Exam 01

Please attempt all of the following problems before the due date. Your grade on this exam will be calculated from your four best answers.

Problem x01.1

A two-dimensional manifold is described by two coordinate charts that map parts of it onto $C^2$, the space of complex numbers. The transition function from one complex coordinate $z_1(P)$ of a point $P$ to the other complex coordinate $z_1(P)$ is given by

$$z_2 = \frac{1}{z_1}$$

a) Describe the range and domain of each chart.
b) What is this manifold?

Solution 1  a) Each chart has the entire complex plane $C^2$ as its range. Each chart omits the point that the other chart maps to zero, so the domain of each chart is the manifold minus one point. b) It is a sphere with one pole mapped to zero and the other to infinity.

Problem x01.2

Given the vector fields

$$u = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

$$v = y \frac{\partial}{\partial y} - x \frac{\partial}{\partial x}$$

and the functions

$$r = \sqrt{x^2 + y^2}$$

$$f = xy$$

$$g = x^2 - y^2$$

Calculate the functions

$$\nabla_u f =$$

$$\nabla_{ru} f =$$

$$\nabla_v f =$$

$$(\mathcal{L}_u v) f =$$

$$\mathcal{L}_u (vf) =$$
Solution 2

\[ \nabla_u f = uf = \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) f = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = xy + yx = 2xy \]

\[ \nabla_{ru} f = ruf = 2xy \sqrt{x^2 + y^2} = 2xyr \]

\[ \nabla_{ur} f = urf = u( rf ) = (ur) f + ru f \]

\[ ur = \]

\[ \frac{\partial \sqrt{x^2 + y^2}}{\partial x} = \frac{x}{r} \]

\[ \frac{\partial r}{\partial y} = \frac{y}{r} \]

\[ ur = \frac{x}{r} + \frac{y}{r} = r \]

\[ \nabla_{ru} r f = r f + ruf = rxy + 2xyr = 3xyr \]

\[ (\mathcal{L}_v u f) = [u, v] f = uf v - vuf \]

\[ = \left( x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right) \left( x \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial x} \right) - \left( x \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial x} \right) \left( x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right) \]

\[ = x \frac{\partial f}{\partial x} \left( x \frac{\partial f}{\partial y} \right) - x \frac{\partial f}{\partial x} \left( y \frac{\partial f}{\partial x} \right) + y \frac{\partial f}{\partial y} \left( x \frac{\partial f}{\partial y} \right) - y \frac{\partial f}{\partial y} \left( y \frac{\partial f}{\partial x} \right) \]

\[ = x \frac{\partial^2 f}{\partial y \partial x} + x^2 \frac{\partial^2 f}{\partial y^2} - xy \frac{\partial^2 f}{\partial x^2} + yx \frac{\partial^2 f}{\partial y^2} - y \frac{\partial^2 f}{\partial x^2} + y \frac{\partial^2 f}{\partial x \partial y} \]

\[ -x^2 \frac{\partial^2 f}{\partial x \partial y} - x \frac{\partial f}{\partial y} - xy \frac{\partial^2 f}{\partial y^2} + y \frac{\partial f}{\partial x} + yx \frac{\partial^2 f}{\partial x^2} + y \frac{\partial^2 f}{\partial x \partial y} \]

\[ (\mathcal{L}_v u f) = x \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial x} - x \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x} = 0 \]

\[ L_a (vf) = (\mathcal{L}_v u f) + v \mathcal{L}_a f = vuf \]

\[ = \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) (2xy) = 2 (x^2 - y^2) = 2g \]

Problem x01.3

2
A two-dimensional manifold is described by a complex coordinate chart that maps the point $P$ into $z(P)$ where

$$z = x^1 + ix^2$$

Within this chart, the manifold has metric

$$ds^2 = \frac{dz dz^*}{(1 + zz^*)^2}$$

where

$$z^* = x^1 - ix^2$$

a) Find the connection coefficients for the real coordinate basis $e_i = \frac{\partial}{\partial x^i}$.

**Solution 3** The non-zero metric components are

$$g_{11} = g_{22} = \frac{1}{(1 + r^2)^2}, \quad r^2 = (x^1)^2 + (x^2)^2$$

Note that

$$\frac{\partial r^2}{\partial x^i} = 2x^i$$

The non-zero first derivatives are

$$g_{11,i} = g_{22,i} = -4 \frac{1}{(1 + r^2)^3} x^i$$

The connection coefficients of the second kind are then

$$\Gamma_{abd} = \frac{1}{2} (g_{ab,d} - g_{bd,a} + g_{ad,b})$$

$$\Gamma_{111} = \frac{1}{2} g_{11,1} = -2 \frac{1}{(1 + r^2)^3} x^1$$

$$\Gamma_{112} = \Gamma_{121} = \frac{1}{2} (g_{11,2} - g_{12,1} + g_{12,1}) = \frac{1}{2} g_{11,2} = -2 \frac{1}{(1 + r^2)^3} x^2$$

$$\Gamma_{122} = \frac{1}{2} (g_{12,2} - g_{22,1} + g_{12,2}) = -\frac{1}{2} g_{22,1} = 2 \frac{1}{(1 + r^2)^3} x^1$$

The metric is unchanged if the coordinates are exchanged, so we also have

$$\Gamma_{222} = -2 \frac{1}{(1 + r^2)^3} x^2$$

$$\Gamma_{221} = \Gamma_{212} = -2 \frac{1}{(1 + r^2)^3} x^1$$

$$\Gamma_{211} = 2 \frac{1}{(1 + r^2)^3} x^2$$

3
Raise the first index using the inverse metric components

\[ g^{11} = g^{22} = (1 + r^2)^2 \]

\[
\begin{align*}
\Gamma^1_{11} &= -2 \frac{1}{1 + r^2} x^1 \\
\Gamma^1_{12} &= \Gamma^1_{21} = -2 \frac{1}{1 + r^2} x^2 \\
\Gamma^1_{22} &= 2 \frac{1}{1 + r^2} x^1 \\
\Gamma^2_{22} &= -2 \frac{1}{1 + r^2} x^2 \\
\Gamma^2_{21} &= \Gamma^2_{12} = -2 \frac{1}{1 + r^2} x^1 \\
\Gamma^2_{11} &= 2 \frac{1}{1 + r^2} x^1
\end{align*}
\]

b) Find the form that Laplace’s equation

\[ \nabla^2 f = g^{ab} D_a D_b f = g^{ab} f_{;ab} \]

takes on this manifold (for an arbitrary function \( f \)).

**Solution 4**

\[
\begin{align*}
f_{;ab} &= f_{,ab} - f_{,k} \Gamma^k_{ab} \\
\nabla^2 f &= g^{ab} f_{;ab} - f_{,k} \Gamma^k_{ab} g^{ab} = (1 + r^2)^2 (f_{,11} + f_{,22}) - f_{,k} \Gamma^k \\
\Gamma^k &= \Gamma^k_{ab} g^{ab} = (1 + r^2)^2 (\Gamma^k_{11} + \Gamma^k_{22}) \\
\Gamma^1 &= (1 + r^2)^2 (\Gamma^1_{11} + \Gamma^1_{22}) = (1 + r^2)^2 \left( -2 \frac{1}{1 + r^2} x^1 + 2 \frac{1}{1 + r^2} x^1 \right) = 0 \\
\Gamma^2 &= 0 \\
\nabla^2 f &= (1 + r^2)^2 (f_{,11} + f_{,22})
\end{align*}
\]

**Problem x01.4**

Let \( x, y, z \) be the usual Cartesian coordinates on space with metric tensor

\[ ds^2 = dx^2 + dy^2 + dz^2 \]
Assume the usual metric-compatible connection so that the basis vectors $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial x$ are auto-parallel and show that the vector fields

$$\Omega_z = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$$

$$\Omega_x = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}$$

$$\Omega_y = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}$$

are Killing vector fields.

