Exercise 14

Please attempt all of the following problems before the due date. Your grade on this assignment will be calculated from the best *three* answers.

**Problem 14.1**

Use the method of Pfaffian differential forms to solve the differential equation

\[
\frac{dy}{dx} = \frac{y}{x}
\]

for the function \( y(x) \). (Note: This equation is pretty easy to solve directly, but humor me and use the method.)

**Answer 14.1**

First write the equation as a form equal to zero:

\[ xdy = ydx \text{ or } \alpha = ydx - xdy = 0 \]

Next, find a function \( h \) such that

\[
d(h\alpha) = 0
\]

\[ d(h(ydx - xdy)) = 0 \]

The choice \( h = \frac{1}{xy} \) gives \( h\alpha = \frac{1}{xy} (ydx - xdy) = \frac{dx}{x} - \frac{dy}{y} \). Now find a function \( f \) such that

\[
h\alpha = df.
\]

In this case, we see by inspection that \( f = \ln x - \ln y = \ln \frac{x}{y} \) will work, so the original differential equation becomes

\[
d \left( \ln \frac{x}{y} \right) = 0
\]

or

\[
\ln \frac{x}{y} = A = \text{constant.}
\]

which yields

\[
\frac{x}{y} = e^A
\]

\[
y(x) = e^{-A}x
\]

The solutions are straight lines of the form

\[
y = bx.
\]
Problem 14.2

Integrate the form

$$\alpha = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$$

over the sphere $\Sigma$ defined by $x^2 + y^2 + z^2 = 1$.

Answer 14.2

Use spherical coordinates $\theta, \varphi$ with

$$
\begin{align*}
z &= \cos \theta \\
x &= \sin \theta \cos \varphi \\
y &= \sin \theta \sin \varphi
\end{align*}
$$

so that

$$
\begin{align*}
dz &= -\sin \theta d\theta \\
dx &= \cos \theta \cos \varphi d\theta - \sin \theta \sin \varphi d\varphi \\
dy &= \\
\end{align*}
$$

Now evaluate the form $\alpha$

$$
\begin{align*}
dy \wedge dz &= (\cos \theta \sin \varphi d\theta + \sin \theta \cos \varphi d\varphi) \wedge (-\sin \theta d\theta) \\
&= \cos \theta \sin \varphi d\theta \wedge (-\sin \theta d\theta) + \sin \theta \cos \varphi d\varphi \wedge (-\sin \theta d\theta) \\
&= -\sin \theta \cos \theta \sin \varphi d\theta \wedge d\theta - \sin^2 \theta \cos \varphi d\varphi \wedge d\theta = -\sin^2 \theta \cos \varphi d\varphi \wedge d\theta = \\
&= \sin^2 \theta \cos \varphi d\theta \wedge d\varphi \\
dz \wedge dx &= (-\sin \theta d\theta) \wedge (\cos \theta \cos \varphi d\theta - \sin \theta \sin \varphi d\varphi) = \sin \theta d\theta \wedge \sin \theta \sin \varphi d\varphi \\
&= \sin^2 \theta \sin \varphi d\theta \wedge d\varphi \\
dx \wedge dy &= (\cos \theta \cos \varphi d\theta \wedge \sin \theta \sin \varphi d\varphi) \wedge (\cos \theta \sin \varphi d\theta + \sin \theta \cos \varphi d\varphi) \\
&= \cos \theta \cos \varphi d\theta \wedge \sin \theta \sin \varphi d\varphi - \sin \theta \sin \varphi d\varphi \wedge \cos \theta \sin \varphi d\theta \\
&= \sin \theta \cos \theta \cos^2 \varphi d\theta \wedge d\varphi - \sin \theta \cos \theta \sin^2 \varphi d\theta \wedge d\varphi \\
&= \sin \theta \cos \theta \cos^2 \varphi d\theta \wedge d\varphi + \sin \theta \cos \theta \sin^2 \varphi d\theta \wedge d\varphi \\
&= \sin \theta \cos \theta (\cos^2 \varphi + \sin^2 \varphi) d\theta \wedge d\varphi = \sin \theta \cos \theta d\theta \wedge d\varphi
\end{align*}
$$

