Exercise 07

Please attempt all of the following problems before the due date. Your grade on this assignment will be calculated from the best three answers.

Problem 07.1

Substitute the expansions $\alpha = \alpha_i \omega^i$, $u = u^j e_j$, $v = v^k e_k$ into $T(\alpha, u, v)$ and show that

$$T(\alpha, u, v) = T^i_{jk} \alpha_i u^j v^k$$

where the tensor components $T^i_{jk}$ are defined by

$$T^i_{jk} := T(\omega^i, e_j, e_k).$$

Answer 07.1

$$T(\alpha, u, v) = T(\alpha_i \omega^i, u^j e_j, v^k e_k) = T(\omega^i, e_j, e_k) \alpha_i u^j v^k = T^i_{jk} \alpha_i u^j v^k$$

Problem 07.2

Show that the tensor product $u \otimes u$ of a vector with itself is a symmetric tensor.

Answer 07.2

$$u \otimes u(\alpha, \beta) = u(\alpha) u(\beta) = u(\beta) u(\alpha) = u \otimes u(\beta, \alpha)$$

Problem 07.3

Suppose that the tensor $A$ is antisymmetric and that $U$ is symmetric. Show that the contracted tensor with components $U^{irs} A_{rsk}$ is zero. Hint: Repeated indexes can be re-named since they are summed over.

Answer 07.3

From the index symmetries: $U^{irs} A_{rsk} = -U^{irs} A_{rsk} = -U^{isr} A_{rsk}$. Rename the indexes $s$ and $r$. $U^{irs} A_{rsk} = -U^{isr} A_{rsk} = -U^{irs} A_{rsk}$

$U^{irs} A_{rsk} = -U^{irs} A_{rsk}$

which implies that it is zero.
Problem 07.4

Show that, for an $n$-dimensional space, the components of the totally anti-symmetric rank-$n$ tensor $\varepsilon$ defined by

$$\varepsilon (\omega^1, \omega^2, \omega^3, ... \omega^n) = 1$$

are given by the expression

$$\varepsilon^{i_1 i_2 i_3 ... i_n} = \begin{cases} 0 ; & i_a = i_b \text{ for any } a, b \\ 1 ; & i_1 i_2 i_3 ... i_n \text{ an even permutation of } 123...n. \\ -1 ; & i_1 i_2 i_3 ... i_n \text{ an odd permutation of } 123...n. \end{cases}$$

in the text.

Answer 07.4

$$\varepsilon (\omega^1, \omega^2, \omega^3, ... \omega^n) = \varepsilon^{123...n} = 1$$

so $\varepsilon^{i_1 i_2 i_3 ... i_n} = 1$ for $i_1 i_2 i_3 ... i_n = 123...n$.

Each reversal of two indexes changes the sign, so the value is 1 for an even number of reversals and -1 for an odd number.

If $i_a = i_b$ for any $a, b$ then reversing the indexes $i_a$ and $i_b$ does not change anything but must reverse the sign. That is possible only if the value is zero.