Final Exam

Please attempt all of the following problems before the due date. All problems count the same even though some are more complex than others.

Bad Tensor Analysis

Problem 1

Index notation, with the Einstein Summation Convention is extremely flexible and can express any kind of tensor operation. However it can be **too** flexible because it can express operations that do not exist. Here are a few examples of "Bad Tensor Analysis." Explain what is wrong with each example. Do more than just state what rule is violated and indicate why the operation cannot arise from valid geometrical operations.

a. $A^i + B^j$

Answer 1a

The violated rule here is that you have different free indexes in different terms. That corresponds to adding different components of vectors together — the x-component of one to the y-component of another — so it cannot happen.

b. $A^{ij} B_j C_{ji}$

Answer 1b

Here the index j is repeated more than twice. Legitimate tensor operations always involve just two indexes, one up and one down. The two types of indexes transform in opposite ways so the result in then invariant when the basis vectors change. The expression given here would depend on which basis is used.

c. $A^{ijk} B_k + C^{ijk} B_j$

Answer 1c

The difficulty here is actually just the same as in part a. The free indexes on the first term are i, j with k summed over while the free indexes on the second term are i, k with j summed over. That corresponds to adding different tensor components together and cannot happen.

d. $A^{abc}K_c = B^{abc}J_n$

Answer 1d

Here we have different free indexes and also have different rank tensors set equal to one another. That cannot happen.

Metric Tensor Follies

The following problems refer to the following situation:

Start with the spacetime line element

$$ds^2 = -e^{2a(r)}\,dt^2 + 2B\,(r)\;dt\,d\varphi + e^{-2a}\,dr^2 + R\,(r)^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)$$

with coordinates

$$x^0 = t, \qquad x^1 = r, \qquad x^2 = \theta, \qquad x^3 = \varphi.$$

Notice that this metric is specified by three functions a, b, R, of the single variable, r.

The world line of a particular observer is given by

$$\begin{aligned} t &= e^{-a}\lambda \\ r &= r_0 \\ \theta &= \frac{\pi}{2} \\ \varphi &= p\lambda \end{aligned}$$

where λ is the curve parameter and r_0 and p are constants.

Problem 2:

a. Use the line element to calculate how much proper time elapses for this observer for each unit of coordinate time t.

Answer 2a

The elapsed proper time is given by the integral

$$\begin{aligned} \Delta\tau &= \int_{t=0}^{t=1} d\lambda \left|\frac{ds}{d\lambda}\right| \\ &= \int_{t=0}^{t=1} d\lambda \sqrt{\left(e^{2a}\left(\frac{dt}{d\lambda}\right)^2 - 2B\left(\frac{dt}{d\lambda}\frac{d\varphi}{d\lambda}\right) - e^{-2a}\left(\frac{dr}{d\lambda}\right)^2 - R^2\left(\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2\right)\right)} \end{aligned}$$

Since r and θ are not changing and $\theta = \frac{\pi}{2}$ all along the world-line, the expression simplifies to

$$\Delta\tau = \int_{t=0}^{t=1} d\lambda \sqrt{\left(e^{2a}\left(\frac{dt}{d\lambda}\right)^2 - 2B\left(\frac{dt}{d\lambda}\frac{d\varphi}{d\lambda}\right) - R^2\left(\frac{d\varphi}{d\lambda}\right)^2\right)}$$

Evaluate the derivatives from the equations of the world-line.

$$\frac{dt}{d\lambda} = e^{-a}, \qquad \frac{d\varphi}{d\lambda} = p$$

$$\Delta\tau = \int_{t=0}^{t=1} d\lambda \sqrt{1 - 2Be^{-a}p - R^2p^2}$$

and

$$d\lambda = e^a dt.$$

Nothing is changing as a function of t because the world-line was at constant r, so the integral is trivial.

$$\begin{aligned} \Delta\tau &= e^a \int_{t=0}^{t=1} dt \sqrt{1 - 2Be^{-a}p - R^2p^2} \\ &= e^a \sqrt{1 - 2Be^{-a}p - R^2p^2} \end{aligned}$$

b. Display the metric tensor components $g_{\mu\nu}$ that are indicated by this line element.

Answer 2b

The line element is given by the doubly summed expression:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Write the double sum out to see what we have here:

$$\begin{aligned} ds^2 = \; & g_{00}dt^2 + g_{01}dtdr + g_{02}dtd\theta + g_{03}dtd\varphi \\ & + g_{10}drdt + g_{11}dr^2 + g_{12}drd\theta + g_{13}drd\varphi \\ & + g_{20}d\theta dt + g_{21}d\theta dr + g_{22}d\theta^2 + g_{23}d\theta d\varphi \\ & + g_{30}d\varphi dt + g_{31}d\varphi dr + g_{32}d\theta d\varphi + g_{33}d\varphi^2 \end{aligned}$$

Notice that the cross terms all appear twice, so we get

$$\begin{aligned} ds^2 = \; & g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\varphi^2 \\ & + 2g_{01}dtdr + 2g_{02}dtd\theta + 2g_{03}dtd\varphi + \ldots \end{aligned}$$