Putting the pieces together,

$$\alpha = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$$

$$\begin{align*}
&= x \sin^2 \theta \cos \varphi d\theta \wedge d\varphi + y \sin^2 \theta \cos \varphi d\theta \wedge d\varphi + z \sin \theta \cos \theta d\theta \wedge d\varphi \\
&= (x \sin \theta \cos \varphi + y \sin \theta \sin \varphi + z \cos \theta) \sin \theta d\theta \wedge d\varphi \\
&= (\sin \theta \cos \varphi \sin \theta \cos \varphi + \sin \theta \sin \varphi \sin \theta \sin \varphi + \cos \theta \cos \varphi \cos \theta) \sin \theta d\theta \wedge d\varphi \\
&= \sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi + \cos^2 \theta) \sin \theta d\theta \wedge d\varphi = \sin \theta d\theta \wedge d\varphi
\end{align*}
$$

Now do the integral

$$
\begin{align*}
\int \alpha &= \int \sin \theta d\theta \wedge d\varphi = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin \theta \\
&= 2\pi \int_0^\pi \sin \theta d\theta = 4\pi
\end{align*}
$$
Problem 14.3

Use \(c = 1\) units in this problem. Recall that the Special Relativistic Lorentz Force Law in component form is

\[
\frac{du^\mu}{d\tau} = \frac{e}{m} F^\mu_{\nu} u^\nu.
\]

For a field that is purely electric in a reference frame \([e]\), the field tensor \(F\) has the form

\[F = (e_0 \otimes E - E \otimes e_0)\]

Use these expressions to find the acceleration of a charged particle that is initially at rest with respect to the frame \([e]\). The main point of the exercise is to see that the signs work out.

Answer 14.3

The non-zero components of the given field tensor are

\[F^{0i} = E^i, \quad F^{i0} = -E^i\]

The components needed for the Force Law are the mixed-rank components:

\[F^i_{0} = g_{0\rho} F^{\rho i} = g_{00} F^{i0} = -(E^i) = E^i\]

The acceleration components are then

\[a^i = \frac{du^i}{d\tau} = \frac{e}{m} F^i_{\nu} u^\nu\]

Since the particle is initially at rest, \(u^i = 0\) and \(u^0 = 1\).

\[a^i = \frac{e}{m} F^{i0} u^0 = \frac{e}{m} E^i\]

or,

\[\vec{a} = \frac{e}{m} \vec{E}\]

Problem 14.4

Find an expression for the electromagnetic field tensor \(F\) if it is purely electric in the reference frame of an observer with four-velocity \(u\).

Answer 14.4
The formula given in problem 14.3

\[ F = e_0 \otimes E - E \otimes e_0 \]

expresses a field tensor that is purely electric in the reference frame \([e]\). The four-velocity of an observer at rest in that frame is \(e_0\). Thus, we can replace \(e_0\) by \(u\).

\[ F = u \otimes E - E \otimes u \]

We are not quite done yet because the electric field \(E\) is supposed to be purely spacelike in the observer’s frame. Using an arbitrary four-vector for \(E\) in the expression for \(F\) is OK because the time components cancel but the electric field vector is now

\[ \vec{E} = E + (E \cdot u) u. \]

To check this expression, note that if \(E = Au\), then \(\vec{E} = Au + (Au \cdot u) u = Au + A (-1) u = 0\).

**Problem 14.5**

Find an expression for the electromagnetic field two-form \(f\) if it is purely magnetic in the reference frame of an observer with four-velocity \(u\).

Hint: Check the example given in the notes.

**Answer 14.5**

The expression that we need from the notes is

\[ f = e_{0\land} \ast B \]

for a field two-form that is purely magnetic in the frame \([e]\) where \(B\) can be any four-vector. As noted in the notes, only the space components of \(B\)

\[ \vec{B} = B + (B \cdot e_0) e_0 = B^i e_i \]

affect the answer. As in problem 14.4, the four-velocity of the observer at rest in frame \([e]\) is \(e_0\) so we just have to replace \(e_0\) by \(u\).

\[ f = u_{\land} \ast B \]

and note that the magnetic field seen by this observer is

\[ \vec{B} = B + (B \cdot u) u. \]