x}) + d * ((* A_T) \wedge k e^{ik \cdot x}) = 0
\]
or
\[
\delta (A_T \wedge k e^{ik \cdot x}) + d * ((* A_T) \wedge k e^{ik \cdot x}) = 0
\]
or
\[
* d * (A_T \wedge k e^{ik \cdot x}) + d * (* A_T \wedge k e^{ik \cdot x}) = 0
\]
The second term is really zero because of the \( \delta a_T = 0 \) constraint. However, it turns out to be useful to leave it there and work out what it is. The star operator acts on the basis one-forms according to
\[
* \omega^0 = * (g^{-1} (\omega^0)) = -* e_0 = e_{0,T} (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3) = -\omega^1 \wedge \omega^2 \wedge \omega^3
\]
\[
* \omega^1 = * (g^{-1} (\omega^1)) = * e_1 = e_{1,T} (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3) = -\omega^2 \wedge \omega^3 \wedge \omega^0
\]
\[
* \omega^2 = * (g^{-1} (\omega^2)) = * e_2 = e_{2,T} (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3) = -\omega^3 \wedge \omega^1 \wedge \omega^0
\]
\[
* \omega^3 = * (g^{-1} (\omega^3)) = * e_3 = e_{3,T} (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3) = -\omega^1 \wedge \omega^2 \wedge \omega^0
\]
so that
\[
* A_T \wedge k = A_T (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3)
\]
\[
= A_T (\omega^0 \wedge \omega^1 \wedge \omega^2 + A_T \omega^3 \wedge \omega^0)
\]
\[
= -A_T (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3)
\]
\[
= -A_T (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3)
\]
\[
* (a_T \wedge k e^{ik \cdot x}) = - (A_T \cdot k) e^{ik \cdot x} \]
* \( (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3) = -* (e_0 \wedge e_1 \wedge e_2 \wedge e_3)
\]
\[
= -1/2 e_0 \wedge e_1 \wedge e_2 \wedge e_3 (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3)
\]
\[
= -(e_0 \otimes e_1 \otimes e_2 \otimes e_3) (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3) = -1
\]
\[
* (A_T \wedge k e^{ik \cdot x}) = (A_T \cdot k) e^{ik \cdot x}
\]
\[ d \star (A_T \wedge ke^{ik_x}) = d((A_T \cdot k)e^{ik_x}) = (A_T \cdot k)de^{ik_x} = (A_T \cdot k)ke^{ik_x} \]

\[= (-A_{T0}k_0 + A_{T1}k_1 + A_{T2}k_2 + A_{T3}k_3)k_\alpha \omega^\alpha \]

To work the other term out, we need to calculate how the star operator acts on the basis two forms in a spacetime. From the notes,

\[\star (\omega^0 \wedge \omega^1) = \star (g^{-1}(\omega^0 \wedge \omega^1))\]
\[= -\star (e_0 \wedge e_1)\]
\[= - \frac{1}{2} (e_0 \wedge e_2)_{\wedge_1} (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3)\]
\[= - (e_0 \otimes e_1)_{\wedge} (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3)\]
\[= (e_0 \otimes e_2)_{\wedge} (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3)\]
\[= \omega^1 \wedge \omega^3\]

Similarly

\[\star (\omega^0 \wedge \omega^2) = \star (g^{-1}(\omega^0 \wedge \omega^2))\]
\[= -\star (e_0 \wedge e_1)\]
\[= - \frac{1}{2} (e_0 \wedge e_2)_{\wedge_1} (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3)\]
\[= - (e_0 \otimes e_1)_{\wedge} (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3)\]
\[= (e_0 \otimes e_2)_{\wedge} (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3)\]
\[= -\omega^2 \wedge \omega^3 = \omega^3 \wedge \omega^2\]

and

\[\star (\omega^1 \wedge \omega^2) = \star (g^{-1}(\omega^1 \wedge \omega^2))\]
\[= \star (e_1 \otimes e_2)\]
\[= \frac{1}{2} (e_1 \wedge e_2)_{\wedge_1} (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3)\]
\[= (e_1 \otimes e_2)_{\wedge} (\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3)\]
\[= - (e_1 \otimes e_2)_{\wedge} (\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^0)\]
\[= -\omega^3 \wedge \omega^0 = \omega^0 \wedge \omega^3\]

The others work out the same way.
For the \( p = 2 \) forms, the results are:

\[
\begin{align*}
\star (\omega^0 \wedge \omega^1) &= -\omega^4 \wedge \omega^4 & \text{& cyclic in 1, 2, 3} \\
\star (\omega^1 \wedge \omega^2) &= \omega^0 \wedge \omega^3 & \text{& cyclic in 1, 2, 3}
\end{align*}
\]

Now work out \( *d \star (A_T \wedge ke^{ik_x}) \).

