Exam 01

Please attempt all of the following problems before the due date. Your grade on this exam will be calculated from your four best answers.

Problem x01.1

Describe a circle $S$ using two charts $\varphi_1$ and $\varphi_2$ that map subsets of the circle into the interval $I = \{-\pi < \theta < \pi\}$.

a) Describe the range and domain of each chart.

b) Describe a transition function between these charts in terms of the charts and their inverses. Give an example of such a transition function.

Answer x01.1

a) $\varphi_1 : U_1 \rightarrow I$, and $\varphi_2 : U_2 \rightarrow I$ where $U_1$ and $U_2$ are different open subsets of $S$.

b) For a transition function, we want a map that goes from $I$ to $S$ and then from $S$ to $I$. For a map from $I$ to $S$, use $(\varphi_1)^{-1} : I \rightarrow U_1 \subset S$. For a map from $S$ to $I$, use $\varphi_2 : U_2 \rightarrow I$. The composite map $((\varphi_1)^{-1} \circ \varphi_2) : I \rightarrow I$ is a transition function.

An example for a circle around the origin of the $x$-$y$ plane, would be $\varphi_1(P) =$ angle counterclockwise from the positive $x$-axis. $\varphi_2(P) =$ angle counterclockwise from the negative $x$-axis. Then

$$\varphi_2(P) = \begin{cases} \varphi_1(P) + \pi & \text{for } -\pi < \varphi_1(P) < 0 \\ \varphi_1(P) - \pi & \text{for } 0 < \varphi_1(P) < \pi \end{cases}$$

or, in terms of the transition function

$$\varphi_2(P) = \varphi_2((\varphi_1)^{-1}(\varphi_1(P))) = \begin{cases} \varphi_1(P) + \pi & \text{for } -\pi < \varphi_1(P) < 0 \\ \varphi_1(P) - \pi & \text{for } 0 < \varphi_1(P) < \pi \end{cases}$$

Define $\varphi_1(P) = \theta_1$

$$\varphi_2((\varphi_1)^{-1}(\theta_1)) = \begin{cases} \theta_1 + \pi & \text{for } -\pi < \theta_1 < 0 \\ \theta_1 - \pi & \text{for } 0 < \theta_1 < \pi \end{cases}$$

or

$$\theta_2 = ((\varphi_1)^{-1} \circ \varphi_2)(\theta_1) = \begin{cases} \theta_1 + \pi & \text{for } -\pi < \theta_1 < 0 \\ \theta_1 - \pi & \text{for } 0 < \theta_1 < \pi \end{cases}$$

is the actual transition function. It is defined everywhere on $I$ except at zero.
The next four problems are set in the space $\mathbb{R}^2$ of real number pairs $(x, y)$, and use the vector fields

$$u = \frac{x}{\partial x} + \frac{y}{\partial y}$$
$$v = \frac{x}{\partial y} - \frac{y}{\partial x}$$

and the functions

$$r = \sqrt{x^2 + y^2}$$
$$f = xy$$
$$g = x^2 - y^2$$

Problem x01.2

Calculate the functions $ur, vr, uf, vf, ug, vg$.

Answer x01.2

$$ur = \left(\frac{x}{\partial x} + \frac{y}{\partial y}\right) \sqrt{x^2 + y^2} = \frac{x}{\partial x} \frac{\partial \sqrt{x^2 + y^2}}{\partial x} + \frac{y}{\partial y} \frac{\partial \sqrt{x^2 + y^2}}{\partial y} = \frac{x^2}{\sqrt{x^2 + y^2}} + \frac{y^2}{\sqrt{x^2 + y^2}}$$

$$ur = r$$

$$vr = x \frac{\partial \sqrt{x^2 + y^2}}{\partial y} - y \frac{\partial \sqrt{x^2 + y^2}}{\partial x} = 0$$

$$vr = 0$$

$$uf = \left(\frac{x}{\partial x} + \frac{y}{\partial y}\right) xy = x \frac{\partial (xy)}{\partial x} + y \frac{\partial (xy)}{\partial y} = 2xy$$

$$uf = 2f$$

$$vf = x \frac{\partial (xy)}{\partial y} - y \frac{\partial (xy)}{\partial x} = x^2 - y^2$$

$$vf = g$$

$$ug = x \frac{\partial (x^2 - y^2)}{\partial x} + y \frac{\partial (x^2 - y^2)}{\partial y} = 2x^2 - 2y^2$$

$$ug = 2g$$

$$vg = x \frac{\partial (x^2 - y^2)}{\partial y} - y \frac{\partial (x^2 - y^2)}{\partial x} = -4xy$$

$$vg = 4f$$

Problem x01.3
Calculate the Lie derivative of the vector field \( u \) in the direction of \( v \).

