Hour Exam No.1

Please attempt all of the following problems before the due date. All problems count the same even though some are more complex than others. Assume that $c = 1$ units are used throughout.

Problem 1

A photon with frequency f in a particular "laboratory reference frame" is observed from a space probe that is moving at 20% of the speed of light in the laboratory frame. The photon is moving in the direction of the spacelike unit vector $\hat{n}$ while the probe is moving in the direction of the spacelike unit vector $\hat{b}$.

a. Find the four-momentum components of the photon in the laboratory frame.

Answer 1a

$$p = hf\,(e_0 + \hat{n})$$

where e_0 is the time unit vector in the lab frame.

b. Find the components of the space probe's four-velocity in the laboratory frame.

Answer 1b

$$u = \frac{e_0 + \vec{v}}{\sqrt{1 - v^2}}$$

Note that the probe is moving in the direction of the unit vector $\hat{b}$ so the velocity vector is just $\vec{v} = v\hat{b}$ and

$$u = \frac{e_0 + v\hat{b}}{\sqrt{1 - v^2}} = \frac{e_0 + v\hat{b}}{\sqrt{1 - v^2}}$$

At 20% of the speed of light, $v = 1/5$ so the numerical answer is

$$u = \frac{e_0 + \frac{1}{5}\hat{b}}{\sqrt{1 - \left(\frac{1}{5}\right)^2}} = \frac{1}{\sqrt{1 - \left(\frac{1}{5}\right)^2}}\left(e_0 + \frac{1}{5}\hat{b}\right) = \frac{5}{12}\sqrt{6}\left(e_0 + \frac{1}{5}\hat{b}\right)$$

C. Use results (a) and (b) to find an expression for the photon frequency that is observed from the space probe.

Answer 1c

In the probe's reference frame, the photon four-momentum takes the form

$$p = hf'\left(e_{0'} + \hat{n}'\right)$$

The timelike unit vector used by the probe is just the probe's 4-velocity

$$e_{0'} = u = \frac{5}{12}\sqrt{6}\left(e_0 + \frac{1}{5}\hat{b}\right).$$

Pick out the term that we want by using the dot product:

$$p \cdot e_{0'} = -hf'$$

$$f' = -\frac{1}{h}p \cdot u = -\frac{1}{h}\left(hf\left(e_0 + \hat{n}\right)\right) \cdot \frac{5}{12}\sqrt{6}\left(e_0 + \frac{1}{5}\hat{b}\right)$$

$$\begin{aligned} f' &= -\frac{5}{12}\sqrt{6}f\left(e_0 + \hat{n}\right)\left(e_0 + \frac{1}{5}\hat{b}\right) \\ &= -\frac{5}{12}\sqrt{6}f\left(e_0 \cdot e_0 + \frac{1}{5}\hat{n} \cdot \hat{b}\right) \\ &= -\frac{5}{12}\sqrt{6}f\left(-1 + \frac{1}{5}\hat{n} \cdot \hat{b}\right) \end{aligned}$$

$$f' = \frac{5}{12}\sqrt{6}f\left(1 - \frac{1}{5}\cos\theta\right)$$

where θ is the angle between the photon direction and the probe direction as seen from the lab frame.

Problem 2

Show, using a (-+++) signature spacetime metric,

a. that the dot product of the four-momenta q, p of any two particles (massive or massless) must obey the inequality

$$q \cdot p \leq 0.$$

Answer 2a

If one one of the particles is massive, its four-momentum take the form

$$p = mu$$

where u is its four-velocity. But, in the rest-frame of that particle,

$$u = e_0$$

and

$$\begin{aligned} q \cdot p &= \left(q^0 e_0 + q^i e_i\right) \cdot m e_0 \\ &= m q^0 e_0 \cdot e_0 = -m q^0 \end{aligned}$$

The time component q^0 of a particle's 4-momentum is its energy, which will always be positive. The mass is also positive, so in this case we have

$$q \cdot p < 0.$$

The only other possible case is where both particles are massless. Their four-momenta then take the forms

$$\begin{aligned} p &= hf\left(e_0 + \hat{n}\right) \\ q &= hf'\left(e_0 + \hat{n}'\right) \end{aligned}$$

where $\hat{n}$ and $\hat{n}'$ are unit vectors in the direction of motion of each particle. In that case,

$$\begin{aligned} q \cdot p &= hf'\left(e_0 + \hat{n}'\right) \cdot hf\left(e_0 + \hat{n}\right) \\ &= h^2 f f'\left(e_0 \cdot e_0 + \hat{n}' \cdot \hat{n}\right) \\ &= h^2 f f'\left(-1 + \cos\theta\right) \end{aligned}$$

where θ is the angle between the two unit vectors. This expression is negative or zero for all values of θ so

$$q \cdot p \leq 0$$

b. that a collection of massive particles cannot decay into a single photon.

