Exercise 03

Please attempt all of the following problems before the due date. Your grade on this assignment will be calculated from the best two answers.

Problem 03.1

(Module 009) Recall the map \( \$ : X \to \hat{X} \) defined for any \( v \in X \) by
\[
\$ (v) = v^\$ 
\]
where the form \( v^\$ \) acts on any form \( \alpha \) according to:
\[
v^\$ (\alpha) = \alpha (v) .
\]

Prove the assertion in the notes that this map is onto — that every element of \( \hat{X} \) is equal to \( v^\$ \) for some vector \( v \) so long as \( X \) is a finite dimensional vector space.

Also try doing the proof when \( X \) has an infinite, but countable set of basis vectors.

Answer 03.1

The need for a finite dimensional vector space is a clue that we need to consider basis sets. Assume a basis set \([e]\) for \( X \) and the corresponding dual basis set \([\omega]\) for \( \hat{X} \) that may be defined by the relations
\[
\omega^i (e_j) = \delta^i_j 
\]
The spaces \( \hat{X} \) and \( \hat{\hat{X}} \) stand in the same relation to each other as \( X \) and \( \hat{X} \). Corresponding to the basis \([\omega]\) there is dual basis \([E]\) of forms on forms defined by
\[
E_k (\omega^j) = \delta^j_k .
\]
Any object in any of these spaces can be expressed uniquely as a sum of these basis objects. Now work out how the \( \$ \) map acts on the basis objects.
\[
\$ (e_k) = e_k^\$ 
\]
where
\[
e_k^\$ (\alpha) = \alpha (e_k) 
\]
Compare the objects \( E_k \) with the objects \( e_i^\$ \). These objects are completely defined by how they act on the basis forms \( \omega^i \)
\[
e_k^\$ (\omega^i) = \omega^i (e_k) = \delta^i_k 
\]
Since they act in EXACTLY the same way on those basis forms, they are the same:

\[ E_k = e_k \$ \]

The \$ map yields a one to one and onto correspondence between the basis vectors and that, in turn, provides a one-to-one correspondence between the spaces.

This same argument can work in the infinite dimensional case, but you need to separately restrict both \( X \) and \( \hat{X} \) to have countable basis sets. Having one countable does not usually imply that the other one is countable.

**Problem 03.2**

(Module 010) Assume basis vectors \( e_i \) and dual basis forms \( \omega^j \). Given a vector

\[ v = \cos \theta e_1 + \sin \theta e_2 + e_3 \]

and a form

\[ \alpha = \cos \theta \omega^1 + \sin \theta \omega^2 - \omega^3 \]

write out the calculation of the dot product \( \alpha \cdot v \) in two different ways:

1. Algebraically starting from \( \alpha \cdot v = (\cos \theta \omega^1 + \sin \theta \omega^2 - \omega^3) \cdot (\cos \theta e_1 + \sin \theta e_2 + e_3) \) and using the defining property of the dual basis.

2. In terms of components, starting from \( \alpha \cdot v = v^i \alpha_i \).

**Answer 03.2**

\[
(\cos \theta \omega^1 + \sin \theta \omega^2 - \omega^3) \cdot (\cos \theta e_1 + \sin \theta e_2 + e_3) = \cos \theta \omega^1 \cdot \cos \theta e_1 + \cos \theta \omega^1 \cdot \sin \theta \omega^2 - \cos \theta \omega^1 \cdot \omega^3 + \sin \theta \omega^2 \cdot \cos \theta e_1 + \sin \theta \omega^2 \cdot \sin \theta e_2 - \sin \theta \omega^2 \cdot \omega^3 - \omega^3 \cdot \cos \theta e_1 - \omega^3 \cdot \sin \theta e_2 - \omega^3 \cdot e_3
\]

Use the defining expression

\[ \omega^i \cdot e_j = \delta_j^i \]

and get

\[
(\cos \theta \omega^1 + \sin \theta \omega^2 - \omega^3) \cdot (\cos \theta e_1 + \sin \theta e_2 + e_3) = \cos^2 \theta + 0 - 0 + 0 + \sin^2 \theta - 0 - 0 - 1
\]

or

\[
(\cos \theta \omega^1 + \sin \theta \omega^2 - \omega^3) \cdot (\cos \theta e_1 + \sin \theta e_2 + e_3) = \cos^2 \theta + \sin^2 \theta - 1 = 0.
\]
In terms of components, which you can read directly from the expressions,

\[
\alpha_1 = \cos \theta, \quad v^1 = \cos \theta \\
\alpha_2 = \sin \theta, \quad v^2 = \sin \theta \\
\alpha_3 = -1, \quad v^3 = 1
\]

\[
\alpha \cdot v = v^i \alpha_i = v^1 \alpha_1 + v^2 \alpha_2 + v^3 \alpha_3 \\
= \cos^2 \theta + \sin^2 \theta - 1 = 0
\]

**Problem 03.3**

(Module 010) Use Einstein’s Summation Convention to write \((\alpha \cdot v)^2\) as a sum of products of components of the form \(\alpha\) and the vector \(v\).

**Answer 03.3**

\[
\alpha \cdot v = v^i \alpha_i \\
(\alpha \cdot v)^2 = (\alpha \cdot v)(\alpha \cdot v)
\]

Express each dot product in index notation, but use a different dummy index for each.

\[
(\alpha \cdot v)^2 = (v^i \alpha_i)(v^k \alpha_k) = v^i v^k \alpha_i \alpha_k
\]

As an explicit sum, we get

\[
(\alpha \cdot v)^2 = \sum_i \sum_k v^i v^k \alpha_i \alpha_k.
\]