Exercise 12

Please attempt all of the following problems before the due date. Your grade on this assignment will be calculated from the best two answers.

In all of the following problems, the "harmonic operator" on a scalar function \( \phi \) takes the form

\[
\Delta \phi = g^{ab} D_{e_a} D_{e_b} \phi
\]

where \( e_a \) are the tangent-space basis vector fields and \( g^{ab} = g^{-1} (\omega^a, \omega^b) \) are the components of the inverse metric tensor.

Problem 12.1

Show that this operator does not depend on the choice of basis-vector fields.

Answer 12.1

Use the fact that \( D_u D_v \phi \) is locally linear in the vector fields \( u, v \). If the basis vector fields change to

\[
e^a_{\prime} = U^a_r e_r
\]

then

\[
e_a = U^r_a e_r,
\omega^a = U^a_r \omega^r
\]

where

\[
U^r_a U^b_r = \delta^b_a, \quad U^r_a U^a_r = \delta^b_b.
\]

For the metric components, use the local linearity of \( g^{-1} \) to get the old frame representation in terms of the new one.

\[
g^{ab} = g^{-1} (\omega^a, \omega^b) = g^{-1} \left( U^a_j \omega^j, U^b_k \omega^k \right) = U^a_j U^b_k g^{-1} \left( \omega^j, \omega^k \right) = U^a_j U^b_k g^{j'k'}.
\]

The harmonic operator is then

\[
\Delta \phi = g^{ab} D_{e_a} D_{e_b} \phi
= U^a_j U^b_k g^{j'k'} D_{U^r_j e_{r'}} D_{U^s_k e_{s'}} \phi
\]

Use local linearity to bring the coefficients outside:

\[
D_{U^r_j e_{r'}} D_{U^s_k e_{s'}} \phi = U^r_a U^s_b D_{e_{r'}} D_{e_{s'}} \phi
\]
so that

\[ \Delta \phi = U_j^a U_b^k g^{i'k'} U_{i'}^a U_b^k D_{e_{i'}} D_{e_{k'}} \phi \]
\[ = U_j^a U_b^k U_{i'}^a U_b^k g^{i'k'} D_{e_{i'}} D_{e_{k'}} \phi \]
\[ = \delta_j^i \delta_k^l g^{i'k'} D_{e_{i'}} D_{e_{k'}} \phi = g^{i'k'} D_{e_{i'}} D_{e_{k'}} \phi \]

and the expression is exactly the same in the new basis.

**Problem 12.2**

Find the form of the harmonic operator on spacetime when it is evaluated at the origin of a local Minkowski coordinate system.

**Answer 12.2**

In this case, all of the connection coefficients are zero and

\[ D_{e_a} D_{e_b} \phi = \nabla_{e_a} \nabla_{e_b} \phi - \nabla_{e_a e_b} \phi \]
\[ = \nabla_{e_a} e_b \phi \]
\[ = e_a e_b \phi \]
\[ = \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} \phi = \frac{\partial^2 \phi}{\partial x^a \partial x^b} \]

The metric components are non-zero only for the diagonal terms, so the double sum over indexes \( a, b \) simplifies.

\[ \Delta \phi = g^{ab} D_{e_a} D_{e_b} \phi \]
\[ = g^{00} \frac{\partial^2 \phi}{\partial x^0 \partial x^0} + g^{11} \frac{\partial^2 \phi}{\partial x^1 \partial x^1} + g^{22} \frac{\partial^2 \phi}{\partial x^2 \partial x^2} + g^{33} \frac{\partial^2 \phi}{\partial x^3 \partial x^3} \]
\[ = - \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \]
\[ \Delta \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \frac{\partial^2 \phi}{\partial t^2} \]

which is the wave operator.

**Problem 12.3**

Express the harmonic operator on a general manifold in terms of partial derivatives and the connection coefficients of a holonomic basis frame field. Use this expression and the connection coefficients found earlier to find the polar coordinate form of the harmonic operator on two dimensional flat space.
The harmonic operator then becomes
\[ \Delta \phi = g^{ab} D_a D_b \phi \]
\[ = g^{ab} \frac{\partial^2 \phi}{\partial x^a \partial x^b} - g^{ab} \Gamma^k_{ba} \frac{\partial \phi}{\partial x^k} \]
and the particular combinations of connection coefficients that we need are just.
\[ \Gamma^k = g^{ab} \Gamma^k_{ba} \]

Now take the basis vectors and forms to be
\[ e_1 = \frac{\partial}{\partial r}, \quad e_2 = \frac{\partial}{\partial \theta} \]
\[ \omega^1 = dr, \quad \omega^2 = d\theta \]

and note, from homework No. 10 that the corresponding metric components are
\[ \begin{bmatrix} \ g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}. \]

From homework No.11, the non-zero connection coefficients are
\[ \Gamma^{21} = \Gamma^{22} = \frac{1}{r} \]
\[ \Gamma^{12} = -r \]
and the combinations that we are concerned with become
\[ \Gamma^1 = g^{ab} \Gamma^1_{ba} = g^{22} \Gamma^1_{22} = -\frac{1}{r^2} r = -\frac{1}{r} \]
\[ \Gamma^2 = g^{ab} \Gamma^2_{ba} = 0 \]

The harmonic operator is then
\[ \Delta \phi = g^{ab} \frac{\partial^2 \phi}{\partial x^a \partial x^b} - g^{ab} \Gamma^k_{ba} \frac{\partial \phi}{\partial x^k} \]
\[ = g^{ab} \frac{\partial^2 \phi}{\partial x^a \partial x^b} - \Gamma^k \frac{\partial \phi}{\partial x^k} \]
\[ = g^{11} \frac{\partial^2 \phi}{\partial r^2} + g^{22} \frac{\partial^2 \phi}{\partial \theta^2} - \Gamma^1 \frac{\partial \phi}{\partial r} \]
\[ \Delta \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \]