Exercise 04

Please attempt all of the following problems before the due date. Your grade on this assignment will be calculated from the best two answers.

Problem 04.1
Show that, for two vectors $u, v$, the tensor product $u \otimes v$ has components:

$$u \otimes v (\omega^r, \omega^s) = u^r v^s$$

Answer 04.1
This goes pretty fast:

$$u \otimes v (\omega^r, \omega^s) = u (\omega^r) v (\omega^s) = \omega^r (u) \omega^s (v) = u^r v^s$$

Problem 04.2
Show that the generalized trace or tensor contraction defined in the notes gives us yet another way to write the result of a form $\alpha$ acting on a vector $v$.

$$\text{Tr} (\alpha \otimes v) = \alpha (v) = \alpha \cdot v = v \cdot \alpha = v (\alpha) = [\alpha] [v].$$

Answer 04.2
The problem is actually to work out $\text{Tr}(\alpha \otimes v)$, starting with the definition of the trace. There is only one vector argument and one form argument, so it is simple.

$$\text{Tr} (\alpha \otimes v) = (\alpha \otimes v) (e_i, \omega^j)$$

Now the definition of the tensor product as well as the form and vector components gives

$$\text{Tr}(\alpha \otimes v) = \alpha (e_i) v (\omega^j) = \alpha_i v^i = [\alpha] [v].$$

Problem 04.3
Consider a second rank covariant tensor $K$ and find a way to obtain the number $K(u, v)$ by contracting the fourth rank tensor

$$K \otimes u \otimes v.$$  
Write this contraction using the generalized trace symbol $\text{Tr}^m_n$ (with the indexes omitted when they equal one) and then indicate what sort of contraction would be needed to obtain $K(u, v)$ from the tensor $K \otimes v \otimes u$.

Answer 04.3

First, figure out what $K(u, v)$ looks like in terms of components

$$K(u, v) = K(u^i e_i, v^j e_j) = u^i v^j K_{ij} = K_{ij} u^i v^j$$

The indexes tell us what we need to do. The first argument of $K$ is contracted with the argument of $u$ while the second argument of $K$ is contracted with the argument of $v$. Check that

$$K_{ij} u^i v^j = (K \otimes u \otimes v) \left(e_i, e_j, \omega^i, \omega^j\right)$$

Now describe this in terms of the generalized trace operation. Notice that we contract the first vector argument with the first form argument. The tensor that is left has only one vector argument and one form argument, so we can express the result as

$$K(u, v) = \text{Tr} \left(\text{Tr} \left(K \otimes u \otimes v\right)\right)$$

You could just as well do the sums in the other order, yielding

$$K(u, v) = \text{Tr} \left(\text{Tr}_2 \left(K \otimes u \otimes v\right)\right).$$

For the tensor in the other order, note that

$$K_{ij} v^i u^j = K_{ij} v^j u^i = (K \otimes v \otimes u) \left(e_i, e_j, \omega_i, \omega^j\right) = \text{Tr} \left(\text{Tr}_1 \left(K \otimes v \otimes u\right)\right)$$

The main point of this exercise is to show just how confusing the generalized trace notation can get. Normally we just write things in terms of components with indexes on them to avoid the confusion.