Semidefiniteness Without Real Symmetry

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Abstract Let A be an n-by-n matrix with real entries. We show that a necessary and sufficient condition for A to have positive semidefinite or negative semidefinite symmetric part $H(A) = \frac{1}{2}(A + A^T)$ is that rank$[H(A)X] \leq$ rank$[X^TAX]$, for all $X \in M_n(\mathbf{R})$. Further, if A has positive semidefinite or negative semidefinite symmetric part, and A^2 has positive semidefinite symmetric part, then rank$[AX] =$ rank$[X^TAX]$, for all $X \in M_n(\mathbf{R})$. This result implies the usual row and column inclusion property for positive semidefinite matrices. Finally, we show that if $A, A^2, \ldots, A^k$ $(k \geq 2)$ all have positive semidefinite symmetric part then rank$[AX] =$ rank$[X^TAX] = \cdots =$ rank$[X^TA^{k-1}X]$, for all $X \in M_n(\mathbf{R})$.

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Let A be an n-by-n matrix with real entries (i.e. $A \in M_n(\mathbf{R})$). The symmetric part of A is defined by $H(A) = \frac{1}{2}(A + A^T)$. The principal submatrix of A lying in rows and columns $\alpha \subseteq \{1, \ldots, n\}$ will be denoted by $A[\alpha]$, and the submatrix lying in rows α and columns β is denoted $A[\alpha, \beta]$. A matrix $A \in M_n(\mathbf{R})$ is called positive semidefinite if it is symmetric ($A^T = A$) and $\mathbf{x}^TA\mathbf{x} \geq 0$ for each $\mathbf{x} \in \mathbf{R}^n$. $A \in M_n(\mathbf{R})$ is said to have positive semidefinite symmetric part if $H(A)$ is positive semidefinite (Other names have been used for such matrices in [GV], [WC].) We say that a matrix $A \in M_n(\mathbf{R})$ is semidefinite, in the event that it is either positive semidefinite or negative semidefinite (i.e. $\mathbf{x}^TA\mathbf{x} \geq 0$, for all $\mathbf{x} \in \mathbf{R}^n$; or $\mathbf{x}^TA\mathbf{x} \leq 0$, for all $\mathbf{x} \in \mathbf{R}^n$). We will consider some familiar properties for positive semidefinite matrices and ask to what extent they hold for matrices with positive semidefinite symmetric part, i.e. without the symmetry assumption. This extends the study of such matrices in [J2,J3,J4,J5,JN]. Not all of the results herein extend directly to complex matrices with positive semidefinite Hermitian part. It is perhaps worth noting that $\mathbf{x}^T A\mathbf{x} = \frac{1}{2}\mathbf{x}^T(A + A^T)\mathbf{x}$.

A matrix $A \in M_n(\mathbf{R})$ satisfies row (column) inclusion if $A[\{i\}, \alpha]$ lies in the row space of $A[\alpha]$ for each $i = 1, \ldots, n$ (if $A[\alpha, \{j\}]$ lies in the column space of $A[\alpha]$ for each $j = 1, \ldots, n$) and each $\alpha \subseteq \{1, \ldots, n\}$. The three properties of positive semidefinite matrices we wish to consider are

(I) $\mathbf{x}^T A\mathbf{x} = 0$ implies $A\mathbf{x} = \mathbf{0}$, for $\mathbf{x} \in \mathbf{R}^n$.

(II) $\operatorname{rank}[AX] = \operatorname{rank}[X^T AX]$, for all $X \in M_n(\mathbf{R})$ (which generalizes the condition that $\operatorname{rank}[Y] = \operatorname{rank}[Y^T Y]$, for all $Y \in M_n(\mathbf{R})$).

(III) A satisfies both row and column inclusion.

None of these properties hold for general matrices $A \in M_n(\mathbf{R})$. (I) is a special case of (II). (III) (found in [BJL], [J1], for example) will be seen to follow easily from (II). We first focus upon properties (I) and (II).

Theorem 1 For $A \in M_n(\mathbf{R})$, the following statements are equivalent:

(a) A has semidefinite symmetric part;

(b) $\operatorname{rank}[H(A)X] \leq \operatorname{rank}[X^T AX]$, for all $X \in M_n(\mathbf{R})$;

(c) $\mathbf{x}^T A\mathbf{x} = 0$ implies $H(A)\mathbf{x} = 0$, for $\mathbf{x} \in \mathbf{R}^n$.

