

## Exercise 06

Please attempt all of the following problems before the due date. Your grade on this assignment will be calculated from the best two answers.

### Problem 06.1

The kinetic energy of a moving object (in three dimensional space) takes the general form

$$K.E. = \frac{1}{2}mg(v, v)$$

where  $v$  is the object's velocity (tangent vector to the curve parameterized by time) and  $g$  is a second rank covariant tensor. Ordinarily we take  $g$  to be a symmetric tensor. Einstein considered theories in which  $g$  is replaced by a tensor  $g + a$  where  $a$  is antisymmetric. Show that the antisymmetric term does not affect the kinetic energy expression.

#### Answer 06.1

The simplest method is to just write

$$K.E. = \frac{1}{2}m(g(v, v) + a(v, v))$$

where

$$a(u, v) = -a(v, u)$$

or

$$a(u, v) + a(v, u) = 0$$

so that, with  $u = v$

$$2a(v, v) = 0$$

and we get back

$$K.E. = \frac{1}{2}m(g(v, v) + a(v, v)) = \frac{1}{2}mg(v, v)$$

### Problem 06.2

A two dimensional coordinate system is transformed from coordinates  $x, y$  to new coordinates  $X(x, y), Y(x, y)$ . Express the new basis forms  $dX$  and  $dY$  in terms of the old basis forms  $dx$  and  $dy$  and then express the wedge product  $dX \wedge dY$  in terms of the wedge product  $dx \wedge dy$ .

## Answer 06.2

The simplest way to organize the calculation is to use the expression

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

and just write

$$\begin{aligned} dX &= \frac{\partial X}{\partial x} dx + \frac{\partial X}{\partial y} dy \\ dY &= \frac{\partial Y}{\partial x} dx + \frac{\partial Y}{\partial y} dy \end{aligned}$$

An alternative way to get the same thing is to express the new basis vectors in terms of the old

$$\begin{aligned} \frac{\partial}{\partial X} &= \frac{\partial x}{\partial X} \frac{\partial}{\partial x} + \frac{\partial y}{\partial X} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial Y} &= \frac{\partial x}{\partial Y} \frac{\partial}{\partial x} + \frac{\partial y}{\partial Y} \frac{\partial}{\partial y} \end{aligned}$$

and note that the dual basis transforms according to the inverse matrix.

Now use the definition of the wedge product:

$$\begin{aligned} dX \wedge dY &= \left( \frac{\partial X}{\partial x} dx + \frac{\partial X}{\partial y} dy \right) \wedge \left( \frac{\partial Y}{\partial x} dx + \frac{\partial Y}{\partial y} dy \right) \\ &= \frac{\partial X}{\partial x} \frac{\partial Y}{\partial x} dx \wedge dx + \frac{\partial X}{\partial x} \frac{\partial Y}{\partial y} dx \wedge dy \\ &\quad + \frac{\partial X}{\partial y} \frac{\partial Y}{\partial x} dy \wedge dx + \frac{\partial X}{\partial y} \frac{\partial Y}{\partial y} dy \wedge dy \end{aligned}$$

The antisymmetric products of forms with themselves all vanish and

$$dy \wedge dx = -dx \wedge dy$$

so that

$$dX \wedge dY = \left( \frac{\partial X}{\partial x} \frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y} \frac{\partial Y}{\partial x} \right) dx \wedge dy$$

which may also be written as

$$dX \wedge dY = \begin{vmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{vmatrix} dx \wedge dy$$

and we recognize the Jacobian determinant of the coordinate transformation. These wedge products transform exactly like area elements under a surface integral.

## Problem 06.3

The three dimensional vector cross product can be defined in tensor index notation as

$$(B \times C)^i = \varepsilon^{ijk} B_j C_k$$

where  $\varepsilon^{ijk}$  is the totally antisymmetric Levi-Civita symbol. Notice that the triple product  $A \cdot (B \times C)$  is then just

$$A \cdot (B \times C) = A_i (B \times C)^i$$

Use the connection between the Levi-Civita symbol and the determinant of a matrix to derive the identity

$$A \cdot (B \times C) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}.$$

## Answer 06.3

This is mostly a notation problem. To make the notation look like the definition of the determinant that we are using, define

$$\begin{aligned} U_i^{1'} &= A_i \text{ for } i = 1, 2, 3 \\ U_i^{2'} &= B_i \text{ for } i = 1, 2, 3 \\ U_i^{3'} &= C_i \text{ for } i = 1, 2, 3 \end{aligned}$$

so that the matrix  $U$  is just the matrix

$$U = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

that we are interested in. Now write the triple-product as

$$\begin{aligned} A_i (B \times C)^i &= A_i \varepsilon^{ijk} B_j C_k \\ &= \varepsilon^{ijk} A_i B_j C_k \\ &= \varepsilon^{ijk} U_i^{1'} U_j^{2'} U_k^{3'} \end{aligned}$$

The last expression is our definition of  $\det U$ , so we are done.