**Solution 5** The coordinates can be cyclically permuted, so we just have to do one of these. The components of $\Omega_z$ are

$$(\Omega_z)_1 = -y, \quad (\Omega_z)_2 = x$$

The components of its covariant derivative are then

$$(\Omega_z)_{1,a} = (\Omega_z)_{1,a}, \quad (\Omega_z)_{2,a} = (\Omega_z)_{2,a}$$

$$\begin{align*}
(\Omega_z)_{1:1} &= 0, \quad (\Omega_z)_{1:2} = -1 \\
(\Omega_z)_{2:1} &= 1, \quad (\Omega_z)_{2:2} = 0
\end{align*}$$

so the vector field solves Killing’s equation

$$(\Omega_z)_{a:b} + (\Omega_z)_{b:a} = 0$$

**Problem x01.5**

For flat three-dimensional space in spherical coordinates, the metric tensor is

$$ds^2 = dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2\right)$$

We have learned two ways to obtain the connection coefficients for the basis fields

$$e_1 = \frac{\partial}{\partial r}, \quad e_2 = \frac{\partial}{\partial \theta}, \quad e_3 = \frac{\partial}{\partial \varphi}$$

a very tedious method that expresses everything in the autoparallel Cartesian bases, and a somewhat less tedious method that uses the general expression for a metric-compatible connection.

For this problem, use a third method: Write the equation for a straight line (described by functions $r(\lambda), \theta(\lambda), \varphi(\lambda)$ of a curve parameter $\lambda$) in terms of the connection coefficients and compare it to what you get when you obtain...
the equation by varying the curve and setting the variation of the path length to zero.

\[ L = \int ds \]
\[ D_F L = \delta L = 0 \]

This method can be very fast because it produces only the non-zero coefficients.

BIG HINT: Use an arbitrary curve parameter \( \lambda \) when you first take the derivative and switch to using the path length \( \lambda = s \) as the parameter before you do any more manipulation of the result.

**Solution 6** The equation for a straight line is

\[ D_u u = 0 \]

where

\[ u = \frac{d}{ds} = \frac{dr}{d\lambda} \frac{\partial}{\partial r} + \frac{d\theta}{ds} \frac{\partial}{\partial \theta} + \frac{d\varphi}{ds} \frac{\partial}{\partial \varphi} \]

Label the coordinates

\[ r = x^1, \quad \theta = x^2, \quad \varphi = x^3 \]

and obtain the equation in the form

\[ \frac{d^2 x^a}{ds^2} + \Gamma^{ab}_{\;\,c} \frac{dx^a}{ds} \frac{dx^b}{ds} = 0 \]

The path length is

\[ L = \int_{1}^{2} ds = \int_{1}^{2} \sqrt{ds^2} = \int_{1}^{2} d\lambda \sqrt{\left( \frac{dr}{d\lambda} \right)^2 + r^2 \left( \frac{d\theta}{d\lambda} \right)^2 + \sin^2 \theta \left( \frac{d\varphi}{d\lambda} \right)^2} \]

Now calculate \( D_F L \cdot (\delta r, 0, 0) \). It is easiest to split this up and start with calculating \( D_F L \cdot (\delta r, 0, 0) \) by replacing \( r \) by \( r + \varepsilon \delta r \) and differentiating with respect to \( \varepsilon \).

\[ D_F L \cdot (\delta r, 0, 0) = \frac{d}{d\varepsilon} \left[ \int_{1}^{2} d\lambda \sqrt{\left( \frac{dr}{d\lambda} + \varepsilon \frac{d\delta r}{d\lambda} \right)^2 + (r + \varepsilon \delta r)^2 \left( \frac{d\theta}{d\lambda} \right)^2 + \sin^2 \theta \left( \frac{d\varphi}{d\lambda} \right)^2} \right]_{\varepsilon=0} \]

\[ = \int_{1}^{2} d\lambda \frac{d}{d\varepsilon} \left[ \sqrt{\left( \frac{dr}{d\lambda} + \varepsilon \frac{d\delta r}{d\lambda} \right)^2 + (r + \varepsilon \delta r)^2 \left( \frac{d\theta}{d\lambda} \right)^2 + \sin^2 \theta \left( \frac{d\varphi}{d\lambda} \right)^2} \right]_{\varepsilon=0} \]
\[
\frac{d}{d\varepsilon} \sqrt{\left( \frac{dr}{d\lambda} + \varepsilon \frac{d\delta r}{d\lambda} \right)^2 + (r + \varepsilon \delta r)^2 \left( \frac{d\theta}{d\lambda} \right)^2 + \sin^2 \theta \left( \frac{d\varphi}{d\lambda} \right)^2} \bigg|_{\varepsilon = 0}
\]

\[
= \frac{\left( \frac{dr}{d\lambda} \right)^2 + r \delta r \left( \frac{d\theta}{d\lambda} \right)^2 + \sin^2 \theta \left( \frac{d\varphi}{d\lambda} \right)^2}{\sqrt{\left( \frac{dr}{d\lambda} \right)^2 + r^2 \left( \frac{d\theta}{d\lambda} \right)^2 + \sin^2 \theta \left( \frac{d\varphi}{d\lambda} \right)^2}}
\]

\[
= \int_1^2 d\lambda \frac{\left( \frac{dr}{d\lambda} \right)^2 + r \delta r \left( \frac{d\theta}{d\lambda} \right)^2 + \sin^2 \theta \left( \frac{d\varphi}{d\lambda} \right)^2}{ds/d\lambda}
\]

The integral is invariant under a change in curve parameter, so just choose \( \lambda = s \)

\[
D_F L \cdot (\delta r, 0, 0) = \int_1^2 ds \left[ \left( \frac{dr}{ds} \right)^2 \frac{d\delta r}{ds} + r \delta r \left( \frac{d\theta}{ds} \right)^2 + \frac{\sin^2 \left( \frac{d\varphi}{ds} \right)^2}{ds} \right] \delta r
\]

To get an extremal length, we need to hold the variation fixed at the end points, so the first term vanishes. The variation function \( \delta r \) is otherwise arbitrary, so the condition

\[
D_F L \cdot (\delta r, 0, 0) = 0
\]

requires

\[
- \frac{d^2 r}{ds^2} + r \left( \frac{d\theta}{ds} \right)^2 + \sin^2 \theta \left( \frac{d\varphi}{ds} \right)^2 = 0
\]

Now work on the condition \( D_F L \cdot (0, \delta \theta, 0) = 0 \). We can do this faster because we know how it is going to go.

\[
D_F L \cdot (0, \delta \theta, 0) = \int_1^2 d\lambda \frac{d}{d\varepsilon} \sqrt{\left( \frac{dr}{d\lambda} \right)^2 + r^2 \left( \frac{d\theta}{d\lambda} + \varepsilon \frac{d\delta \theta}{d\lambda} \right)^2 + \sin^2 \left( \theta + \varepsilon \delta \theta \right) \left( \frac{d\varphi}{d\lambda} \right)^2} \bigg|_{\varepsilon = 0}
\]

\[
= \int_1^2 d\lambda \left[ \frac{2 \frac{d\theta}{d\lambda} \frac{d\delta \theta}{d\lambda} + r^2 \cos \theta \sin \theta \left( \frac{d\varphi}{d\lambda} \right)^2}{ds/d\lambda} \right] \delta \theta
\]

\[
= \int_1^2 ds \left[ - \frac{d}{ds} \left( \frac{r^2 \frac{d\theta}{ds} + \frac{d\varphi}{ds} \sin \theta}{d\lambda} \right) \delta \theta + \frac{r^2 \cos \theta \sin \theta \left( \frac{d\varphi}{ds} \right)^2}{ds/d\lambda} \right] \delta \theta
\]

\[
= \int_1^2 ds \left[ \frac{d}{ds} \left( \frac{r^2 \frac{d\theta}{ds} + \frac{d\varphi}{ds} \sin \theta}{d\lambda} \right) + \frac{r^2 \cos \theta \sin \theta \left( \frac{d\varphi}{ds} \right)^2}{ds/d\lambda} \right] \delta \theta
\]
\[ -\frac{d}{ds} \left( r^2 \frac{d\theta}{ds} \right) + r^2 \cos \theta \sin \theta \left( \frac{d\varphi}{ds} \right)^2 = 0 \]

or

\[ -\frac{d^2 \theta}{ds^2} - \frac{2}{r} \left( \frac{dr}{ds} \right) \left( \frac{d\theta}{ds} \right) + \cos \theta \sin \theta \left( \frac{d\varphi}{ds} \right)^2 = 0 \]