Moreover, if (a), (b) or (c) is true then we have equality in (b) if and only if $\operatorname{rank}[A + A^T] = \operatorname{rank}[A]$.

Proof We first show that (a) implies (b). Since $H(A)$ is positive semidefinite or negative semidefinite we can write $A + A^T = \pm B^T B$, for some $B \in M_n(\mathbf{R})$. Let $\mathbf{u} \in \ker[X^T AX]$. $X^T AX\mathbf{u} = \mathbf{0}$ implies that $\mathbf{u}^T X^T AX\mathbf{u} = 0$ and $\mathbf{u}^T X^T A^T X\mathbf{u} = 0$. We have then $0 = \mathbf{u}^T X^T(A + A^T)X\mathbf{u} = \pm\mathbf{u}^T X^T B^T BX\mathbf{u} = \pm(BX\mathbf{u})^T BX\mathbf{u}$, so that $BX\mathbf{u} = \mathbf{0}$. But then $B^T BX\mathbf{u} = \mathbf{0}$, and so $(A + A^T)X\mathbf{u} = \mathbf{0}$. Thus, $\mathbf{u} \in \ker[(A + A^T)X]$. We have just shown that $\ker[X^T AX] \subseteq \ker[(A+A^T)X]$. This implies $n - \operatorname{rank}[X^T AX] \leq n - \operatorname{rank}[(A+A^T)X]$, which in turn implies $\operatorname{rank}[(A + A^T)X] \leq \operatorname{rank}[X^T AX]$.

For (b) implies (c), just take $X \in M_n(\mathbf{R})$ with first column $\mathbf{x}$ and all zeros in the remaining columns.

Suppose (c) is true and (a) is false. Then there exist $\mathbf{x}$, $\mathbf{y} \in \mathbf{R}^n$ such that $\mathbf{x}^T H(A)\mathbf{x} > 0$ and $\mathbf{y}^T H(A)\mathbf{y} < 0$. Now consider the quadratic $(s\mathbf{x}+\mathbf{y})^T H(A)(s\mathbf{x}+\mathbf{y}) = s^2\mathbf{x}^T H(A)\mathbf{x} + s(\mathbf{x}^T H(A)\mathbf{y} + \mathbf{y}^T H(A)\mathbf{x}) + \mathbf{y}^T H(A)\mathbf{y}$, with $s \in \mathbf{R}$. This has discriminant $(\mathbf{x}^T H(A)\mathbf{y} + \mathbf{y}^T H(A)\mathbf{x})^2 - 4(\mathbf{x}^T H(A)\mathbf{x})(\mathbf{y}^T H(A)\mathbf{y}) > 0$, so the quadratic has two unequal roots s_1 and s_2 in $\mathbf{R}$. Then $(s_1\mathbf{x} + \mathbf{y})^T H(A)(s_1\mathbf{x} + \mathbf{y}) = (s_1\mathbf{x} + \mathbf{y})^T A(s_1\mathbf{x} + \mathbf{y}) = 0$, which implies

$H(A)(s_1\mathbf{x}+\mathbf{y}) = 0$, and similarly $H(A)(s_2\mathbf{x}+\mathbf{y}) = 0$. Then $s_1H(A)\mathbf{x}+H(A)\mathbf{y} = \mathbf{0}$ and $s_2H(A)\mathbf{x}+H(A)\mathbf{y} = \mathbf{0}$, which on subtracting becomes $(s_1-s_2)H(A)\mathbf{x} = \mathbf{0}$. So $H(A)\mathbf{x} = \mathbf{0}$ (since $s_1 \neq s_2$) implying $\mathbf{x}^TH(A)\mathbf{x} = 0$, contradiction.