Compare this to the line element that we have:

$$ds^2 = -e^{2a}\, dt^2 + 2B\, dt\, d\varphi + e^{-2a}\, dr^2 + R^2 d\theta^2 + R^2 \sin^2\theta\, d\varphi^2$$

and get the non-zero metric components:

$$\begin{aligned} g_{00} &= -e^{2a}, \qquad g_{11} = e^{-2a}, \qquad g_{22} = R^2, \qquad g_{33} = R^2 \sin^2\theta \\ g_{03} &= B \end{aligned}$$

C. Find the four-velocity vector of this observer.

Answer 2c

Method 1:

First find a tangent vector to the world line by just using the chain rule:

$$\frac{d}{d\lambda} = \frac{dt}{d\lambda}\frac{\partial}{\partial t} + \frac{dr}{d\lambda}\frac{\partial}{\partial r} + \frac{d\theta}{d\lambda}\frac{\partial}{\partial \theta} + \frac{d\varphi}{d\lambda}\frac{\partial}{\partial \varphi}$$

From the equations for the world-line,

$$\frac{\partial t}{\partial \lambda} = e^{-a}, \qquad \frac{\partial r}{\partial \lambda} = 0, \qquad \frac{\partial \theta}{\partial \lambda} = 0, \qquad \frac{\partial \varphi}{\partial \lambda} = p$$

so that a tangent vector is

$$\frac{d}{d\lambda} = e^{-a}\frac{\partial}{\partial t} + p\frac{\partial}{\partial \varphi}$$

The four-velocity will be in the direction of this tangent vector, but it must be normalized so that

$$u \cdot u = -1$$

Take

$$u = N\frac{\partial}{\partial \lambda}$$

so that

$$\begin{aligned} u \cdot u &= N\left(e^{-a}\frac{\partial}{\partial t} + p\frac{\partial}{\partial \varphi}\right) \cdot N\left(e^{-a}\frac{\partial}{\partial t} + p\frac{\partial}{\partial \varphi}\right) \\ &= N^2\left(e^{-2a}\frac{\partial}{\partial t}\cdot\frac{\partial}{\partial t} + 2pe^{-a}\frac{\partial}{\partial t}\cdot\frac{\partial}{\partial \varphi} + p^2\frac{\partial}{\partial \varphi}\cdot\frac{\partial}{\partial \varphi}\right) \end{aligned}$$

But

$$\frac{\partial}{\partial t}\cdot\frac{\partial}{\partial t} = g\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0}\right) = g_{00}$$

and so forth, so

$$\begin{aligned} -1 &= N^2\left(e^{-2a}g_{00} + 2pe^{-a}g_{03} + p^2 g_{33}\right) \\ &= N^2\left(\left(e^{-2a}\right)\left(-e^{2a}\right) + 2pe^{-a}B + p^2R^2\sin^2\theta\right) \\ &= N^2\left(-1 + 2pe^{-a}B + p^2R^2\sin^2\theta\right) \end{aligned}$$

and thus (with $\theta = \pi/2$)

$$N = \frac{1}{\sqrt{1 - 2pe^{-a}B - p^2R^2}}$$

The four velocity vector is then

$$u = \frac{e^{-a}}{\sqrt{1 - 2pe^{-a}B - p^2R^2}}\frac{\partial}{\partial t} + \frac{p}{\sqrt{1 - 2pe^{-a}B - p^2R^2}}\frac{\partial}{\partial \varphi}$$

Method 2:

Just plug straight into the definition

$$u = \frac{d}{d\tau} = \frac{d\lambda}{d\tau}\frac{d}{d\lambda}$$

Use

$$\frac{d}{d\lambda} = e^{-a}\frac{\partial}{\partial t} + p\frac{\partial}{\partial \varphi}$$

and

$$\begin{aligned} d\tau^2 &= -ds^2 = -\left(-e^{2a}\,dt^2 + 2B\,dt\,d\varphi + e^{-2a}\,dr^2 + R^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)\right) \\ &= e^{2a}\,dt^2 - 2B\,dt\,d\varphi - e^{-2a}\,dr^2 - R^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) \\ &= \left(e^{2a}\left(\frac{dt}{d\lambda}\right)^2 - 2B\frac{dt}{d\lambda}\frac{d\varphi}{d\lambda} - e^{-2a}\left(\frac{dr}{d\lambda}\right)^2 - R^2\left(\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2\right)\right)d\lambda^2 \end{aligned}$$

From the equations of motion for the particle,

$$\frac{dt}{d\lambda} = e^{-a}, \qquad \frac{d\varphi}{d\lambda} = p, \qquad \frac{dr}{d\lambda} = \frac{d\theta}{d\lambda} = 0$$

so

$$\begin{aligned} d\tau^2 &= \left(e^{2a}e^{-2a} - 2Be^{-a}p - R^2\sin^2\theta p^2\right)d\lambda^2 \\ &= \left(1 - 2Be^{-a}p - R^2\sin^2\theta p^2\right) \end{aligned}$$

and

$$\frac{d\lambda}{d\tau} = \frac{1}{\sqrt{1 - 2Be^{-a}p - R^2\sin^2\theta p^2}}$$

which gives the final result:

$$u = \frac{1}{\sqrt{1 - 2Be^{-a}p - R^2\sin^2\theta p^2}}\left(e^{-a}\frac{\partial}{\partial t} + p\frac{\partial}{\partial \varphi}\right)$$

Problem 3

a. What motions leave this metric unchanged? (Note: There is nothing to calculate here. Just look at the metric.)