Answer x01.3

\[
\mathcal{L}_v u = [v, u] = vu - uv
\]

\[
\mathcal{L}_v u = \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) - \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)
\]

Remember that \( \mathcal{L}_v u \) is a vector field, so it is actually an operator on functions. It is sometimes helpful to show it acting on a function.

\[
(\mathcal{L}_v u) F = \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) F - \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) F
\]

Now go through the terms one by one.

\[
\begin{align*}
\left( x \frac{\partial}{\partial y} \right) x \frac{\partial F}{\partial x} &= x \frac{\partial (x \frac{\partial F}{\partial x})}{\partial y} = x \frac{\partial^2 F}{\partial x \partial y} \\
\left( x \frac{\partial}{\partial y} \right) y \frac{\partial F}{\partial y} &= x \frac{\partial F}{\partial y} + xy \frac{\partial^2 F}{\partial y \partial x} \\
\left( -y \frac{\partial}{\partial x} \right) x \frac{\partial F}{\partial x} &= -y \frac{\partial F}{\partial x} - yx \frac{\partial^2 F}{\partial x \partial y} \\
\left( -y \frac{\partial}{\partial x} \right) y \frac{\partial F}{\partial y} &= -y \frac{\partial^2 F}{\partial y \partial x} \\
\left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \left( x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} \right) &= (x^2 - y^2) \frac{\partial^2 F}{\partial x \partial y} + xy \left( \frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x^2} \right) + x \frac{\partial F}{\partial y} - y \frac{\partial F}{\partial x}
\end{align*}
\]

\[
\begin{align*}
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \left( x \frac{\partial F}{\partial x} - y \frac{\partial F}{\partial y} \right) &= (x^2 - y^2) \frac{\partial^2 F}{\partial y \partial x} + yx \left( \frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x^2} \right) + x \frac{\partial F}{\partial y} - y \frac{\partial F}{\partial x}
\end{align*}
\]

Subtract the second expression from the first and get

\[
\mathcal{L}_v u = 0
\]
The following two problems use coordinates
\[ x^1 = x, \quad x^2 = y \]
and metric tensor components in the corresponding holonomic basis
\[ g_{ij} = \Psi^i \delta_{ij} \]
where \( \delta_{ij} \) is the Kronecker delta.

Problem x01.4

Calculate the covariant derivative of the vector field \( u \) in the direction of \( v \). In other words, calculate \( D_v u \).

Expand the vectors in terms of components and the basis vectors \( e_i = \frac{\partial}{\partial x^i} \)

\[
D_v u = D_v e_i (u^j e_j) \\
= v^d D_e e_i (u^j e_j) \\
= v^d (D_e e_i (u^k) e_k + v^d u^j (D_e e_j))
\]

Note the dummy index change from \( j \) to \( k \) in the first term. Now write out each term.

\[
D_e e_i (u^k) = u^k, d = \frac{\partial u^k}{\partial x^d} \\
D_e e_i = \Gamma^k_{jd} e_k
\]

and obtain

\[
D_v u = v^d \left( \frac{\partial u^k}{\partial x^d} + u^j \Gamma^k_{jd} \right) e_k
\]

Alternatively, you could just write the component form of the derivative

\[
D_v u = v^d u^k, d e_k
\]

where

\[
u^k, d = \frac{\partial u^k}{\partial x^d} + u^j \Gamma^k_{jd}
\]

Next, calculate the connection coefficients \( \Gamma^k_{jd} \) for this metric from the formula

\[
\Gamma^k_{jd} = \frac{1}{2} g^{ka} \left( \frac{\partial g_{aj}}{\partial x^d} - \frac{\partial g_{dj}}{\partial x^a} + \frac{\partial g_{ad}}{\partial x^j} \right)
\]

\[ g_{ij} = \Psi^i \delta_{ij} \]
\[ g^{kb} = \Psi^{-4} \delta^{kb} \]

\[ \Gamma^k_{jd} = \frac{1}{2} \Psi^{-4} \delta^{ka} g^{k\alpha} \left( \frac{\partial \Psi}{\partial x^d} \delta_{aj} - \frac{\partial \Psi}{\partial x^a} \delta_{dj} + \frac{\partial \Psi}{\partial x^j} \delta_{ad} \right) \]

\[ = 2\Psi^{-1} \left( \frac{\partial \Psi}{\partial x^d} \delta^{ka} \delta_{aj} - \frac{\partial \Psi}{\partial x^a} \delta^{ka} \delta_{dj} + \frac{\partial \Psi}{\partial x^j} \delta^{ka} \delta_{ad} \right) \]

\[ = 2\Psi^{-1} \left( \frac{\partial \Psi}{\partial x^d} \delta^k - \frac{\partial \Psi}{\partial x^a} \delta_{dj} + \frac{\partial \Psi}{\partial x^j} \delta^k \right) \]