Finally, we prove the "equality" part of the theorem. The "only if" part of this is clear. For the "if" part, notice that $A\mathbf{u} = \mathbf{0}$ implies $\mathbf{u}^TA\mathbf{u} = 0$, so $\mathbf{u}^T(A+A^T)\mathbf{u} = 0$, and hence $(A+A^T)\mathbf{u} = \mathbf{0}$. Thus $\ker[A] \subseteq \ker[A+A^T]$, but then since $\operatorname{rank}[A+A^T] = \operatorname{rank}[A]$ we have that $\ker[A] = \ker[A+A^T]$. Also, we saw in the proof of Theorem 1 that $\ker[X^TAX] \subseteq \ker[(A+A^T)X]$. Suppose that $(A+A^T)X\mathbf{u} = \mathbf{0}$. This implies $X\mathbf{u} \in \ker[A+A^T] = \ker[A]$, and then $AX\mathbf{u} = \mathbf{0}$, i.e. $\ker[(A+A^T)X] \subseteq \ker[AX]$. We can now conclude with $\operatorname{rank}[X^TAX] \leq \operatorname{rank}[AX] \leq \operatorname{rank}[(A+A^T)X] \leq \operatorname{rank}[X^TAX]$, as required. ∎

Remarks Notice that if A is skew-symmetric (i.e. $A^T = -A$), then A has semidefinite symmetric part, and we generally have strict inequality in the inequality of statement (b). In fact, if A has all diagonal entries equal to zero and A satisfies inequality (b) then A is skew-symmetric (to see this take $X = E_{ii}$, for $1 \leq i \leq n$).

Corollary 2 If $A \in M_n(\mathbf{R})$ is symmetric, the following statements are equivalent:
(a) A is semidefinite;
(b) $\operatorname{rank}[AX] = \operatorname{rank}[X^TAX]$, for all $X \in M_n(\mathbf{R})$;
(c) $\mathbf{x}^TA\mathbf{x} = 0$ implies $A\mathbf{x} = 0$, for $\mathbf{x} \in \mathbf{R}^n$.
Proof Corollary 2 follows from Theorem 1, where statement (b) is just a consequence of $\operatorname{rank}[AX] \geq \operatorname{rank}[X^TAX]$. ∎

We can now consider property (III).
Corollary 3 Let $A \in M_n(\mathbf{R})$ be symmetric. If A is semidefinite then A satisfies both row and column inclusion.
Proof In the equality $\operatorname{rank}[AX] = \operatorname{rank}[X^TA] = \operatorname{rank}[X^TAX]$, take X as the diagonal matrix with 1's in the (i_1,i_1), $(i_2,i_2),\ldots,$ (i_k,i_k) positions, and all zeros elsewhere. ∎

The row and column inclusion property (III) has been used in the completion theory of positive semidefinite matrices [BJL], and is also easily proved by factoring $A = R^TR$. Row and column inclusion has been shown to hold more generally [J1]. A similar fact is known for distance matrices [HRW] (see the proof of their Theorem 3.1), and for a class of matrices in [CF] that are closely related to positive semidefinite matrices (usually called almost positive semidefinite matrices).

We next turn to consideration of where a matrix $A \in M_n(\mathbf{R})$ has powers which have positive semidefinite symmetric part, and row and column inclusion. We note that row and column inclusion fails if we simply drop the symmetry assumption. For example, consider $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. Then $H(A)$ is positive semidefinite but both row and column inclusion fail for A.

We now prove an analog of Theorem 1, to arrive closer to the symmetric case of Corollary 2. This will then lead in Corollary 5 to row and column inclusion for matrices which are not necessarily symmetric.

Theorem 4 Let $A \in M_n(\mathbf{R})$. Given the following statements:

(a) A has semidefinite symmetric part and A^2 has positive semidefinite symmetric part;

(b) $\operatorname{rank}[AX] = \operatorname{rank}[X^TAX]$, for all $X \in M_n(\mathbf{R})$;

(c) $\mathbf{x}^TA\mathbf{x} = 0$ implies $A\mathbf{x} = 0$, for $\mathbf{x} \in \mathbf{R}^n$.

Then (a) implies (b), (b) implies (c), and (c) implies A has semidefinite symmetric part.