Answer 2a

Let the particles have momenta $p_1, p_2, ..., p_n$ and suppose that they decay into a photon with momentum q. Four-momentum conservation then requires

$$p_1 + p_2 + ... + p_n = q$$

Dot both sides of this equation with q.

$$q \cdot p_1 + q \cdot p_2 + ... + q \cdot p_n = q \cdot q$$

Every one of the terms on the left is negative definite (not zero) because they each involve a massive particle four-momentum. The right-hand side is zero, so the requirement is impossible.

Problem 3

Consider three linear mappings of a vector space V into itself:

$$\begin{aligned} A &: V \to V \\ B &: V \to V \\ C &: V \to V \end{aligned}$$

The vector space is spanned by a set of n basis vectors, $\{e_i\}$ and its dual space $\hat{V}$ is spanned by the dual basis forms $\{\omega^k\}$.

a. Regard these mappings as tensors and write expressions for their tensor components in terms of the basis vectors and forms.

Answer 3a

Take the basis vectors to be e_i and the basis forms to be ω^j and associate with A the tensor that assigns the value $\alpha(A(v)) = A(v) \cdot \alpha$ to the pair (v, α)

$$A(v, \alpha) = \alpha(A(v))$$

The components of this tensor are then

$$A_j{}^i = A\left(e_j, \omega^i\right) = A(e_j) \cdot \omega^i$$

Simlarly,

$$\begin{aligned} B_j{}^i &= B\left(e_j, \omega^i\right) = B(e_j) \cdot \omega^i \\ C_j{}^i &= C\left(e_j, \omega^i\right) = C(e_j) \cdot \omega^i \end{aligned}$$

b. Regard the composite mapping $C \circ B \circ A$ as a tensor and find an expression for its components in terms of the components of A, B, C.

Answer 3b

We need to find the components

$$\begin{aligned} K_s{}^r &= \left((C \circ B \circ A) e_s\right) \cdot \omega^r \\ &= \omega^r \left(A\left(B\left(C\left(e_s\right)\right)\right)\right) \end{aligned}$$

Start from the inside and note that the components of the vector $C(e_s)$ are

$$C(e_s) \cdot \omega^i = C_s{}^i$$

and expand that vector in terms of the basis

$$C(e_s) = C_s{}^i e_i$$

Now insert that expansion into the expression and pull out the coefficients:

$$\begin{aligned} K_s{}^r &= \omega^r \left(A \left(B \left(C_s{}^i e_i\right)\right)\right) \\ &= C_s{}^i \omega^r \left(A \left(B \left(e_i\right)\right)\right) \end{aligned}$$

Next, expand $B(e_i)$ in terms of its components

$$B(e_i) = B_i{}^j e_j$$

and insert that expansion into the expression and pull out the coefficients:

$$\begin{aligned} K_s{}^r &= C_s{}^i \omega^r \left(A \left(B_i{}^j e_j\right)\right) \\ &= C_s{}^i B_i{}^j \omega^r \left(A \left(e_j\right)\right) \\ &= C_s{}^i B_i{}^j A_j{}^r \end{aligned}$$

This expression is, of course, a matrix product. Using [] to denote a square array of components, we have found that

$$[C \circ B \circ A] = [C]\,[B]\,[A]$$

Problem 4

Suppose that an electrical current I runs along the z axis of a Minkowski coordinate system.

a. Express the resulting magnetic field components as functions of the Minkowski coordinates, t, x, y, z. It is OK to consult an E&M book here.