Finally evaluate \( D_F L \cdot (0, 0, \delta \varphi) = 0 \)

\[
D_F L \cdot (0, 0, \delta \varphi) = \int_1^2 d\lambda \frac{d}{d\varepsilon} \left[ \left( \frac{dr}{d\lambda} \right)^2 + r^2 \left( \left( \frac{d\theta}{d\lambda} \right)^2 + \sin^2 \theta \left( \frac{d\varphi}{d\lambda} + \varepsilon \frac{d\delta \varphi}{d\lambda} \right)^2 \right) \right]_{\varepsilon=0} \\
= \int_1^2 d\lambda r^2 \sin^2 \theta \left( \frac{d\varphi}{d\lambda} \frac{d\delta \varphi}{d\lambda} \right) = \int_1^2 ds r^2 \sin^2 \theta \left( \frac{d\varphi}{ds} \frac{d\delta \varphi}{ds} \right) \\
= -\int_1^2 ds \frac{d}{ds} \left( r^2 \sin^2 \theta \frac{d\varphi}{ds} \right) \delta \varphi \\
so that \\
\[
-\frac{d}{ds} \left( r^2 \sin^2 \theta \frac{d\varphi}{ds} \right) = 0 \\
\]

or

\[
-2r^2 \frac{dr}{ds} \sin^2 \theta \frac{d\varphi}{ds} - 2r \cos \theta \sin \theta \frac{d\theta}{ds} \frac{d\varphi}{ds} - r^2 \sin^2 \theta \frac{d^2 \varphi}{ds^2} = 0 \\
\frac{d^2 \varphi}{ds^2} - \frac{2}{r} \frac{dr}{ds} \frac{d\varphi}{ds} - \frac{2}{r} \cot \theta \frac{d\theta}{ds} \frac{d\varphi}{ds} = 0 \\
\]

Now collect the results and express them in terms of indexed coordinates

\[
\frac{d^2 x^1}{ds^2} - r \left( \frac{dx^1}{ds} \right)^2 - r \sin^2 \theta \left( \frac{dx^3}{ds} \right)^2 = 0 \\
\frac{d^2 x^2}{ds^2} + \frac{2}{r} \left( \frac{dx^1}{ds} \right) \left( \frac{dx^2}{ds} \right) - \cos \theta \sin \theta \left( \frac{dx^3}{ds} \right)^2 = 0 \\
\frac{d^2 x^3}{ds^2} + \frac{2}{r} \frac{dx^1}{ds} \frac{dx^3}{ds} + \frac{2}{r} \cot \theta \frac{dx^2}{ds} \frac{dx^3}{ds} = 0 \\
\]

Compare these to

\[
\Gamma^{1\,ab}_{\phantom{1\,ab}11} \frac{dx^a}{ds} \frac{dx^b}{ds} = 0 \\
\Gamma^{2\,ab}_{\phantom{2\,ab}12} \frac{dx^a}{ds} \frac{dx^b}{ds} = 0 \\
\Gamma^{3\,ab}_{\phantom{3\,ab}13} \frac{dx^a}{ds} \frac{dx^b}{ds} = 0 \\
\]

and read off the non-zero terms

\[
\Gamma_{11}^{1} = -r, \quad \Gamma_{33}^{1} = -r \sin^2 \theta \\
\Gamma_{12}^{2} = \frac{1}{r}, \quad \Gamma_{33}^{2} = -\cos \theta \sin \theta \\
\Gamma_{13}^{3} = \frac{1}{r}, \quad \Gamma_{23}^{3} = \frac{1}{r} \cot \theta \\
\]

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