Exercise 02

Please attempt all of the following problems before the due date. Your grade on this assignment will be calculated from the best two answers.

Problem 02.1

(Module 008) Recall that a form \( \alpha \) is completely described by the numbers

\[ \alpha_i = \alpha(e_i) \]

that it gives when acting on a set of basis vectors of a vector space. Show that these numbers, together with the dual basis, can be used to provide a unique representation of the form as

\[ \alpha = \alpha_1 \omega^1 + \alpha_2 \omega^2 + \alpha_3 \omega^3 + \ldots + \alpha_n \omega^n \]

where \( n \) is the dimensionality of the vector space.

Answer 02.1

First show that the representation works by calculating the resulting numbers

\[ \alpha(e_i) = \alpha_1 \omega^1(e_i) + \alpha_2 \omega^2(e_i) + \alpha_3 \omega^3(e_i) + \ldots + \alpha_n \omega^n(e_i) \]

\[ = \alpha_1 \delta_i^1 + \alpha_2 \delta_i^2 + \alpha_3 \delta_i^3 + \ldots + \alpha_n \delta_i^n \]

\[ = \alpha_i \]

so it does give the right numbers for each basis vector.

Next show that the representation is unique by supposing that there is some other set of coefficients \( \beta_i \) such that

\[ \alpha = \beta_1 \omega^1 + \beta_2 \omega^2 + \beta_3 \omega^3 + \ldots + \beta_n \omega^n \]

that gives the same numbers \( \alpha_i = \alpha(e_i) \). The same calculation as before then gives

\[ \alpha_i = \beta_i \]

so the coefficients are not actually different.

An alternative way to arrange the proof is to show that the dual basis forms are linearly independent so that the difference expression

\[ (\alpha_1 - \beta_1) \omega^1 + (\alpha_2 - \beta_2) \omega^2 + (\alpha_3 - \beta_3) \omega^3 + \ldots + (\alpha_n - \beta_n) \omega^n = 0 \]

implies that the coefficients all vanish.

This unique representation of forms in terms of the dual basis forms shows that the dual basis actually is a basis for the vector space of forms.
For the following problems, use the basis vectors

\[ e_1 = \hat{i} \]
\[ e_2 = \hat{j} \]
\[ e_2 = \hat{k} \]

that point along the \( x, y, z \) axes of a Cartesian coordinate system. Denote the dual basis forms by

\[ \omega^1 = dx \]
\[ \omega^2 = dy \]
\[ \omega^2 = dz \]

**Problem 02.2**

(Module 008) Calculate the numbers that the form

\[ \alpha = dx - dy + dz \]

assigns to each of the vectors:

\[ u = \hat{i} + \hat{j} \]
\[ v = \hat{j} - \hat{k} \]
\[ w = \hat{i} + \hat{j} + \hat{k}. \]

**Answer 02.2**

\[ \alpha (u) = (dx - dy + dz) \cdot (\hat{i} + \hat{j}) \]
\[ = dx (\hat{i} + \hat{j}) - dy (\hat{i} + \hat{j}) + dz (\hat{i} + \hat{j}) \]
\[ = dx (\hat{i}) + dx (\hat{j}) - dy (\hat{i}) - dy (\hat{j}) + dz (\hat{i}) + dz (\hat{j}) \]

From the definition of the dual basis forms

\[ dx (\hat{i}) = 1, \quad dx (\hat{j}) = 0, \quad dx (\hat{j}) = 0 \]

so

\[ \alpha (u) = 1 + 0 - 1 + 0 + 0 = 0 \]

Similarly,

\[ \alpha (v) = (dx - dy + dz) \cdot (\hat{j} - \hat{k}) \]
\[ = dx \cdot \hat{j} - dx \cdot \hat{k} - dy \cdot \hat{j} + dy \cdot \hat{k} + dz \cdot \hat{j} - dz \cdot \hat{k} \]
\[ = 0 - 0 - 1 + 0 + 0 - 1 = -2 \]
and
\[ \alpha(w) = (dx - dy + dz) \cdot (i + j + k) \]
\[ = dx \cdot i + dx \cdot j + dx \cdot k - dy \cdot i - dy \cdot j - dy \cdot k + dz \cdot i + dz \cdot j + dz \cdot k \]
\[ = 1 + 0 + 0 - 1 - 0 + 0 + 0 + 1 = 1 \]