Answer 3a

Since nothing depends on t, translations in the t coordinate leave the metric unchanged. The same thing works for the φ coordinate. With the $dtd\varphi$ term present, a particular axis for polar coordinates is picked out, so we do not have spherical symmetry.

b. Note the vector fields whose integral curves correspond to the motions described in part a. (Again, nothing to calculate. Just write down the answer.)

Answer 3b

The metric is invariant under translations along the integral curves of the vector fields

$$\frac{\partial}{\partial t} \text{ and } \frac{\partial}{\partial \varphi}$$

c. Is this spacetime static or stationary?

Answer 3c

It is stationary because it is invariant under a time translation. However the $dtd\varphi$ term prevents it from being invariant under a time reversal, so it is not static.

Problem 4

a. The coordinate (or holonomic) basis vectors $\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}$ are not orthonormal. Find a set of basis vectors e_μ that are orthonormal

Answer 4a

First notice that the only pair of vectors that fail to be orthogonal to each other are $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi}$ so start by finding linear combinations of those two vectors that are orthogonal to each other.

Try

$$v = \frac{\partial}{\partial t} + n\frac{\partial}{\partial \varphi}$$

and require it to be orthogonal to $\frac{\partial}{\partial \varphi}$. This is an arbitrary choice, by the way. We could just as easily find a combination orthogonal to $\frac{\partial}{\partial t}$. That would give a different, but equally valid orthonormal frame.

$$\begin{aligned} v \cdot \frac{\partial}{\partial \varphi} &= 0 \\ \left(\frac{\partial}{\partial t} + n\frac{\partial}{\partial \varphi}\right) \cdot \frac{\partial}{\partial \varphi} &= 0 \\ g_{03} + n g_{33} &= 0 \end{aligned}$$

That gives the result

$$n = -g_{03}/g_{33} = -B/\left(R^2 \sin^2 \theta\right)$$

or

$$v = \frac{\partial}{\partial t} - \frac{B}{R^2 \sin^2 \theta}\frac{\partial}{\partial \varphi}$$

The vectors $v, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}$ are all orthogonal to each other now and we just need to find their norms to get them normalized.

$$\begin{aligned} v \cdot v &= \left(\frac{\partial}{\partial t} - \frac{B}{R^2 \sin^2 \theta}\frac{\partial}{\partial \varphi}\right) \cdot \left(\frac{\partial}{\partial t} - \frac{B}{R^2 \sin^2 \theta}\frac{\partial}{\partial \varphi}\right) \\ &= \frac{\partial}{\partial t} \cdot \frac{\partial}{\partial t} - 2\frac{B}{R^2 \sin^2 \theta}\frac{\partial}{\partial t} \cdot \frac{\partial}{\partial \varphi} + \frac{B^2}{R^4 \sin^4 \theta}\frac{\partial}{\partial \varphi} \cdot \frac{\partial}{\partial \varphi} \\ &= g_{00} - 2\frac{B}{R^2 \sin^2 \theta} g_{03} + \frac{B^2}{R^4 \sin^4 \theta} g_{33} \\ &= -e^{2a} - 2\frac{B^2}{R^2 \sin^2 \theta} + \frac{B^2}{R^4 \sin^4 \theta} R^2 \sin^2 \theta \\ &= -e^{2a} - 2\frac{B^2}{R^2 \sin^2 \theta} + \frac{B^2}{R^2 \sin^2 \theta} \\ &= -e^{2a} - \frac{B^2}{R^2 \sin^2 \theta} \end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial r}\cdot\frac{\partial}{\partial r} &= g_{11} = e^{-2a} \\
\frac{\partial}{\partial \theta}\cdot\frac{\partial}{\partial \theta} &= g_{22} = R^2 \\
\frac{\partial}{\partial \varphi}\cdot\frac{\partial}{\partial \varphi} &= g_{33} = R^2 \sin^2\theta
\end{aligned}
$$

so our orthonormal frame vectors are

$$
\begin{aligned}
e_0 &= \frac{1}{\sqrt{e^{2a} + \frac{B^2}{R^2\sin^2\theta}}}\left(\frac{\partial}{\partial t} - \frac{B}{R^2\sin^2\theta}\frac{\partial}{\partial \varphi}\right) \\
e_1 &= e^a \frac{\partial}{\partial r} \\
e_2 &= \frac{1}{R}\frac{\partial}{\partial \theta} \\
e_3 &= \frac{1}{R\sin\theta}\frac{\partial}{\partial \varphi}
\end{aligned}
$$

b. The coordinate (or holonomic) basis forms $dt, dr, d\theta, d\varphi$ are not orthonormal. Find a set of basis forms ω^μ that are orthonormal.

Answer 4b

We have a tranformation from the holonomic basis to the orthonormal basis that has the matrix form

$$[e] = M\left[\frac{\partial}{\partial x}\right]$$

so the dual basis will be given by the inverse matrix

$$[\omega] = [dx]\, M^{-1}$$

In detail, we have

$$
\begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} =
\begin{pmatrix}
\frac{1}{\sqrt{e^{2a}+\frac{B^2}{R^2\sin^2\theta}}} & 0 & 0 & -\frac{B}{R^2\sin^2\theta}\frac{1}{\sqrt{e^{2a}+\frac{B^2}{R^2\sin^2\theta}}} \\
0 & e^a & 0 & 0 \\
0 & 0 & \frac{1}{R} & 0 \\
0 & 0 & 0 & \frac{1}{R\sin\theta}
\end{pmatrix}
\begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \varphi} \end{pmatrix}
$$

so that the row of orthonormal basis forms is given by

$$\begin{pmatrix} \omega^0 & \omega^1 & \omega^2 & \omega^3 \end{pmatrix} =$$

$$= \begin{pmatrix} dt & dr & d\theta & d\varphi \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{e^{2a}+\frac{B^2}{R^2\sin^2\theta}}} & 0 & 0 & -\frac{B}{R^2\sin^2\theta}\frac{1}{\sqrt{e^{2a}+\frac{B^2}{R^2\sin^2\theta}}} \\ 0 & e^a & 0 & 0 \\ 0 & 0 & \frac{1}{R} & 0 \\ 0 & 0 & 0 & \frac{1}{R\sin\theta} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} dt & dr & d\theta & d\varphi \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{R^2\sin^2\theta}\left(B^2+e^{2a}R^2\sin^2\theta\right)} & 0 & 0 & \frac{B}{R\sin\theta} \\ 0 & \frac{1}{e^a} & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} dt\sqrt{\frac{1}{R^2\sin^2\theta}\left(B^2+e^{2a}R^2\sin^2\theta\right)}, & dre^{-a}, & Rd\theta, & Rd\varphi\sin\theta + dt\frac{B}{R\sin\theta} \end{pmatrix}$$

or

$$\begin{aligned} \omega^0 &= dt\sqrt{e^{2a}+\frac{B^2}{R^2\sin^2\theta}} \\ \omega^1 &= e^{-a}\,dr \\ \omega^2 &= R\,d\theta \\ \omega^3 &= \frac{B}{R\sin\theta}\,dt + R\sin\theta\,d\varphi \end{aligned}$$

The Conformal Trick

The spacetime line element

$$ds^2 = -dt^2 + \psi^4 \left(dx^2 + dy^2 + dz^2\right)$$

where ψ is a function of the space coordinates x, y, z is not a solution of Einstein's equations, but it does incorporate a valuable trick – the conformal factor. Why the fourth power? For one thing it avoids flipping either the signature of the metric or the orientation of an orthonormal basis. The following problems explore this metric.

Problem 5

For this spacetime,

a. What would be the effective Newtonian gravitational potential governing the motion of slow-moving objects?

Answer 5a

From the equivalence principal alone, the motion of slow objects is governed entirely by the metric component g_{00}. Here that component is the same as for Minkowski spacctime, so there is no gravitational potential at all and no Newtonian acceleration of slow moving objects.

b. What spacetime motions would leave this spacetime metric unchanged? Is this spacetime static or stationary?

Answer 5b

Since the function ψ is an arbitrary function of the space coordinates, there are not spatial motions that leave the metric unchanged. However nothing depends on t so translation and in t will leave the metric unchanged. That invariance means the spacetime is at least stationary. There are no cross-terms in the metric to spoil time reversal, to the metric is also static.

Problem 6

Calculate the connection coefficients, the Riemann tensor components, the Ricci tensor components, and the Einstein tensor components for this spacetime. (Hint: You can save a lot of effort by expressing things in terms of Kronecker deltas like δ_{ij} instead of writing out components.)

What happens when you try to write Einstein's equations with a perfect fluid stress-energy tensor?

Answer 6

The straightforward procedure is to just plug into the formulas.

Start by giving the non-zero metric tensor components

$$\begin{aligned} g_{00} &= -1, \qquad g_{ij} = \psi^4 \delta_{ij} \\ g^{00} &= -1, \qquad g^{ij} = \psi^{-4} \delta^{ij} \end{aligned}$$

It is useful to let Latin indexes range from 1 to 3 and use the Kronecker delta for the flat space metric components.

Next, give the derivatives $g_{\mu\nu,\delta}$. All of the ones with any of the indexes equal to zero vanish, leaving just

$$g_{mn,d} = 4\psi^3 \psi_{,d} \delta_{mn}$$

Now work out the connection coefficients from the formula

$$\Gamma^{\alpha}{}_{\beta\delta} = \frac{1}{2} g^{\alpha\rho} \left(\partial_\delta g_{\rho\beta} + \partial_\beta g_{\rho\delta} - \partial_\rho g_{\beta\delta} \right)$$

All of the terms with any zero indexes vanish, leaving just the spatial connection coefficients

$$\begin{aligned} \Gamma^{a}{}_{bd} &= \frac{1}{2} g^{ar} \left(\partial_d g_{rb} + \partial_b g_{rd} - \partial_r g_{bd} \right) \\ &= \frac{1}{2} \psi^{-4} \delta^{ar} \left(4\psi^3 \psi_{,d} \delta_{rb} + 4\psi^3 \psi_{,b} \delta_{rd} - 4\psi^3 \psi_{,r} \delta_{bd} \right) \\ &= \frac{2}{\psi} \delta^{ar} \left(\psi_{,d} \delta_{rb} + \psi_{,b} \delta_{rd} - \psi_{,r} \delta_{bd} \right) \\ &= 2\psi^{-1} \left(\psi_{,d} \delta_{ab} + \psi_{,b} \delta_{ad} - \psi_{,a} \delta_{bd} \right) \end{aligned}$$

That wasn't so hard.

