Final Exam

Please attempt all of the following problems before the due date. All problems count the same even though some are more complex than others. Assume that $c = 1$ units are used throughout.

Problem 1

Consider a complex three-dimensional vector space $\mathbb{C}P$ with basis vectors $b_1, b_2, b_3$ and metric tensor components

$$
\begin{bmatrix}
g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 1 \\
-1 & 0 & 1 \\
-1 & -1 & 0
\end{bmatrix}
$$

An antilinear conjugation operator $C$ generates a conjugate space $\mathbb{C}_P$ that is spanned by basis vectors $b_\overline{A} = C(b_A)$

The direct sum space $D_P = \mathbb{C}_P \oplus \mathbb{C}_P$ consists of six component vectors of the form

$$
\chi = \chi^1 b_1 + \chi^2 b_2 + \chi^3 b_3 + \chi^4 \overline{b}_1 + \chi^5 \overline{b}_2 + \chi^6 \overline{b}_3
$$

Within this space, the metric tensor components are given by

$$
g_{A\overline{B}} = g_{A\overline{B}} = 0 \\
g_{\overline{A}B} = g_{A\overline{B}}.$$

a. A vector $\chi$ in $D_P$ is considered to be real if it is self-conjugate: $C(\chi) = \chi$.

Find the conditions that the complex coefficients $\chi^A$ and $\chi^{\overline{A}}$ must satisfy if $\chi$ is real.

Answer 1a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

b. Calculate the inner products of the real second rank basis tensors

$$
E_1 = \frac{1}{\sqrt{2}} (b_1 \otimes b_1 + b_1 \otimes b_1) \\
E_2 = \frac{1}{\sqrt{2}} (b_2 \otimes b_2 + b_2 \otimes b_2) \\
E_3 = \frac{1}{\sqrt{2}} (b_3 \otimes b_3 + b_3 \otimes b_3)
$$
and thus display the array of metric components for the three dimensional space that is spanned by these vectors.

Answer 1b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

C. Use the Scientific Notebook command Compute|Matrices|Eigenvalues to determine the signature of the pseudo-Riemannian space described in part b above.

Answer 1c

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.
Problem 2

For a scalar function $f$, vector fields

\[ u = u^\alpha \frac{\partial}{\partial x^\alpha}, \quad w = w^\alpha \frac{\partial}{\partial x^\alpha}, \]

and the definitions used in this course, express the following quantities in terms of partial derivatives and the connection coefficients $\Gamma^\beta_{\alpha\beta}$.

a. $uf =$

Answer 2a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

b. $\nabla_w u =$

Answer 2b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

c. $\mathring{D}_w u =$

Answer 2c

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

d. $D_w u =$

Answer 2d

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.
e. \( D_wD_u u - \nabla_w \nabla_u u = \)

Answer 2e

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

Problem 3

Use the mass-energy conservation equations

\[ T^{\alpha\rho} ;_{\rho} = 0 \]

to find the equations that are obeyed by the matter density \( \rho \) of a fluid

a. when the pressure and density are related by

\[ p = -\rho \]

Answer 3a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

b. when the pressure and density are related by

\[ p = \rho \]

Answer 3b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.
Problem 4

For an electromagnetic field tensor with components $F_{\alpha\beta}$ in a holonomic frame-field and a connection with zero torsion, express each of the following quantities in terms of the partial derivatives $F_{\alpha\beta,\delta}$ and the connection coefficients $\Gamma^\gamma_{\mu\nu}$. Take full advantage of the index symmetries of these objects to cancel terms when possible.

a. $F_{\alpha\beta;\delta} =$

Answer 4a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

b. $F_{\alpha\beta;\delta} + F_{\delta\alpha;\beta} + F_{\beta\delta;\alpha} =$

Answer 4b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

c. $F^{\alpha\beta}_{\delta;\gamma} =$

Answer 4c

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

d. $F^{\alpha\beta}_{\gamma;\rho} =$
Answer 4d

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.
Problem 5

Consider a spacetime with the metric tensor
\[ g = -dt \otimes dt + t (dx \otimes dx + dy \otimes dy + dz \otimes dz) \]
This metric is a special case of the one we considered in homework No.14. You can use the solution of that homework as a starting point here.

a. Calculate all of the non-zero components of the full Riemann curvature tensor.

Answer 5a

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

b. Calculate all of the non-zero components of the Ricci curvature tensor.

Answer 5b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

c. Calculate the scalar curvature.

Answer 5b

Put all of your calculations here. When you have completed all of the problems, wrap the resulting file and e-mail it to me at rgowdy@saturn.vcu.edu.

d. Calculate the Einstein tensor and find the matter density and pressure that would exist in this spacetime.