or

\[ D_v u = v^d \left( \frac{\partial u^k}{\partial x^d} + 2\Psi^{-1} u^j \left( \frac{\partial \Psi}{\partial x^d} \delta^k \delta_{aj} - \frac{\partial \Psi}{\partial x^a} \delta^k \delta_{dj} + \frac{\partial \Psi}{\partial x^j} \delta^k \delta_{ad} \right) \right) e_k \]

\[ = v^d \left( \frac{\partial u^k}{\partial x^d} + 2\Psi^{-1} \left( u^j \frac{\partial \Psi}{\partial x^d} \delta^k \delta_{aj} - u^d \frac{\partial \Psi}{\partial x^d} \delta^k \delta_{dj} + u^j \frac{\partial \Psi}{\partial x^d} \delta^k \delta_{ad} \right) \right) e_k \]

or in component form

\[ u^k : d = \frac{\partial u^k}{\partial x^d} + 2\Psi^{-1} \left( u^k \frac{\partial \Psi}{\partial x^d} - u^d \frac{\partial \Psi}{\partial x^d} + u^j \frac{\partial \Psi}{\partial x^d} \delta^k \delta_{ad} \right) \]

Notice that we get one violation of the summation convention in the form

\[ v^d u^d = \sum_d v^d u^d \]

**Problem x01.5**

For a curve that maps parameter value \( t \) to the coordinates \( x(t), y(t) \) find the conditions that the functions \( x(t), y(t) \) must satisfy if the curve is to be a straight line in this geometry.

**Answer x01.5**

For a straight line, you want the tangent vector

\[ \mathbf{u} = \frac{\partial}{\partial t} = \frac{dx^i}{dt} \frac{\partial}{\partial x^i} = \dot{x}^i e_i \]

to be constant along the curve so that its derivative in the tangent direction is zero.

\[ D_u u = 0 \]
Now just set \( v = u \) in the previous result and obtain

\[
D_u u = u^d \left( \frac{\partial u^k}{\partial x^d} + 2\Psi^{-1} \left( u^k \frac{\partial \Psi}{\partial x^d} - u^d \frac{\partial \Psi}{\partial x^k} + u^j \frac{\partial \Psi}{\partial x^j} \delta^k_d \right) \right) e_k
\]

\[
= u^d \frac{\partial u^k}{\partial x^d} + 2\Psi^{-1} u^d \left( u^k \frac{\partial \Psi}{\partial x^d} - u^d \frac{\partial \Psi}{\partial x^k} + u^j \frac{\partial \Psi}{\partial x^j} \delta^k_d \right) e_k
\]

\[
= u^d \frac{\partial u^k}{\partial x^d} + 2\Psi^{-1} \left( u^d u^k \frac{\partial \Psi}{\partial x^d} - u^d u^k \frac{\partial \Psi}{\partial x^d} + u^d u^j \frac{\partial \Psi}{\partial x^j} \delta^k_d \right) e_k
\]

\[
= u^d \frac{\partial u^k}{\partial x^d} e_k + 2\Psi^{-1} \left( u^d u^k \frac{\partial \Psi}{\partial x^d} - \left( \sum_d u^d u^d \right) \frac{\partial \Psi}{\partial x^d} + u^d u^j \frac{\partial \Psi}{\partial x^j} \delta^k_d \right) e_k
\]

Notice that we get the first term into a more familiar form

\[
u_d \frac{\partial u^k}{\partial x^d} = \frac{dx^d}{dt} \frac{\partial u^k}{\partial x^d} = \frac{d}{dt} \left( u^k \right) = \frac{d^2 x^k}{dt^2} = \ddot{x}^k
\]

The equation of a straight line then becomes

\[
\dddot{x}^k + 2\Psi^{-1} \left( \dot{x}^k \dot{x}^d \frac{\partial \Psi}{\partial x^d} - \left( \sum_d \dot{x}^d \dot{x}^d \right) \frac{\partial \Psi}{\partial x^k} + \dot{x}^j \frac{\partial \Psi}{\partial x^j} \delta^k_d \right) = 0
\]

\[
\dddot{x}^k + 2\Psi^{-1} \left( \dot{x}^k \dot{x}^d \frac{\partial \Psi}{\partial x^d} - \left( \sum_d \dot{x}^d \dot{x}^d \right) \frac{\partial \Psi}{\partial x^k} + \dot{x}^j \dot{x}^d \frac{\partial \Psi}{\partial x^j} \right) = 0
\]

Two terms are just the same except for dummy indexes.

\[
\dddot{x}^k + 2\Psi^{-1} \left( 2\dot{x}^k \dot{x}^d \frac{\partial \Psi}{\partial x^d} - \left( \sum_d \dot{x}^d \dot{x}^d \right) \frac{\partial \Psi}{\partial x^k} \right) = 0
\]

We end up with a pair of second order ordinary differential equations. Notice that, if the conformal factor \( \Psi \) is a constant, the equations become just

\[
\dddot{x}^k = 0
\]