Now work out the Riemann tensor components from the standard formula:

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}{}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}{}_{\beta\mu} + \Gamma^{\alpha}{}_{\rho\mu} \Gamma^{\rho}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\rho\nu} \Gamma^{\rho}{}_{\beta\mu}$$

Components with zero indexes any where will vanish, so we just have the space components

$$R^{a}{}_{bmn} = \partial_m \Gamma^{a}{}_{bn} - \partial_n \Gamma^{a}{}_{bm} + \Gamma^{a}{}_{rm} \Gamma^{r}{}_{bn} - \Gamma^{a}{}_{rn} \Gamma^{r}{}_{bm}$$

Work on the derivative terms first.

$$\begin{aligned}\partial_m \Gamma^a{}_{bn} &= \left(2\psi^{-1}\left(\psi_{,n}\delta_{ab} + \psi_{,b}\delta_{an} - \psi_{,a}\delta_{bn}\right)\right)_{,m} \\ &= -2\psi^{-2}\psi_{,m}\left(\psi_{,n}\delta_{ab} + \psi_{,b}\delta_{an} - \psi_{,a}\delta_{bn}\right) \\ &\quad +2\psi^{-1}\left(\psi_{,nm}\delta_{ab} + \psi_{,bm}\delta_{an} - \psi_{,am}\delta_{bn}\right)\end{aligned}$$

$$\begin{aligned}\partial_n \Gamma^a{}_{bm} &= \left(2\psi^{-1}\left(\psi_{,m}\delta_{ab} + \psi_{,b}\delta_{am} - \psi_{,a}\delta_{bm}\right)\right)_{,n} \\ &= -2\psi^{-2}\psi_{,n}\left(\psi_{,m}\delta_{ab} + \psi_{,b}\delta_{am} - \psi_{,a}\delta_{bm}\right) \\ &\quad +2\psi^{-1}\left(\psi_{,nm}\delta_{ab} + \psi_{,bn}\delta_{am} - \psi_{,an}\delta_{bm}\right)\end{aligned}$$

and the difference:

$\partial_m \Gamma^a{}_{bn} - \partial_n \Gamma^a{}_{bm} =$
$-2\psi^{-2}\psi_{,m}\left(\psi_{,n}\delta_{ab} + \psi_{,b}\delta_{an} - \psi_{,a}\delta_{bn}\right) + 2\psi^{-2}\psi_{,n}\left(\psi_{,m}\delta_{ab} + \psi_{,b}\delta_{am} - \psi_{,a}\delta_{bm}\right)$
$+2\psi^{-1}\left(\psi_{,nm}\delta_{ab} + \psi_{,bm}\delta_{an} - \psi_{,am}\delta_{bn}\right) - 2\psi^{-1}\left(\psi_{,nm}\delta_{ab} + \psi_{,bn}\delta_{am} - \psi_{,an}\delta_{bm}\right)$

$= -2\psi^{-2}\left(\psi_{,m}\psi_{,n}\delta_{ab} + \psi_{,m}\psi_{,b}\delta_{an} - \psi_{,m}\psi_{,a}\delta_{bn}\right) + 2\psi^{-2}\left(\psi_{,n}\psi_{,m}\delta_{ab} + \psi_{,n}\psi_{,b}\delta_{am} - \psi_{,n}\psi_{,a}\delta_{bm}\right)$
$+2\psi^{-1}\left(\psi_{,nm}\delta_{ab} + \psi_{,bm}\delta_{an} - \psi_{,am}\delta_{bn} - \psi_{,nm}\delta_{ab} - \psi_{,bn}\delta_{am} + \psi_{,an}\delta_{bm}\right)$

$= 2\psi^{-2}\left(\psi_{,n}\psi_{,m}\delta_{ab} + \psi_{,n}\psi_{,b}\delta_{am} - \psi_{,n}\psi_{,a}\delta_{bm} - \psi_{,m}\psi_{,n}\delta_{ab} - \psi_{,m}\psi_{,b}\delta_{an} + \psi_{,m}\psi_{,a}\delta_{bn}\right)$
$+2\psi^{-1}\left(\psi_{,nm}\delta_{ab} + \psi_{,bm}\delta_{an} - \psi_{,am}\delta_{bn} - \psi_{,nm}\delta_{ab} - \psi_{,bn}\delta_{am} + \psi_{,an}\delta_{bm}\right)$

$= 2\psi^{-2}\left(\psi_{,n}\psi_{,b}\delta_{am} - \psi_{,n}\psi_{,a}\delta_{bm} - \psi_{,m}\psi_{,b}\delta_{an} + \psi_{,m}\psi_{,a}\delta_{bn}\right)$
$+2\psi^{-1}\left(\psi_{,bm}\delta_{an} - \psi_{,am}\delta_{bn} - \psi_{,bn}\delta_{am} + \psi_{,an}\delta_{bm}\right)$

Now pull out a factor so that we can compare and cancel terms more easily.
$\frac{1}{2}\psi^2\left(\partial_m \Gamma^a{}_{bn} - \partial_n \Gamma^a{}_{bm}\right) =$
$\psi_{,n}\psi_{,b}\delta_{am} - \psi_{,n}\psi_{,a}\delta_{bm} - \psi_{,m}\psi_{,b}\delta_{an} + \psi_{,m}\psi_{,a}\delta_{bn}$
$+\psi\left(\psi_{,bm}\delta_{an} - \psi_{,am}\delta_{bn} - \psi_{,bn}\delta_{am} + \psi_{,an}\delta_{bm}\right)$