Proof Similar arguments are used to those in the proof of Theorem 1. We first show that (a) implies (b). Write $A + A^T = \pm B^TB$. Let $\mathbf{u} \in \ker[X^TAX]$. Then $X^TAX\mathbf{u} = \mathbf{0}$, so $\mathbf{u}^TX^TB^TBX\mathbf{u} = 0$, which implies $BX\mathbf{u} = \mathbf{0}$, and $B^TBX\mathbf{u} = \mathbf{0}$. Then $(A + A^T)X\mathbf{u} = \mathbf{0}$, so $AX\mathbf{u} = -A^TX\mathbf{u}$, and $A^TAX\mathbf{u} = -(A^T)^2X\mathbf{u}$, in which case $\mathbf{u}^TX^TA^TAX\mathbf{u} = -\mathbf{u}^TX^T(A^T)^2X\mathbf{u}$. But $\mathbf{u}^TX^TA^TAX\mathbf{u} = (AX\mathbf{u})^T(AX\mathbf{u}) \geq 0$, and so $\mathbf{u}^TX^T(A^T)^2X\mathbf{u} \leq 0$. This implies $\mathbf{u}^TX^TA^2X\mathbf{u} \leq 0$, from which we have that $\mathbf{u}^TX^T(A^2 + (A^2)^T)X\mathbf{u} \leq 0$. But since $\mathbf{u}^TX^T(A^2 + (A^2)^T)X\mathbf{u} \geq 0$, we must have that $\mathbf{u}^TX^T(A^2)^TX\mathbf{u} = 0$, and so $(AX\mathbf{u})^T(AX\mathbf{u}) = 0$, i.e. $AX\mathbf{u} = \mathbf{0}$, which implies $\mathbf{u} \in \ker[AX]$. We have just shown that $\ker[X^TAX] \subseteq \ker[AX]$, which implies $n - \operatorname{rank}[X^TAX] \leq n - \operatorname{rank}[AX]$, so $\operatorname{rank}[AX] \leq \operatorname{rank}[X^TAX]$, and clearly $\operatorname{rank}[AX] \geq \operatorname{rank}[X^TAX]$.

That (b) implies (c) is proved in the way same as in Theorem 1.

For the last part of the theorem we will assume (c). In order to obtain a contradiction suppose there exist $\mathbf{x},\ \mathbf{y} \in \mathbf{R}^n$ such that $\mathbf{x}^TA\mathbf{x} > 0$ and $\mathbf{y}^TA\mathbf{y} < 0$. Then we use an argument which is similar to that used in Theorem 1 (albeit in Theorem 1 we were working with $H(A)$, whereas here we are working with A) in showing that (c) implies (a). Thus $(s\mathbf{x} + \mathbf{y})^TA(s\mathbf{x} + \mathbf{y}) = 0$ for two unequal real values $s = s_1$ and $s = s_2$. This time $(s_1\mathbf{x}+\mathbf{y})^TA(s_1\mathbf{x}+\mathbf{y}) = 0$ implies $A(s_1\mathbf{x}+\mathbf{y}) = 0$, and similarly $A(s_2\mathbf{x}+\mathbf{y}) = 0$. In which case, $(s_1 - s_2)A\mathbf{x} = \mathbf{0}$, so $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x}^TA\mathbf{x} = 0$, contradiction. ■

Remarks It is not enough to only assume that A has semidefinite symmetric part in (a), in order to conclude that (b) holds. What we saw for $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ is that row and column inclusion fail. It is also not enough to assume just that A has a semidefinite symmetric part in (a), in order to conclude that (c) holds. This may be seen by taking $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

This matrix A has positive semidefinite symmetric part, but for $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{x}^T A\mathbf{x} = 0$, although $A\mathbf{x} \neq \mathbf{0}$. Finally, (c) does not imply (a), because when $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ it is routine to check that (b) holds, but A^2 does not have positive semidefinite symmetric part.

Corollary 5 Let $A \in M_n(\mathbf{R})$. If A has semidefinite symmetric part, and A^2 has positive semidefinite symmetric part, then A satisfies both row and column inclusion.
Proof A having semidefinite symmetric part and A^2 having positive semidefinite symmetric part implies that $\operatorname{rank}[AX] = \operatorname{rank}[X^T AX]$. But it also implies that A^T has semidefinite symmetric part and $(A^T)^2$ has positive semidefinite symmetric part. Then $\operatorname{rank}[A^T X] = \operatorname{rank}[X^T A^T X]$ from Theorem 4, and this implies $\operatorname{rank}[X^T A] = \operatorname{rank}[X^T AX]$. The rest is the same as in the proof of Corollary 3. ∎

