An Inequality for Nonnegative Matrices and the Inverse Eigenvalue Problem

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Abstract. We present two versions of the same inequality, relating the maximal diagonal entry of a nonnegative matrix to its eigenvalues. We demonstrate a matrix factorization of a companion matrix, which leads to a solution of the nonnegative inverse eigenvalue problem (denoted the *nniep*) for 4×4 matrices of trace zero, and we give some sufficient conditions for a solution to the nniep for 5×5 matrices of trace zero. We also give a necessary condition on the eigenvalues of a 5×5 trace zero nonnegative matrix in lower Hessenberg form. Finally, we give a brief discussion of the nniep in restricted cases.
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An Inequality for Nonnegative Matrices

An $n \times n$ matrix with real entries is said to be nonnegative if all of its entries are nonnegative. A nonnegative matrix is said to irreducible if under similarity with a permutation matrix, it cannot be written in the form

$$\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square matrices of order less than n.

The Perron-Frobenius theorem states that if A is a nonnegative matrix, then it has a real eigenvalue r (known as the Perron root) which is greater than or equal to the modulus (or absolute value) of each of the other eigenvalues, and also A has an eigenvector v associated with r such that each of its entries are nonnegative. Further, if A is irreducible then r is positive and the entries of v are positive [L-T].

If A is irreducible, then knowing that the vector $v = (v_1, v_2, \ldots, v_n)^T$ just mentioned has positive entries, we can write $v = Dw$, where $D = \mathrm{diag}(v_1, \ldots, v_n)$ and $w = (1, 1, \ldots, 1)^T$, and then $Av = rv$ can become $D^{-1}ADw = rw$, i.e. under similarity an r-eigenvector is $w = (1, 1, \ldots, 1)^T$.

It was Brauer [B] in 1952 who first showed that if the spectrum of an arbitrary $n \times n$ matrix A is (the arbitrary set) $\sigma = \{r, \lambda_2, \ldots, \lambda_n\}$, if $v = (v_1, v_2, \ldots, v_n)^T$ is an eigenvector associated with r, and we take the $n \times n$ matrix B, the i^{th} column of which is $\alpha_i v$ for each i, $1 \leq i \leq n$, then $A + B$ has eigenvalues $r + \sum_{i=1}^{n} \alpha_i v_i, \lambda_2, \ldots, \lambda_n$.

To see this, let P be an $n \times n$ invertible matrix which upper triangularizes A, and choose the first column of P as v, so that

$$P^{-1}AP = \begin{pmatrix} r & * & * & \ldots \\ 0 & \lambda_2 & * & \ldots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & 0 & \lambda_n \end{pmatrix},$$

then

$$P^{-1}(A+B)P = P^{-1}AP + \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ 0 & 0 & \ldots & 0 \\ 0 & \vdots & \ldots & 0 \\ \vdots & & \ldots & 0 \end{pmatrix} P,$$

$$= \begin{pmatrix} r + \sum_{i=1}^{n} \alpha_i v_i & * & \ldots & * \\ 0 & \lambda_2 & * & * \\ 0 & 0 & \ddots & * \\ \vdots & \vdots & & \end{pmatrix},$$

and we're done.

We come now to our first result.

Theorem 1: Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with spectrum $\sigma = \{r, \lambda, \lambda_3, \ldots, \lambda_n\}$, where r is the Perron root, λ is real and $n \geq 3$, then

$$\max_{1 \leq i \leq n} a_{ii} \geq \frac{1}{n-2} \sum_{j=3}^{n} \lambda_j.$$

Proof: Suppose (for now) that A is irreducible. We will assume without loss of generality that A has eigenvector $w = (1, 1, \ldots, 1)^T$ corresponding to the Perron root r. Under permutation similarity we can also assume that a_{nn} is maximal among the diagonal entries of A. Now consider the matrix $C = A + B$, where the j^{th} column of B is $\beta_j w$ for each j, $1 \leq j \leq n$. We know from Brauer's theorem that C has eigenvalues $r' = r + \sum_{j=1}^{n} \beta_j, \lambda, \lambda_3, \ldots, \lambda_n$. Take now $\beta_j = a_{nn} - a_{jj}$, $1 \leq j \leq n$, so that C is nonnegative, has all its diagonal entries equal, and has eigenvector w (still) corresponding to the Perron root r'.

From Geršgorin's theorem applied to $C = (c_{ij})$ we have that

$$|\lambda - c_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} = r' - c_{ii}, \text{ for some } i, \; 1 \leq i \leq n.$$

But C has all its diagonal entries equal to a_{nn}, so rewriting this inequality as

$$-r' + a_{nn} \leq \lambda - a_{nn} \leq r' - a_{nn},$$

then just using the left-hand inequality, and knowing r', we obtain

$$\begin{aligned} 2a_{nn} \leq r' + \lambda = r + \sum_{j=1}^{n} (a_{nn} - a_{jj}) + \lambda, \\ = r + na_{nn} - \sum_{j=1}^{n} a_{jj} + \lambda, \\ = r + na_{nn} - [r + \lambda + \sum_{j=3}^{n} \lambda_j] + \lambda, \\ = na_{nn} - \sum_{j=3}^{n} \lambda_j, \end{aligned}$$

so that $\sum_{j=3}^{n} \lambda_j \leq (n-2)a_{nn}$, which proves the theorem when A is irreducible.

In case A is reducible, consider the matrix $C = A + B$, where this time each entry of B is $\epsilon > 0$ (ϵ small), then C is irreducible and has eigenvalues $r + n\epsilon, \lambda, \lambda_3, \ldots, \lambda_n$. Now applying the result just proved we have that $\max_{1 \leq i \leq n}(a_{ii} + \epsilon) \geq \frac{1}{n-2} \sum_{j=3}^{n} \lambda_j$. But this is true for any ϵ arbitrarily small, so it is true for $\epsilon = 0$ and the theorem is proved.

Another version of the inequality in Theorem 1 is the following.

Theorem 2: Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with spectrum $\sigma = \{r, a+ib, a-ib, \lambda_4, \dots, \lambda_n\}$, where r is the Perron root and $n \geq 4$, then

$$b^2 \leq [(n-2) \max_{1\leq i\leq n} a_{ii} - a - \sum_{j=4}^{n} \lambda_j][n \max_{1\leq i\leq n} a_{ii} - 3a - \sum_{j=4}^{n} \lambda_j].$$

Proof: We reason as before, applying Brauer's theorem and then Geršgorin's theorem, to arrive at the inequality

$$\begin{aligned}
(a-a_{nn})^2 + b^2 &\leq (r' - a_{nn})^2, \\
&\leq (r + \sum_{j=1}^{n}(a_{nn} - a_{jj}) - a_{nn})^2, \\
&\leq (r + (n-1)a_{nn} - \sum_{j=1}^{n} \lambda_j)^2, \\
&\leq (r + (n-1)a_{nn} - (r + 2a + \sum_{j=4}^{n} \lambda_j))^2,
\end{aligned}$$

so that

$$\begin{aligned}
b^2 &\leq ((n-1)a_{nn} - 2a - \sum_{j=4}^{n} \lambda_j) - (a - a_{nn})^2, \\
&\leq ((n-2)a_{nn} - a - \sum_{j=4}^{n} \lambda_j)(na_{nn} - 3a - \sum_{j=4}^{n} \lambda_j).
\end{aligned}$$

The Nonnegative Inverse Eigenvalue Problem

Let $\sigma = \{\lambda_1, ..., \lambda_n\} \subset \mathbf{C}$. The nonnegative inverse eigenvalue problem is to find necessary and sufficient conditions that σ is the set of eigenvalues of an $n \times n$ nonnegative matrix A (say) (this well-known problem is currently unsolved except in restricted cases, see [B-P], [M], [B-H]). By $\overline{\sigma}$ we shall mean the complex conjugate of each of the entries of the set σ. The necessary conditions $\overline{\sigma} = \sigma$ and $s_k = \lambda_1^k + \cdots + \lambda_n^k = \text{trace}(A^k) \geq 0$ for $k = 1, 2, ...$ are easy to see.

Johnson [J], Loewy and London [L-L], have shown that for a nonnegative matrix A with spectrum $\sigma = \{\lambda_1, ..., \lambda_n\}$ and $s_k = \lambda_1^k + \cdots + \lambda_n^k = \text{trace}(A^k)$ then $n^{m-1}s_{km} \geq s_k^m$, for $k, m = 1, 2, ...$. We shall refer to these necessary conditions henceforth as J-L-L.

We now solve the nniep for 4×4 matrices of trace zero.

Theorem 3: Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \subset \mathbf{C}$. If $s_1 = 0,\ s_2 \geq 0,\ s_3 \geq 0$ and $4s_4 \geq s_2^2$, then there exists a nonnegative 4×4 matrix with spectrum σ.

