Hour Exam No.1

Please attempt all of the following problems before the due date. All problems count the same even though some are more complex than others.

Problem 1

Let t, x, y, z be Minkowski coordinates on a spacetime with the metric tensor of Special Relativity. On this spacetime there is a scalar field described by the function

$$\phi(t, x, y, z) = t^2 - x^2 - y^2 - z^2$$

and a particle with four-velocity

$$u = 4t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

Using $c = 1$ units, find

a. $u \cdot d\phi$

Answer 1a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

b. $u \cdot u$

Answer 1b

1. Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

c. $d\phi \cdot d\phi$

Answer 1c

1. Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

Problem 2

In a two dimensional spacetime with coordinates t, x, you are given a vector field v with components

$$\begin{aligned} u^0 &= 2x \\ u^1 &= t \end{aligned}$$

and a form-field α with components

$$\begin{aligned} \alpha_0 &= 3x \\ \alpha_1 &= 4t \end{aligned}$$

as well as a set of connection coefficients

$$\begin{aligned} \Gamma^0{}_{00} &= \Gamma^0{}_{01} = 0 \\ \Gamma^1{}_{10} &= \Gamma^1{}_{11} = 0 \\ \Gamma^0{}_{10} &= x, \qquad \Gamma^0{}_{11} = t \\ \Gamma^1{}_{00} &= -x, \qquad \Gamma^1{}_{01} = -t \end{aligned}$$

Calculate

a. $u^j \alpha_j$.

Answer 2a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

b. $\Gamma^k{}_{ki} u^i$

Answer 2b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

c. $\alpha_j \Gamma^j{}_{ki} u^i$

Answer 2c

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Problem 3

Suppose that the metric tensor on a spacetime has the form

$$g = -dt \otimes dt + dx \otimes dx$$

and you decide to use the coordinates

$$\begin{aligned} \tau &= \sqrt{t^2 - x^2} \\ \psi &= \tanh^{-1} \frac{x}{t} \end{aligned}$$

in the region where these coordinates are real numbers.

a. Express the new holonomic basis vectors $\frac{\partial}{\partial \tau}$ and $\frac{\partial}{\partial \psi}$ in terms of the old basis vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$.

Answer 3a

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b. Express the new holonomic basis forms $d\tau$ and $d\psi$ in terms of the old basis forms dt and dx.

Answer 3b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

C. Express the metric tensor in terms of the new coordinates, τ, ψ.

Answer 3c

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

Problem 4

Assume the ($c = 1$) metric of Special relativity. Suppose that the Minkowski coordinate components of the four-momentum of a particle obey the system of equations:

$$p^j = \frac{e}{m} F^j{}_k p^k$$

Use tensor index manipulation to show that, for this system of equations to be consistent with special relativity, you must require the components of the tensor F to obey

$$F_{ij} = -F_{ji}.$$

Answer 4

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

Problem 5

For the vector field

$$K = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}$$

a. Find the integral curves in the x, t plane.

Answer 5a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

b. Calculate the coordinate transformation that this vector field generates

$$\begin{aligned} x' &= e^{\varepsilon K} x \\ t' &= e^{\varepsilon K} t \end{aligned}$$

You can either use the result of part a or you can consider what happens for very small values of the parameter ε.

Answer 5b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

Problem 6

Represent a two-sphere of radius r with polar coordinates on it by embedding it in three dimensional cartesian space like this:

$$\begin{aligned} x(r,\theta,\varphi) &= r\sin\theta\cos\varphi \\ y(r,\theta,\varphi) &= r\sin\theta\sin\varphi \\ z(r,\theta,\varphi) &= r\cos\theta \end{aligned}$$

Use the fact that the cartesian basis vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are constant to find the connection coefficients for the two-sphere basis vectors

$$e_2 = \frac{1}{r}\frac{\partial}{\partial\theta}, \qquad e_3 = \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

Note that, when you do this, you will have to discard components in the direction

$$e_1 = \frac{\partial}{\partial r}$$

that are perpendicular to the two-sphere.

Answer 6

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.