\[\star (A_T \wedge k) = \star (A_{T\alpha} \omega^\alpha \wedge k_\beta \omega^\beta) = A_{T\alpha}k_\beta \star (\omega^\alpha \wedge \omega^\beta)\]
\[= A_{T0}k_1 \star (\omega^0 \wedge \omega^1) + A_{T0}k_2 \star (\omega^0 \wedge \omega^2) + A_{T0}k_3 \star (\omega^0 \wedge \omega^3)\]
\[ + A_{T1}k_0 \ast (\omega^1 \land \omega^0) + A_{T1}k_2 \ast (\omega^1 \land \omega^2) + A_{T1}k_3 \ast (\omega^1 \land \omega^3) \\
+ A_{T2}k_0 \ast (\omega^2 \land \omega^0) + A_{T2}k_1 \ast (\omega^2 \land \omega^1) + A_{T2}k_3 \ast (\omega^2 \land \omega^3) \\
+ A_{T3}k_0 \ast (\omega^3 \land \omega^0) + A_{T3}k_1 \ast (\omega^3 \land \omega^1) + A_{T3}k_2 \ast (\omega^3 \land \omega^2) \\
= (A_{T0}k_1 - A_{T1}k_0) \ast (\omega^0 \land \omega^1) + (A_{T0}k_2 - A_{T2}k_0) \ast (\omega^0 \land \omega^2) \\
+ (A_{T0}k_3 - A_{T3}k_0) \ast (\omega^0 \land \omega^3) \\
+ (A_{T1}k_2 - A_{T2}k_1) \ast (\omega^1 \land \omega^2) + (A_{T2}k_3 - A_{T3}k_2) \ast (\omega^2 \land \omega^3) \\
+ (A_{T3}k_1 - A_{T1}k_3) \ast (\omega^3 \land \omega^1) \]

\[ \ast (\omega^0 \land \omega^1) = -\omega^2 \land \omega^3 \\
\ast (\omega^0 \land \omega^2) = -\omega^3 \land \omega^1 \\
\ast (\omega^0 \land \omega^3) = -\omega^1 \land \omega^2 \\
\ast (\omega^1 \land \omega^2) = \omega^0 \land \omega^3 \\
\ast (\omega^2 \land \omega^3) = \omega^0 \land \omega^1 \\
\ast (\omega^3 \land \omega^1) = \omega^0 \land \omega^2 \]

\[ \ast (A_T \land k) = (A_{T0}k_1 - A_{T1}k_0) (\omega^3 \land \omega^2) + (A_{T0}k_2 - A_{T2}k_0) (\omega^1 \land \omega^3) \\
+ (A_{T0}k_3 - A_{T3}k_0) (\omega^2 \land \omega^1) \\
+ (A_{T1}k_2 - A_{T2}k_1) (\omega^0 \land \omega^3) + (A_{T2}k_3 - A_{T3}k_2) (\omega^0 \land \omega^1) \\
+ (A_{T3}k_1 - A_{T1}k_3) (\omega^0 \land \omega^2) \]

\[ d \ast (A_T \land ke^{ikx}) = d \ast ((A_{T0}k_1 - A_{T1}k_0) (\omega^3 \land \omega^2 e^{ikx}) + (A_{T0}k_2 - A_{T2}k_0) (\omega^1 \land \omega^3 e^{ikx}) \\
+ (A_{T0}k_3 - A_{T3}k_0) (\omega^2 \land \omega^1 e^{ikx}) \\
+ (A_{T1}k_2 - A_{T2}k_1) (\omega^0 \land \omega^3 e^{ikx}) + (A_{T2}k_3 - A_{T3}k_2) (\omega^0 \land \omega^1 e^{ikx}) \\
+ (A_{T3}k_1 - A_{T1}k_3) (\omega^0 \land \omega^2 e^{ikx}) \]

\[ e^{-ikx}d \ast (A_T \land ke^{ikx}) = (A_{T0}k_1 - A_{T1}k_0) (\omega^3 \land \omega^2 \land k) \\
+ (A_{T0}k_2 - A_{T2}k_0) (\omega^1 \land \omega^3 \land k) \\
+ (A_{T0}k_3 - A_{T3}k_0) (\omega^2 \land \omega^1 \land k) \\
+ (A_{T1}k_2 - A_{T2}k_1) (\omega^0 \land \omega^3 \land k) \\
+ (A_{T2}k_3 - A_{T3}k_2) (\omega^0 \land \omega^1 \land k) \\
+ (A_{T3}k_1 - A_{T1}k_3) (\omega^0 \land \omega^2 \land k) \]
Now take the final Hodge star:

\[ (\omega^3 \wedge \omega^2 \wedge \omega^0) = - * (e_3 \otimes e_2 \otimes e_0, \omega^0) = e_3 \otimes e_2 \otimes e_0 \omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 \]

Now combine the two parts of the expression:

\[-e^{-ik \cdot x} \Delta a_T = e^{-ik \cdot x} \Delta a_T\]
\[-e_\alpha \cdot e^{-ik \cdot x} \Delta a_T = A_{T\alpha} (k \cdot k)\]
so that

\[
\Delta a_T = - (k \cdot k) a_T
\]

and the \(\Delta a_T = 0\) equation simply requires the propagation form \(k\) to be lightlike:

\[
k \cdot k = 0
\]