Next, start on the product terms in the curvature.

$$\begin{aligned}\Gamma^a{}_{rm}\Gamma^r{}_{bn} &= \left(2\psi^{-1}\left(\psi_{,m}\delta_{ar} + \psi_{,r}\delta_{am} - \psi_{,a}\delta_{rm}\right)\left(2\psi^{-1}\left(\psi_{,n}\delta_{rb} + \psi_{,b}\delta_{rn} - \psi_{,r}\delta_{bn}\right)\right)\right) \\ &= 4\psi^{-2}\left(\psi_{,m}\delta_{ar} + \psi_{,r}\delta_{am} - \psi_{,a}\delta_{rm}\right)\left(\psi_{,n}\delta_{rb} + \psi_{,b}\delta_{rn} - \psi_{,r}\delta_{bn}\right)\end{aligned}$$

Pull out a factor so we can see terms to cancel more easily.

$\frac{1}{4}\psi^2 \Gamma^a{}_{rm}\Gamma^r{}_{bn} =$
$= \left(\psi_{,m}\delta_{ar} + \psi_{,r}\delta_{am} - \psi_{,a}\delta_{rm}\right)\left(\psi_{,n}\delta_{rb} + \psi_{,b}\delta_{rn} - \psi_{,r}\delta_{bn}\right)$

$= \psi_{,m}\delta_{ar}\psi_{,n}\delta_{rb} + \psi_{,m}\delta_{ar}\psi_{,b}\delta_{rn} - \psi_{,m}\delta_{ar}\psi_{,r}\delta_{bn}$
$+\psi_{,r}\delta_{am}\psi_{,n}\delta_{rb} + \psi_{,r}\delta_{am}\psi_{,b}\delta_{rn} - \psi_{,r}\delta_{am}\psi_{,r}\delta_{bn}$
$-\psi_{,a}\delta_{rm}\psi_{,n}\delta_{rb} - \psi_{,a}\delta_{rm}\psi_{,b}\delta_{rn} + \psi_{,a}\delta_{rm}\psi_{,r}\delta_{bn}$

$= \psi_{,m}\psi_{,n}\delta_{ab} + \psi_{,m}\psi_{,b}\delta_{an} - \psi_{,m}\psi_{,a}\delta_{bn}$
$+\psi_{,b}\psi_{,n}\delta_{am} + \psi_{,n}\psi_{,b}\delta_{am} - \psi_{,r}\psi_{,r}\delta_{am}\delta_{bn}$
$-\psi_{,a}\psi_{,n}\delta_{mb} - \psi_{,a}\psi_{,b}\delta_{mn} + \psi_{,a}\psi_{,m}\delta_{bn}$

Now put in the other product term and see what cancels.
$\frac{1}{4}\psi^2\left(\Gamma^a{}_{rm}\Gamma^r{}_{bn} - \Gamma^a{}_{rn}\Gamma^r{}_{bm}\right) =$
$= \psi_{,m}\psi_{,n}\delta_{ab} + \psi_{,m}\psi_{,b}\delta_{an} - \psi_{,m}\psi_{,a}\delta_{bn}$
$+\psi_{,b}\psi_{,n}\delta_{am} + \psi_{,n}\psi_{,b}\delta_{am} - \psi_{,r}\psi_{,r}\delta_{am}\delta_{bn}$
$-\psi_{,a}\psi_{,n}\delta_{mb} - \psi_{,a}\psi_{,b}\delta_{mn} + \psi_{,a}\psi_{,m}\delta_{bn}$
$-\psi_{,n}\psi_{,m}\delta_{ab} - \psi_{,n}\psi_{,b}\delta_{am} + \psi_{,n}\psi_{,a}\delta_{bm}$
$-\psi_{,b}\psi_{,m}\delta_{an} - \psi_{,m}\psi_{,b}\delta_{an} + \psi_{,r}\psi_{,r}\delta_{an}\delta_{bm}$
$+\psi_{,a}\psi_{,m}\delta_{nb} + \psi_{,a}\psi_{,b}\delta_{mn} - \psi_{,a}\psi_{,n}\delta_{bm}$

$= \psi_{,n}\psi_{,b}\delta_{am} - \psi_{,r}\psi_{,r}\delta_{am}\delta_{bn} + \psi_{,a}\psi_{,m}\delta_{bn}$
$-\psi_{,m}\psi_{,b}\delta_{an} + \psi_{,r}\psi_{,r}\delta_{an}\delta_{bm} - \psi_{,a}\psi_{,n}\delta_{bm}$

$= \psi_{,n}\psi_{,b}\delta_{am} - \psi_{,m}\psi_{,b}\delta_{an} + \psi_{,a}\psi_{,m}\delta_{bn} - \psi_{,a}\psi_{,n}\delta_{bm} + \psi_{,r}\psi_{,r}\left(\delta_{an}\delta_{bm} - \delta_{am}\delta_{bn}\right)$

