Exercise 08

Please attempt all of the following problems before the due date. Your grade on this assignment will be calculated from the best three answers.

Problem 08.1

Find the components $g_{ij}$ of the metric tensor in a basis that consists of orthonormal vectors. (Yes, it is that easy!)

Answer 08.1

$$g_{ij} = \begin{cases} 
0; i \neq j \\
\pm 1; i = j 
\end{cases} .$$

Problem 08.2

Define the form $g(v)$ by requiring $g(v)(u) = g(v, u)$ for any vector $u$. Expand the vector $v$ in terms of basis vectors and $g(v)$ in terms of the dual basis forms

$$v = v^r e_r$$
$$g(v) = v^r \omega^r$$

and show that the coefficients in the two expansions are connected by

$$v_s = v^r g_{rs} .$$

Answer 08.2

$$g(v)(u) = g(v, u)$$
$$v^r \omega^s (e_s) = g(v^r e_r, e_s)$$
$$v^r \delta^r_s = v^r g_{rs}$$

Problem 08.3

Insert basis vectors and forms into the definitions of the tensors $g$ and $g^{-1}$ and show that the components

$$g^{ij} := g^{-1}(\omega^i, \omega^j)$$
are related to the components \( g_{ij} \) of the metric tensor \( g \) by the system of equations:

\[
g_{ir} g_{rj} = \delta_i^j
\]

so that the matrix of components of \( g^{-1} \) is the matrix inverse of the matrix of components of \( g \).

**Answer 08.3**

First insert basis vectors into the definition of \( g \):

\[
g(u) \cdot v = g(u, v)
g(e_i) \cdot e_r = g(e_i, e_r) = g_{ir}
g(e_i) = \omega^r (g(e_i) \cdot e_r) = \omega^r g_{ir}
\]

Now insert basis vectors into the definition of \( g^{-1} \):

\[
g^{-1}(g(v)) = v\]
\[
g^{-1}(g(e_i)) = e_i\]
\[
g^{-1}(g_{ir} \omega^r) = e_i \quad \text{(Use the basis vector definition of} \ g)\]
\[
g_{ir} g^{-1}(\omega^r) = e_i\]
\[
g_{ir} g^{-1}(\omega^r) \cdot \omega^j = e_i \cdot \omega^j\]
\[
g^{-1}(\omega^r) \cdot \omega^j = g^{rj} \quad \text{Basis forms in the definition of} \ g^{-1}(\alpha, \beta)\]
\[
e_i \cdot \omega^j = \delta_i^j\]
\[
g_{ir} g^{rj} = \delta_i^j
\]

**Problem 08.4**

Find the dot product \( \alpha \cdot \beta \) when the form components are

\[
[\alpha] = \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix}, \quad [\beta] = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}
\]

and the metric tensor components are

\[
[g] = \begin{bmatrix} -\cos \theta & \sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{bmatrix}
\]

(Hint: Just set it up, put the cursor after it, hit CTRL-E and let Maple do all the work.)

**Answer 08.4**

\[
\alpha \cdot \beta = [\alpha]^T [g]^{-1} [\beta]
\]
\[
\begin{bmatrix}
    u_1 \\
v_1 \\
u_2 \\
v_2
\end{bmatrix}^T \begin{bmatrix}
    -\cos \theta & \sin \theta & 0 & 0 \\
    \sin \theta & \cos \theta & 0 & 0 \\
    0 & 0 & \cos \theta & \sin \theta \\
    0 & 0 & \sin \theta & \cos \theta
\end{bmatrix}^{-1} \begin{bmatrix}
    x_1 \\
x_2 \\
y_1 \\
y_2
\end{bmatrix} = \\
( -u_1 \frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta} + u_2 \frac{\sin \theta}{\cos^2 \theta + \sin^2 \theta} ) x_1 + \left( u_1 \frac{\sin \theta}{\cos^2 \theta + \sin^2 \theta} + u_2 \frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta} \right) x_2 \\
+ \left( u_1 \frac{\cos \theta}{\cos^2 \theta - \sin^2 \theta} - u_2 \frac{\sin \theta}{\cos^2 \theta - \sin^2 \theta} \right) y_1 + \left( -u_1 \frac{\sin \theta}{\cos^2 \theta - \sin^2 \theta} + u_2 \frac{\cos \theta}{\cos^2 \theta - \sin^2 \theta} \right) y_2 = \\
( -u_1 \cos \theta + u_2 \sin \theta ) x_1 + ( u_1 \sin \theta + u_2 \cos \theta ) x_2 \\
+ \left( u_1 \frac{\cos \theta}{\cos^2 \theta - \sin^2 \theta} - u_2 \frac{\sin \theta}{\cos^2 \theta - \sin^2 \theta} \right) y_1 + \left( -u_1 \frac{\sin \theta}{\cos^2 \theta - \sin^2 \theta} + u_2 \frac{\cos \theta}{\cos^2 \theta - \sin^2 \theta} \right) y_2
\]