In order to prove our final theorem we need a lemma, which will indicate that A and A^2 having positive semidefinite symmetric part is rather special.
Lemma 6 Let $A \in M_n(\mathbf{R})$. If A and and A^2 each have positive semidefinite symmetric part, then $\ker(A) = \ker(A^m)$, for any positive integer m.
Proof The validity of this lemma is unchanged under orthogonal similarity, so we may assume that the symmetric part of A is $H \oplus O$, in which $H \in M_p(\mathbf{R})$ is positive definite. The skew-symmetric part of A, partitioned conformally, is denoted $\begin{bmatrix} S_{11} & S_{12} \\ -S_{12}^T & S_{22} \end{bmatrix}$, in which $S_{11} \in M_p(\mathbf{R})$, $S_{22} \in M_q(\mathbf{R})$, and $p + q = n$. Then

$$A^2 = \begin{bmatrix} (H + S_{11})^2 - S_{12}S_{12}^T & (H + S_{11})S_{12} + S_{12}S_{22} \\ -S_{12}^T(H + S_{11}) - S_{22}S_{12}^T & S_{22}^2 - S_{12}^T S_{12} \end{bmatrix}.$$

Since A^2 has positive semidefinite symmetric part and $S_{22}^2 - S_{12}^T S_{12}$ is (symmetric) negative semidefinite we must have $S_{22}^2 - S_{12}^T S_{12} = 0$. This means that both $S_{22} = 0$ and $S_{12} = 0$, which implies that $A = \begin{bmatrix} H + S_{11} & 0 \\ 0 & 0 \end{bmatrix}$. Since H is positive definite, $H + S_{11}$ is nonsingular and $\ker(A^m)$ is precisely all vectors of the form $\begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} \in \mathbf{R}^n$, for any $\mathbf{x} \in \mathbf{R}^q$, and for $m = 1, 2, \ldots$. ∎

Remarks It was shown in [J5] that (among other things) for $A \in M_n(\mathbf{C})$ any number of positive integer powers $A, A^2, \ldots, A^k$ could have positive definite Hermitian part, without A^{k+1} having positive definite Hermitian part; however if A^k has positive definite Hermitian part for all positive integer powers k, then A is Hermitian. If the first k consecutive powers have positive semidefinite symmetric part, we may generalize the (a) implies (b) part of Theorem 4 as follows.

Theorem 7 Let $A \in M_n(\mathbf{R})$. If $A, A^2, \ldots, A^k$ $(k \geq 2)$ each have positive semidefinite symmetric part, then

$$\text{rank}[AX] = \text{rank}[X^T AX] = \cdots = \text{rank}[X^T A^{k-1} X], \text{ for all } X \in M_n(\mathbf{R}).$$

Proof It follows from Theorem 4 that the claim is valid for $k = 2$. We verify by induction that it is also valid for all $k \geq 3$. Suppose that we know the claim for $k - 1$, so that $\text{rank}[AX] = \text{rank}[X^T AX] = \cdots = \text{rank}[X^T A^{k-2} X]$, for all $X \in M_n(\mathbf{R})$. Let $X^T A^{k-1} X\mathbf{u} = \mathbf{0}$. Then $\mathbf{u}^T X^T A^{k-1} X\mathbf{u} = 0$, but since we can write $A^{k-1} + (A^T)^{k-1} = C^T C$, we must have $\mathbf{u}^T X^T C^T C X\mathbf{u} = 0$, which implies $CX\mathbf{u} = \mathbf{0}$, so then $\mathbf{0} = C^T CX\mathbf{u} = (A^{k-1} + (A^T)^{k-1})X\mathbf{u}$. $A^{k-1}X\mathbf{u} = -(A^T)^{k-1}X\mathbf{u}$ implies $A^T A^{k-1} X\mathbf{u} = -(A^T)^k X\mathbf{u}$ so $\mathbf{u}^T X^T A^T A^{k-1} X\mathbf{u} = -\mathbf{u}^T X^T (A^T)^k X\mathbf{u}$. That is $(AX\mathbf{u})^T A^{k-2}(AX\mathbf{u}) = -\mathbf{u}^T X^T A^k X\mathbf{u}$, and since both A^k and A^{k-2} have positive semidefinite symmetric part we must have $(AX\mathbf{u})^T A^{k-2}(AX\mathbf{u}) = 0$, so by induction $A(AX\mathbf{u}) = 0$. This means that $X\mathbf{u} \in \ker(A^2)$, which using Lemma 6 implies that $X\mathbf{u} \in \ker(A)$, so $AX\mathbf{u} = \mathbf{0}$. We have just shown that $\ker[X^T A^{k-1} X] \subseteq \ker[AX]$. Finally, as in the proof of Theorem 4, $\text{rank}[X^T A^{k-1} X] = \text{rank}[AX]$, and the induction step is complete. ■

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