Multiply this result by 2 so we get the same factor outside as for the derivative terms.
$\frac{1}{2}\psi^2\left(\Gamma^a{}_{rm}\Gamma^r{}_{bn} - \Gamma^a{}_{rn}\Gamma^r{}_{bm}\right) =$
$2\psi_{,n}\psi_{,b}\delta_{am} - 2\psi_{,m}\psi_{,b}\delta_{an} + 2\psi_{,a}\psi_{,m}\delta_{bn} - 2\psi_{,a}\psi_{,n}\delta_{bm} + 2\psi_{,r}\psi_{,r}\left(\delta_{an}\delta_{bm} - \delta_{am}\delta_{bn}\right)$

and put the derivative and product terms together to get
$\frac{1}{2}\psi^2 R^a{}_{bmn} =$
$\psi_{,n}\psi_{,b}\delta_{am} - \psi_{,n}\psi_{,a}\delta_{bm} - \psi_{,m}\psi_{,b}\delta_{an} + \psi_{,m}\psi_{,a}\delta_{bn} + \psi\left(\psi_{,bm}\delta_{an} - \psi_{,am}\delta_{bn} - \psi_{,bn}\delta_{am} + \psi_{,an}\delta_{bm}\right)$
$+2\psi_{,n}\psi_{,b}\delta_{am} - 2\psi_{,m}\psi_{,b}\delta_{an} + 2\psi_{,a}\psi_{,m}\delta_{bn} - 2\psi_{,a}\psi_{,n}\delta_{bm} + 2\psi_{,r}\psi_{,r}\left(\delta_{an}\delta_{bm} - \delta_{am}\delta_{bn}\right)$

$\frac{1}{2}\psi^2 R^a{}_{bmn} = 3\psi_{,n}\psi_{,b}\delta_{am} - 3\psi_{,n}\psi_{,a}\delta_{bm} - 3\psi_{,m}\psi_{,b}\delta_{an} + 3\psi_{,m}\psi_{,a}\delta_{bn}$
$+\psi\left(\psi_{,bm}\delta_{an} - \psi_{,am}\delta_{bn} - \psi_{,bn}\delta_{am} + \psi_{,an}\delta_{bm}\right) + 2\psi_{,r}\psi_{,r}\left(\delta_{an}\delta_{bm} - \delta_{am}\delta_{bn}\right)$

so that our result for the non-zero Riemann tensor components is:

$$\begin{aligned}\frac{1}{2}\psi^2 R^a{}_{bmn} &= 3\left(\psi_{,n}\psi_{,b}\delta_{am} - \psi_{,n}\psi_{,a}\delta_{bm} - \psi_{,m}\psi_{,b}\delta_{an} + \psi_{,m}\psi_{,a}\delta_{bn}\right) \\ &+\psi\left(\psi_{,bm}\delta_{an} - \psi_{,am}\delta_{bn} - \psi_{,bn}\delta_{am} + \psi_{,an}\delta_{bm}\right) \\ &+2\psi_{,r}\psi_{,r}\left(\delta_{an}\delta_{bm} - \delta_{am}\delta_{bn}\right)\end{aligned}$$

Now go for the Ricci tensor. Contract on a, m

$$\begin{aligned}\frac{1}{2}\psi^2 R_{bn} &= \\ \frac{1}{2}\psi^2 R^a{}_{ban} &= 3\left(\psi_{,n}\psi_{,b}\delta_{aa} - \psi_{,n}\psi_{,a}\delta_{ba} - \psi_{,a}\psi_{,b}\delta_{an} + \psi_{,a}\psi_{,a}\delta_{bn}\right) \\ &+\psi\left(\psi_{,ba}\delta_{an} - \psi_{,aa}\delta_{bn} - \psi_{,bn}\delta_{aa} + \psi_{,an}\delta_{ba}\right) \\ &+2\psi_{,r}\psi_{,r}\left(\delta_{an}\delta_{ba} - \delta_{aa}\delta_{bn}\right)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\psi^2 R^a{}_{ban} &= 3\left(3\psi_{,n}\psi_{,b} - \psi_{,n}\psi_{,b} - \psi_{,n}\psi_{,b} + \psi_{,r}\psi_{,r}\delta_{bn}\right) \\ &+\psi\left(\psi_{,bn} - \psi_{,rr}\delta_{bn} - 3\psi_{,bn} + \psi_{,bn}\right) \\ &+2\psi_{,r}\psi_{,r}\left(\delta_{nb} - 3\delta_{bn}\right)\end{aligned}$$

$\frac{1}{2}\psi^2 R^a{}_{ban} = 3\left(3\psi_{,n}\psi_{,b} - \psi_{,n}\psi_{,b} - \psi_{,n}\psi_{,b} + \psi_{,r}\psi_{,r}\delta_{bn}\right)$
$+\psi\left(\psi_{,bn} - \psi_{,rr}\delta_{bn} - 3\psi_{,bn} + \psi_{,bn}\right) + 2\psi_{,r}\psi_{,r}\left(\delta_{nb} - 3\delta_{bn}\right)$

$= 3\left(\psi_{,n}\psi_{,b} + \psi_{,r}\psi_{,r}\delta_{bn}\right) + \psi\left(-\psi_{,rr}\delta_{bn} - \psi_{,bn}\right) - 4\psi_{,r}\psi_{,r}\delta_{nb}$
$= 3\psi_{,n}\psi_{,b} + 3\psi_{,r}\psi_{,r}\delta_{bn} - \psi\psi_{,rr}\delta_{bn} - \psi\psi_{,bn} - 4\psi_{,r}\psi_{,r}\delta_{nb}$
$= 3\psi_{,n}\psi_{,b} - \psi_{,r}\psi_{,r}\delta_{bn} - \psi\psi_{,rr}\delta_{bn} - \psi\psi_{,bn}$