Answer 4a

From the E&M book, we learn that the magnetic field at a distance r from the current has magnitude.

$$\|B\| = \frac{\mu_0 I}{2\pi r}$$

Assume a right-handed Cartesian coordinate system x, y, z. If we adopt polar coordinates r and θ in the x, y planes, then

$$x = r\cos\theta, \qquad y = r\sin\theta$$

and the field points tangent to the circles at constant, t, z, r in the direction of increasing θ.

At this point, you could draw a picture and use plane geometry and trig. I prefer to find the unit vector $\hat{\theta}$ tangent to these circles. That unit vector is related to the tangent vector $\frac{\partial}{\partial\theta}$ through some normalization function N

$$\hat{\theta} = N\frac{\partial}{\partial\theta}$$

chosen so that

$$\hat{\theta}\cdot\hat{\theta} = 1$$

$$N^2\frac{\partial}{\partial\theta}\cdot\frac{\partial}{\partial\theta} = 1$$

To figure out the dot product, use the chain rule for partial derivatives to get $\frac{\partial}{\partial\theta}$ in terms of the Cartesian basis vectors since we know their dot products:

$$\frac{\partial}{\partial\theta} = \frac{dx}{d\theta}\frac{\partial}{\partial x} + \frac{dy}{d\theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial\theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$\begin{aligned} \frac{\partial}{\partial \theta} \cdot \frac{\partial}{\partial \theta} &= \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right) \cdot \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right) \\ &= x^2 + y^2 = r^2 \end{aligned}$$

Now find the normalization function

$$N^2 r^2 = 1$$

$$N = \frac{1}{r}$$

and the desired unit vector in the direction of the magnetic field

$$\begin{aligned} \hat{\theta} &= \frac{1}{r} \frac{\partial}{\partial \theta} \\ &= \frac{1}{r} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right) \end{aligned}$$

The magnetic field, in terms of x, y, z, t is then

$$\begin{aligned} B &= \frac{\mu_0 I}{2\pi r} \frac{1}{r} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right) \\ &= \frac{\mu_0 I}{2\pi r^2} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right) \end{aligned}$$

Take the Minkowski coordinate basis vectors to be

$$e_0 = \frac{\partial}{\partial t}, \qquad e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}$$

and find the magnetic field components to be

$$\begin{aligned} B_0 &= B_3 = 0 \\ B_1 &= -\frac{\mu_0 I y}{2\pi \left(x^2 + y^2\right)} \\ B_2 &= \frac{\mu_0 I x}{2\pi \left(x^2 + y^2\right)} \end{aligned}$$

b. Find the equation of motion of a positively charged particle that is moving through this magnetic field in the $x - z$ plane.

Answer 4b

First, figure out what the Maxwell tensor components would be. Since there is no electric field,

$$\begin{aligned} F_{01} &= F_{02} = F_{03} = 0 \\ F_{10} &= F_{20} = F_{30} = 0 \end{aligned}$$

The remaining components are linked to the magnetic components by the relations

$$\begin{aligned} F_{12} &= -F_{21} = B_3 = 0 \\ F_{23} &= -F_{32} = B_1 = -\frac{\mu_0 I y}{2\pi\left(x^2+y^2\right)} \\ F_{31} &= -F_{13} = B_2 = \frac{\mu_0 I x}{2\pi\left(x^2+y^2\right)} \end{aligned}$$

Because these are spacelike components in an orthonormal frame, we can raise or lower the indexes without changing anything.

Next, write out the space components of the Lorentz force law in the form that we had in the notes

$$\frac{dp^m}{dt} = kmF^m{}_s v^s$$

where we found that km must equal the charge on the particle and

$$v^s = \frac{dx^s}{dt}$$

with $t = x^0$. For a particle of charge q, we get

$$\frac{dp^m}{dt} = qF^m{}_s \frac{dx^s}{dt}$$

To finish the job of producing equations of motion, we need to express the space momentum components p^m in terms of the derivatives $\frac{dx^s}{dt}$.

$$p^m = \frac{v^m}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\left(\frac{dx}{dt}\right)^2-\left(\frac{dy}{dt}\right)^2-\left(\frac{dz}{dt}\right)^2}}\frac{dx^m}{dt}$$

Now write out the equations one by one for $m = 1, 2, 3$:

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{\sqrt{1-\left(\frac{dx}{dt}\right)^2-\left(\frac{dy}{dt}\right)^2-\left(\frac{dz}{dt}\right)^2}}\frac{dx}{dt}\right) &= qF^1{}_2\frac{dy}{dt} + qF^1{}_3\frac{dz}{dt} \\ &= -\frac{q\mu_0 I x}{2\pi\left(x^2+y^2\right)}\frac{dz}{dt} \end{aligned}$$

$$\frac{d}{dt}\left(\frac{1}{\sqrt{1-\left(\frac{dx}{dt}\right)^2-\left(\frac{dy}{dt}\right)^2-\left(\frac{dz}{dt}\right)^2}}\frac{dy}{dt}\right) = qF^2{}_1\frac{dx}{dt}+qF^2{}_3\frac{dz}{dt}$$

$$= -\frac{q\mu_0 I y}{2\pi\left(x^2+y^2\right)}\frac{dz}{dt}$$

$$\frac{d}{dt}\left(\frac{1}{\sqrt{1-\left(\frac{dx}{dt}\right)^2-\left(\frac{dy}{dt}\right)^2-\left(\frac{dz}{dt}\right)^2}}\frac{dz}{dt}\right) = qF^3{}_1\frac{dx}{dt}+qF^3{}_2\frac{dy}{dt}$$

$$= \frac{q\mu_0 I x}{2\pi\left(x^2+y^2\right)}\frac{dx}{dt}+\frac{q\mu_0 I y}{2\pi\left(x^2+y^2\right)}\frac{dy}{dt}$$

Problem 5

In spherical coordinates, the metric tensor on Minkowski spacetime has the form

$$g = -dt \otimes dt + dr \otimes dr + r^2 \left(d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi \right)$$

a. Write the components of this tensor and the inverse metric g^{-1} in the holonomic basis that corresponds to the coordinates

$$\begin{aligned} x^0 &= t \\ x^1 &= r \\ x^2 &= \theta \\ x^3 &= \varphi \end{aligned}$$

Answer 5a

There are no cross-terms, so the tensor is diagonal. Compare the tensor in the form

$$g = g_{00} dx^0 \otimes dx^0 + g_{11} dx^1 \otimes dx^1 + g_{22} dx^2 \otimes dx^2 + g_{33} dx^3 \otimes dx^3$$

to the one above with the names of the coordinates changed:

$$g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + r^2 \left(dx^2 \otimes dx^2 + \sin^2 \theta dx^3 \otimes dx^3 \right)$$

and identify

$$\begin{aligned} g_{00} &= -1 \\ g_{11} &= 1 \\ g_{22} &= r^2 \\ g_{33} &= r^2 \sin^2 \theta \end{aligned}$$

or

$$g = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}^{-1}$$

$$g^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{bmatrix}$$

or

$$g^{00} = -1, \quad g^{11} = 1, \quad g^{22} = \frac{1}{r^2}, \quad g^{33} = \frac{1}{r^2 \sin^2 \theta}$$

b. Normalize the basis vectors $\partial_t, \partial_r, \partial_\theta, \partial_\varphi$ and forms $dt, d\theta, d\varphi$ to obtain an orthonormal (i.e. Minkowski) frame at each point. Write out the components of the transformation matrices that express the orthonomal basis vectors and forms in terms of the holonomic basis objects.

Answer 5b

The vectors are already normal to one another and two are already unit vectors, so we just need normalizing factors for the other two:

$$\begin{aligned} e_0 &= \partial_t, \quad e_1 = \partial_r \\ e_2 &= K^{-1}\partial_\theta \\ e_3 &= J^{-1}\partial_\varphi \end{aligned}$$

From the metric,

$$e_2 \cdot e_2 = K^{-2}\partial_\theta \cdot \partial_\theta = K^{-2} g_{22} = K^{-2} r^2 = 1$$

so

$$e_2 = r^{-1}\partial_\theta$$

and

$$e_3 \cdot e_3 = J^{-2}\partial_\varphi \cdot \partial_\varphi = J^{-2} g_{33} = J^{-2} r^2 \sin^2 \theta = 1$$

so

$$e_3 = \frac{1}{r \sin \theta}\partial_\varphi$$

In terms of a transformation matrix,

$$\begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r \sin \theta} \end{bmatrix} \begin{bmatrix} \partial_t \\ \partial_r \\ \partial_\theta \\ \partial_\varphi \end{bmatrix}$$

