

# Final Exam

Please attempt all of the following problems before the due date. All problems count the same even though some are more complex than others.

## Bad Tensor Analysis

### Problem 1

Index notation, with the Einstein Summation Convention is extremely flexible and can express any kind of tensor operation. However it can be **too** flexible because it can express operations that do not exist. Here are a few examples of "Bad Tensor Analysis." Explain what is wrong with each example. Do more than just state what rule is violated and indicate why the operation cannot arise from valid geometrical operations.

a.  $A^i + B^j$

Answer 1a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).

b.  $A^{ij}B_jC_{ji}$

Answer 1b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).

c.  $A^{ijk}B_k + C^{ijk}B_j$

## Answer 1c

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).

d.  $A^{abc}K_c = B^{abc}J_n$

## Answer 1d

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).

# Metric Tensor Follies

The following problems refer to the following situation:

Start with the spacetime line element

$$ds^2 = -e^{2a(r)} dt^2 + 2B(r) dt d\varphi + e^{-2a} dr^2 + R(r)^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

with coordinates

$$x^0 = t, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \varphi.$$

Notice that this metric is specified by three functions  $a, b, R$ , of the single variable,  $r$ .

The world line of a particular observer is given by

$$\begin{aligned} t &= e^{-a} \lambda \\ r &= r_0 \\ \theta &= \frac{\pi}{2} \\ \varphi &= p\lambda \end{aligned}$$

where  $\lambda$  is the curve parameter and  $r_0$  and  $p$  are constants.

## Problem 2:

- a. Use the line element to calculate how much proper time elapses for this observer for each unit of coordinate time  $t$ .

Answer 2a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).

- b. Display the metric tensor components  $g_{\mu\nu}$  that are indicated by this line element.

Answer 2b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).

C. Find the four-velocity vector of this observer.

### Answer 2c

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).

## Problem 3

- a. What motions leave this metric unchanged? (Note: There is nothing to calculate here. Just look at the metric.)

### Answer 3a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).

- b. Note the vector fields whose integral curves correspond to the motions described in part a. (Again, nothing to calculate. Just write down the answer.)

### Answer 3b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).

- c. Is this spacetime static or stationary?

### Answer 3c

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).

## Problem 4

- a. The coordinate (or holonomic) basis vectors  $\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}$  are not orthonormal. Find a set of basis vectors  $e_\mu$  that are orthonormal.

### Answer 4a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).

- b. The coordinate (or holonomic) basis forms  $dt, dr, d\theta, d\varphi$  are not orthonormal. Find a set of basis forms  $\omega^\mu$  that are orthonormal.

### Answer 4b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).

# The Conformal Trick

The spacetime line element

$$ds^2 = -dt^2 + \psi^4(dx^2 + dy^2 + dz^2)$$

where  $\psi$  is a function of the space coordinates  $x, y, z$  is not a solution of Einstein's equations, but it does incorporate a valuable trick – the conformal factor. Why the fourth power? For one thing it avoids flipping either the signature of the metric or the orientation of an orthonormal basis. The following problems explore this metric.

## Problem 5

For this spacetime,

- a.** What would be the effective Newtonian gravitational potential governing the motion of slow-moving objects?

Answer 5a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).

- b.** What spacetime motions would leave this spacetime metric unchanged? Is this spacetime static or stationary?

Answer 5b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).

## Problem 6

Calculate the connection coefficients, the Riemann tensor components, the Ricci tensor components, and the Einstein tensor components for this spacetime. (Hint: You can save a lot of effort by expressing things in terms of Kronecker deltas like  $\delta_{ij}$  instead of writing out components.)

What happens when you try to write Einstein's equations with a perfect fluid stress-energy tensor?

## Answer 6

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at [rgowdy@saturn.vcu.edu](mailto:rgowdy@saturn.vcu.edu).