Behold the Ricci Tensor components:

$$\frac{1}{2}\psi^2 R_{bn} = \frac{1}{2}\psi^2 R^a{}_{ban} = 3\psi_{,n}\psi_{,b} - \psi_{,r}\psi_{,r}\delta_{bn} - \psi\psi_{,rr}\delta_{bn} - \psi\psi_{,bn}$$

Contract again on n, b. Remember that we have to use the **actual inverse metric tensor** $g^{bn} = \psi^{-4}\delta^{bn}$ to raise an index to do this contraction.

$$\frac{1}{2}\psi^2 R = \frac{1}{2}\psi^2 R^a{}_{ban} g^{bn} = \frac{1}{2}\psi^2 R^a{}_{ban}\psi^{-4}\delta^{bn}$$

$$\frac{1}{2}\psi^2 R = -4\psi^{-4}\psi\psi_{,rr}$$

$$R = -\frac{8}{\psi^5}\psi_{,rr} = -8\frac{\nabla^2\psi}{\psi^5}$$

Now go for the Einstein Tensor

$$G_{\beta\nu} = R_{\beta\nu} - \frac{1}{2} g_{\beta\nu} R$$

The timelike components of the tensor are then

$$G_{0\nu} = -\frac{1}{2} g_{0\nu} R$$

or

$$\begin{aligned} G_{00} &= \frac{1}{2} R = -4 \frac{\nabla^2 \psi}{\psi^5} \\ G_{0i} &= 0 \end{aligned}$$

The spacelike components are

$$\begin{aligned} \frac{1}{2}\psi^2 G_{bn} &= \frac{1}{2}\psi^2 \left(R_{bn} - \frac{1}{2} g_{bn} R \right) \\ &= \frac{1}{2}\psi^2 R_{bn} - \frac{1}{4}\psi^2 g_{bn} R \\ &= \frac{1}{2}\psi^2 R_{bn} + \frac{1}{4}\psi^2 \psi^4 \delta_{bn} \left(8 \frac{\nabla^2 \psi}{\psi^5} \right) \\ &= \frac{1}{2}\psi^2 R_{bn} + 2\psi \nabla^2 \psi \delta_{bn} \end{aligned}$$

or

$$\begin{aligned} \frac{1}{2}\psi^2 G_{bn} &= 3\psi_{,n}\psi_{,b} - \psi_{,r}\psi_{,r}\delta_{bn} - \psi\psi_{,rr}\delta_{bn} - \psi\psi_{,bn} + 2\psi\psi_{rr}\delta_{bn} \\ &= 3\psi_{,n}\psi_{,b} - \psi\psi_{,bn} + \psi\psi_{rr}\delta_{bn} - \psi_{,r}\psi_{,r}\delta_{bn} \end{aligned}$$

$$\frac{1}{2}\psi^2 G_{bn} = 3\psi_{,n}\psi_{,b} - \psi\psi_{,bn} + \left(\psi\nabla^2\psi - \nabla\psi \cdot \nabla\psi \right) \delta_{bn}$$

To finish the Einstein equations, note that the contravariant stress-energy tensor components for an isotropic fluid are

$$T^{00} = \rho, \qquad T^{0n} = 0, \qquad T^{bn} = p\delta^{bn}$$

The covariant components are then

$$T_{\beta\nu} = g_{\beta\rho} g_{\nu\sigma} T^{\rho\sigma}$$

or

$$\begin{aligned} T_{00} &= g_{00} g_{00} T^{00} = T^{00} = \rho \\ T_{0n} &= 0 \\ T_{bn} &= g_{br} g_{ns} T^{rs} = \psi^8 \delta_{br} \delta_{ns} \delta_{rs} = \psi^8 p \delta_{bn} \\ \frac{1}{2}\psi^2 T_{bn} &= \frac{1}{2}\psi^{10} p \delta_{bn} \end{aligned}$$

The Einstein equations are then

$$
\begin{aligned}
G_{00} &= -4\frac{\nabla^2\psi}{\psi^5} = 8\pi\rho \\
\frac{1}{2}\psi^2 G_{bn} &= 3\psi_{,n}\psi_{,b} - \psi\psi_{,bn} - \left(\psi\nabla^2\psi - \nabla\psi\cdot\nabla\psi\right)\delta_{bn} = \frac{1}{2}\psi^{10}p\delta_{bn}
\end{aligned}
$$

or

$$
\begin{aligned}
\nabla^2\psi &= -2\pi\psi^5\rho \\
3\psi_{,n}\psi_{,b} - \psi\psi_{,bn} - \left(\psi\nabla^2\psi - \nabla\psi\cdot\nabla\psi\right)\delta_{bn} &= \frac{1}{2}\psi^{10}p\delta_{bn}
\end{aligned}
$$

It is pretty clear that there are no nontrivial solutions of the second set of (six independent) equations, but that is no surprise since our original guess for the metric tensor corresponded to having no Newtonian gravitational field at all.