In index notation,

$$e_\mu = U_{\mu'}{}^{\nu} \partial_\nu$$

so that the components are

$$[U] = \begin{bmatrix} U_{0'}{}^0 & U_{0'}{}^1 & U_{0'}{}^2 & U_{0'}{}^3 \\ U_{1'}{}^0 & U_{1'}{}^1 & U_{1'}{}^2 & U_{1'}{}^3 \\ U_{2'}{}^0 & U_{2'}{}^1 & U_{2'}{}^2 & U_{2'}{}^3 \\ U_{3'}{}^0 & U_{3'}{}^1 & U_{3'}{}^2 & U_{3'}{}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r \sin \theta} \end{bmatrix}$$

The orthonormal basis forms are related to the holonomic ones through the inverse matrix relation

$$\begin{bmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix}^T = \begin{bmatrix} dt \\ dr \\ d\theta \\ d\varphi \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{bmatrix}^{-1}$$

or

$$\begin{bmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix}^T = \begin{bmatrix} dt \\ dr \\ d\theta \\ d\varphi \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{bmatrix}$$

In index form,

$$\omega^\nu = dx^\mu U_\mu{}^{\nu'}$$

so we can read off the components from

$$\begin{bmatrix} U_0{}^{0'} & U_0{}^{1'} & U_0{}^{2'} & U_0{}^{3'} \\ U_1{}^{0'} & U_1{}^{1'} & U_1{}^{2'} & U_1{}^{3'} \\ U_2{}^{0'} & U_2{}^{1'} & U_2{}^{2'} & U_2{}^{3'} \\ U_3{}^{0'} & U_3{}^{1'} & U_3{}^{2'} & U_3{}^{3'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{bmatrix}$$

C. The notes discuss the splitting of the Maxwell Field Tensor components into Electric and Magnetic field components in an orthonormal (Minkowski) frame. Show how this splitting works in a holonomic spherical coordinate frame. In other words, express f_{rs} in terms of E^k and B_j when all of these components are taken using the holonomic bases.

Answer 5b

For the orthonormal components,

$$\begin{aligned} f_{1'2'} &= B_{3'} \\ f_{3'1'} &= B_{2'} \\ f_{2'3'} &= B_{1'} \\ f^{1'}{}_{0'} &= f_{1'0'} = E^{1'} \\ f^{2'}{}_{0'} &= f_{2'0'} = E^{2'} \\ f^{2'}{}_{0'} &= f_{2'0'} = E^{2'} \end{aligned}$$

These components are related to the holonomic components according to

$$\begin{aligned} f_{r's'} &= U_{r'}{}^a U_{s'}{}^b f_{ab}, \qquad f_{r'0'} = U_{r'}{}^a U_{0'}{}^0 f_{a0} = U_{r'}{}^a f_{a0}, \\ B_{j'} &= U_{j'}{}^k B_k, \qquad E^{j'} = E^k U_k{}^{j'} \end{aligned}$$

The transformation matrices are diagonal and we can read the components from the earlier result. The sums collapse to

$$
\begin{aligned}
f_{1'2'} &= U_{1'}{}^{1}U_{2'}{}^{2}f_{12} = U_{2'}{}^{2}f_{12} = \frac{1}{r}f_{12} \\
f_{3'1'} &= U_{3'}{}^{3}U_{1'}{}^{1}f_{31} = U_{3'}{}^{3}f_{31} = \frac{1}{r\sin\theta}f_{31} \\
f_{2'3'} &= U_{2'}{}^{2}U_{3'}{}^{3}f_{23} = \frac{1}{r^2\sin\theta}f_{23} \\
f_{1'0'} &= U_{1'}{}^{1}f_{10} = f_{10} \\
f_{2'0'} &= U_{2'}{}^{2}f_{20} = \frac{1}{r}f_{20} \\
f_{3'0'} &= U_{3'}{}^{3}f_{30} = \frac{1}{r\sin\theta}f_{30} \\
B_{1'} &= U_{1'}{}^{1}B_{1} = B_{1} \\
B_{2'} &= U_{2'}{}^{2}B_{2} = \frac{1}{r}B_{2} \\
B_{3'} &= U_{3'}{}^{3}B_{3} = \frac{1}{r\sin\theta}B_{3} \\
E^{1'} &= E^{1}U_{1}{}^{1'} = E^{1} \\
E^{2'} &= E^{2}U_{2}{}^{2'} = E^{2}r \\
E^{3'} &= E^{3}U_{3}{}^{3'} = E^{3}r\sin\theta
\end{aligned}
$$

