Exercise 11

Please attempt all of the following problems before the due date. Your grade on this assignment will be calculated from the best three answers.

Problem 11.1

Use homogeneous linearity to show that if \( f(p, p) = 0 \) for any \( p \) that satisfies \( p^2 = -m^2c^2 \) for a particular value of \( m \) then \( f(w, w) = 0 \) for any four-vector \( w \) with \( w^2 < 0 \).

Hint: Start by constructing a \( p \) from a \( w \).

Answer 11.1

For any \( w \) with \( w^2 < 0 \) construct the vector \( p = \frac{mc}{\sqrt{-w^2}}w \) and note that
\[
p^2 = \frac{mc}{\sqrt{-w^2}}w \cdot \frac{mc}{\sqrt{-w^2}}w = \frac{m^2c^2}{-w^2}w^2 = -m^2c^2
\]
Therefore \( f(p, p) = 0 \) or \( f \left( \frac{mc}{\sqrt{-w^2}}w, \frac{mc}{\sqrt{-w^2}}w \right) = 0 \)

Use homogeneous linearity to see that
\[
f \left( \frac{mc}{\sqrt{-w^2}}w, \frac{mc}{\sqrt{-w^2}}w \right) = \frac{m^2c^2}{-w^2}f(w, w) = 0
\]
and conclude that
\[
f(w, w) = 0
\]

Problem 11.2

Show that if the four-vectors \( a, b \) each have positive time components \( a^0, b^0 \) in some reference frame, and obey the constraints \( a^2 < 0, b^2 < 0 \) then their sum \( a + b \) obeys the same constraint: \((a + b)^2 < 0\).

Answer 11.2
\[
(a + b)^2 = (a + b) \cdot (a + b) = a^2 + 2a \cdot b + b^2
\]
\[
a^2 = -(a^0)^2 + a^2 < 0 \text{ so } a^0 > |\vec{a}|
\]
Similarly, \( b^0 > |\vec{b}| \)
\[
a \cdot b = -a^0b^0 + \vec{a} \cdot \vec{b} = -a^0b^0 + |\vec{a}| |\vec{b}| \cos \theta
\]
\[
a^0b^0 > |\vec{a}| |\vec{b}| \geq |\vec{a}| |\vec{b}| \cos \theta
\]
\[ a \cdot b < 0. \]
\[ (a + b)^2 = a^2 + 2a \cdot b + b^2 = \text{sum of three negative terms.} \]

**Problem 11.3**

Use the preceding results to show that the tensor \( f \) must obey the constraint

\[ f(a, b) = -f(b, a) \]

for any two four-vectors \( a, b \).

**Answer 11.3**

First, for any two future-pointing vectors \( a, b \) such that \( a^2 < 0, b^2 < 0 \) we have \( (a + b)^2 < 0 \) so that

\[ f(a, a) = 0, \quad f(b, b) = 0 \]

and

\[ f(a + b, a + b) = 0 \]

Use multilinearity to get

\[ f(a + b, a + b) = f(a, a) + f(a, b) + f(b, a) + f(b, b) \]

so that

\[ f(a, b) + f(b, a) = 0 \]

holds for future-pointing timelike vectors. Now consider arbitrary vectors \( k, j \). The vectors \( a + \varepsilon k \) and \( a + \varepsilon j \) will still be future-pointing and timelike if \( \varepsilon \) is sufficiently small.

To check that, note that \( a^0 + \varepsilon k^0 > 0 \) so long as \( \varepsilon < \frac{a^0}{|k^0|} \) and

\[ (a + \varepsilon k)^2 = a^2 + 2\varepsilon a \cdot k + \varepsilon^2 k^2 < 0 \] if we just choose \( \varepsilon \)

so that \( 2\varepsilon a \cdot k + \varepsilon^2 k^2 < |a|^2 \).

Since \( a + \varepsilon k \) and \( a + \varepsilon j \) are both future-pointing timelike, we have

\[ f(a + \varepsilon k, a + \varepsilon j) + f(a + \varepsilon j, a + \varepsilon k) = 0. \]

Use multilinearity to expand this

\[ f(a + \varepsilon k, a + \varepsilon j) = f(a, a) + f(a, \varepsilon j) + f(\varepsilon k, a) + f(\varepsilon k, \varepsilon j) \]

\[ = \varepsilon f(a, j) + \varepsilon f(k, a) + \varepsilon^2 f(k, j) \]

\[ f(a + \varepsilon j, a + \varepsilon k) = \varepsilon f(a, k) + \varepsilon f(j, a) + \varepsilon^2 f(j, k) \]

\[ = \varepsilon f(a, j) + \varepsilon f(k, a) + \varepsilon^2 f(k, j) + \varepsilon f(a, k) + \varepsilon f(j, a) + \varepsilon^2 f(j, k) = 0 \]

Since this expression has to hold for all sufficiently small \( \varepsilon \), the coefficient of each power of \( \varepsilon \) must be zero. In particular, the coefficient of \( \varepsilon^2 \) must be zero and we get

\[ f(k, j) + f(j, k) = 0. \]
Problem 11.4

Use the preceding results to show that there always exist three components $B^b$ such that the space part of the antisymmetric tensor $f$ has the form

$$f_{sr} = \varepsilon_{srb}B^b.$$ 

Answer 11.4

Because $f(a, b) = -f(b, a)$ for any two vectors $a, b$ this relation also holds for the spacelike basis vectors and $f(e_s, e_r) = -f(e_r, e_s)$ or $f_{sr} = -f_{rs}$. The independent components of $f_{rs}$ are just the ones with $r < s$. The components with $r = s$ are all zero and the components with $r > s$ are just the sign reversed ones. The independent components are then just these three: $f_{12}, f_{13}, f_{23}$. Now check the proposed form $\varepsilon_{srb}B^b$ and notice that it is antisymmetric in $s, r$ so all we have to check are the three independent components

$$
\begin{align*}
  f_{12} &= \varepsilon_{123}B^b = \varepsilon_{123}B^3 = B^3 \\
  f_{13} &= \varepsilon_{132}B^b = \varepsilon_{132}B^2 = -B^2 \\
  f_{23} &= \varepsilon_{231}B^b = \varepsilon_{231}B^1 = B^1
\end{align*}
$$

so if we choose

$$
B^1 = f_{23}, \quad B^2 = -f_{13}, \quad B^3 = f_{12}
$$

the relation will hold for all components and we will have

$$f_{sr} = \varepsilon_{srb}B^b.$$