**Problem 02.3**

(Module 008) Define a new set of basis vectors

\[
\begin{bmatrix}
  i' \\
  j' \\
  k'
\end{bmatrix} = \begin{bmatrix}
  \cos \phi & \sin \phi & 0 \\
  -\sin \phi & \cos \phi & 0 \\
  0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  \cos \theta & 0 & \sin \theta \\
  0 & 1 & 0 \\
  -\sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
  i \\
  j \\
  k
\end{bmatrix}
\]

Express the new dual basis forms \( dx', dy' \) and \( dz' \) in terms of \( dx, dy, dz \).

**Answer 02.3**

From the example worked in the notes, we know that the dual basis forms will be related by the inverse matrix:

\[
\begin{bmatrix}
  dx' \\
  dy' \\
  dz'
\end{bmatrix} = \begin{bmatrix}
  dx & dy & dz
\end{bmatrix} \left( \begin{bmatrix}
  \cos \phi & \sin \phi & 0 \\
  -\sin \phi & \cos \phi & 0 \\
  0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  \cos \theta & 0 & \sin \theta \\
  0 & 1 & 0 \\
  -\sin \theta & 0 & \cos \theta
\end{bmatrix} \right)^{-1}
\]

We could just mark the right hand side with the mouse and evaluate it using \( \text{ctrl-E} \). However, that will give a row matrix that is excessively long. Instead, use the transpose:

\[
\begin{bmatrix}
  dx' \\
  dy' \\
  dz'
\end{bmatrix} = \left( \begin{bmatrix}
  dx & dy & dz
\end{bmatrix} \left( \begin{bmatrix}
  \cos \phi & \sin \phi & 0 \\
  -\sin \phi & \cos \phi & 0 \\
  0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  \cos \theta & 0 & \sin \theta \\
  0 & 1 & 0 \\
  -\sin \theta & 0 & \cos \theta
\end{bmatrix} \right)^{-1}
\right)^T
\]

Marking the right-hand side gives a column of results that will fit on your display screen if you scroll, but are still too long to fit on a page.

\[
\begin{bmatrix}
  dx' \\
  dy' \\
  dz'
\end{bmatrix} = \begin{bmatrix}
  dy \left( \cos \phi \right) \\
  -dx \left( \cos \phi \right) + dx \left( \cos \theta \right) \\
  dy \left( \sin \phi \right) - dx \left( \cos \theta \right)
\end{bmatrix}
\]

However, each line can be broken up as follows:
\[\begin{align*}
\frac{dx'}{dx} &= dy \frac{\sin \phi}{\cos^2 \phi + \sin^2 \phi} + dx (\cos \theta) \frac{\cos \phi}{\cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi + \cos^2 \phi \sin^2 \theta + \sin^2 \theta \sin^2 \phi} \\
&\quad + dz (\cos \phi) \frac{\sin \phi}{\cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi + \cos^2 \phi \sin^2 \theta + \sin^2 \theta \sin^2 \phi} \\
\frac{dy'}{dy} &= dy \frac{\cos \phi}{\cos^2 \phi + \sin^2 \phi} - dx (\cos \theta) \frac{\sin \phi}{\cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi + \cos^2 \phi \sin^2 \theta + \sin^2 \theta \sin^2 \phi} \\
&\quad - dz (\sin \theta) \frac{\sin \phi}{\cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi + \cos^2 \phi \sin^2 \theta + \sin^2 \theta \sin^2 \phi} \\
\frac{dz'}{dz} &= dz \frac{\cos \theta \cos^2 \phi + \cos \theta \sin^2 \phi}{\cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi + \cos^2 \phi \sin^2 \theta + \sin^2 \theta \sin^2 \phi} - dx \frac{\sin \theta}{\cos^2 \theta + \sin^2 \theta} 
\end{align*}\]