and the relationships for the magnetic components become

$$
\begin{aligned}
\frac{1}{r}f_{12} &= \frac{1}{r\sin\theta}B_3 \\
\frac{1}{r\sin\theta}f_{31} &= \frac{1}{r}B_2 \\
\frac{1}{r^2\sin\theta}f_{23} &= B_1
\end{aligned}
$$

or

$$
f_{12} = \frac{1}{\sin\theta}B_3, \quad f_{31} = \sin\theta B_2, \quad f_{23} = r^2\sin\theta B_1
$$

For the electric components, the relationships become

$$
\begin{aligned}
f_{10} &= E^{1} \\
f_{2'0'} &= E^{2'} \\
\frac{1}{r}f_{20} &= E^{2}r \\
f_{20} &= r^2E^2 \\
f_{3'0'} &= E^{3'} \\
\frac{1}{r\sin\theta}f_{30} &= E^{3}r\sin\theta \\
f_{30} &= r^2\sin^2\theta E^3
\end{aligned}
$$

Problem 6

For one-forms α, β, γ and a vector v, consider the tensor

$$K = v \otimes (\alpha \otimes \beta \otimes \gamma + \beta \otimes \gamma \otimes \alpha + \gamma \otimes \alpha \otimes \beta)$$

a. Express the components of TrK in terms of the components of α, β, γ, v.

Answer 6a

For any form ζ and vectors u, v, w this tensor assigns the number:

$$K(\zeta, u, v, w) = v(\zeta)[\alpha(u)\beta(v)\gamma(w) + \beta(u)\gamma(v)\alpha(w) + \gamma(u)\alpha(v)\beta(w)]$$

The trace is defined to be

$$\mathrm{Tr}K(v, w) = K(\omega^a, e_a, v, w)$$

and has components

$$\begin{aligned}(\mathrm{Tr}K)_{rs} &= \mathrm{Tr}K(e_r, e_s) = K(\omega^a, e_a, e_r, e_s) \\ &= v(\omega^a)[\alpha(e_a)\beta(e_r)\gamma(e_s) + \beta(e_a)\gamma(e_r)\alpha(e_s) + \gamma(e_a)\alpha(e_r)\beta(e_s)]\end{aligned}$$

Notice that we just replaced ζ by ω^a, u by e_a, v by e_r and w by e_s in the expression for $K(\zeta, u, v, w)$. But $v(\omega^a) = v^a, \quad \alpha(e_a) = \alpha_a$ and so on, are just the components that we want, so

$$(\mathrm{Tr}K)_{rs} = v^a[\alpha_a\beta_r\gamma_s + \beta_a\gamma_r\alpha_s + \gamma_a\alpha_r\beta_s]$$

b. Express the tensor TrK as a combination of basis-independent objects such as dot products and tensor products of the objects α, β, γ, v.

Answer 6b

The answer can be seen pretty directly from the index expression above, or you can get it by writing out the definition of Tr$K(v, w)$

$$\begin{aligned}TrK(v, w) &= v(\omega^a)[\alpha(e_a)\beta(v)\gamma(w) + \beta(e_a)\gamma(v)\alpha(w) + \gamma(e_a)\alpha(v)\beta(w)] \\ &= v^a\alpha_a\beta(v)\gamma(w) + v^a\beta_a\gamma(v)\alpha(w) + v^a\gamma_a\alpha(v)\beta(w) \\ &= (v \cdot \alpha)\beta(v)\gamma(w) + (v \cdot \beta)\gamma(v)\alpha(w) + (v \cdot \gamma)\alpha(v)\beta(w)\end{aligned}$$

$$TrK = (v \cdot \alpha)\beta \otimes \gamma + (v \cdot \beta)\gamma \otimes \alpha + (v \cdot \gamma)\alpha \otimes \beta$$