Proof: We remark that the last inequality among the hypotheses of the theorem is the J-L-L necessary condition with $n = 4,\ k = 2$ and $m = 2$.

Let $p_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4,\ p_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4,\ p_3 = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4,\ p_4 = \lambda_1\lambda_2\lambda_3\lambda_4$, i.e. p_1, p_2, p_3, p_4 are Newton's elementary symmetric polynomials.

Newton's identities for symmetric functions [van der W] state that

$$\begin{aligned} s_1 - p_1 &= 0, \\ s_2 - p_1 s_1 + 2p_2 &= 0, \\ s_3 - p_1 s_2 + p_2 s_1 - 3p_3 &= 0, \\ s_4 - p_1 s_3 + p_2 s_2 - p_3 s_1 + 4p_4 &= 0. \end{aligned}$$

We can write these equations in matrix form with a companion matrix as

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -p_4 & p_3 & -p_2 & p_1 \end{pmatrix} \begin{pmatrix} -4 & 0 & 0 & 0 \\ -s_1 & -3 & 0 & 0 \\ -s_2 & -s_1 & -2 & 0 \\ -s_3 & -s_2 & -s_1 & -1 \end{pmatrix} = \begin{pmatrix} -s_1 & -3 & 0 & 0 \\ -s_2 & -s_1 & -2 & 0 \\ -s_3 & -s_2 & -s_1 & -1 \\ -s_4 & -s_3 & -s_2 & -s_1 \end{pmatrix}.$$

Letting $p_1 = s_1 = 0$ and multiplying both sides on the right by $\operatorname{diag}(-\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}, -1)$ we get

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -p_4 & p_3 & -p_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & \frac{s_2}{3} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & \frac{s_2}{3} & 0 & 1 \\ \frac{s_4}{4} & \frac{s_3}{3} & \frac{s_2}{2} & 0 \end{pmatrix}. \qquad (*)$$

It is easily verified that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & \frac{s_2}{3} & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-s_2}{4} & 0 & 1 & 0 \\ \frac{-s_3}{4} & \frac{-s_2}{3} & 0 & 1 \end{pmatrix},$$

then multiplying on the left of $(*)$ with this matrix we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-s_2}{4} & 0 & 1 & 0 \\ \frac{-s_3}{4} & \frac{-s_2}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -p_4 & p_3 & -p_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & \frac{s_2}{3} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & \frac{s_2}{12} & 0 & 1 \\ \frac{3s_4 - s_2^2}{12} & \frac{s_3}{12} & \frac{s_2}{6} & 0 \end{pmatrix}.$$

Finally, performing a similarity on this matrix we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{s_2}{12} & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & \frac{s_2}{12} & 0 & 1 \\ \frac{3s_4 - s_2^2}{12} & \frac{s_3}{12} & \frac{s_2}{6} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{-s_2}{12} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & 0 & 0 & 1 \\ \frac{4s_4 - s_2^2}{16} & \frac{s_3}{12} & \frac{s_2}{4} & 0 \end{pmatrix},$$

which is nonnegative and similar to the companion matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, as required.

Remark: Notice that the analogue of equation $(*)$ for $n \times n$ matrices gives a factorization of an $n \times n$ companion matrix, even when the trace is not necessarily zero.

The next theorem gives sufficient conditions for the existence of a nonnegative 5×5 matrix of trace zero with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$.

Theorem 4: Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$. If $s_1 = 0,\ s_2 \geq 0,\ s_3 \geq 0,\ 4s_4 \geq s_2^2$ and $2s_5 \geq s_2 s_3$, then there exists a nonnegative 5×5 matrix with spectrum σ.

Proof: We perform the same procedure with 5×5 matrices as in Theorem 3. Beginning with the corresponding equation $(*)$ we have

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ p_5 & -p_4 & p_3 & -p_2 & 0 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{s_2}{5} & 0 & 1 & 0 & 0 \\ \frac{s_3}{5} & \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_4}{5} & \frac{s_3}{4} & \frac{s_2}{3} & 0 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{s_2}{5} & 0 & 1 & 0 & 0 \\ \frac{s_3}{5} & \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_4}{5} & \frac{s_3}{4} & \frac{s_2}{3} & 0 & 1 \\ \frac{s_5}{5} & \frac{s_4}{4} & \frac{s_3}{3} & \frac{s_2}{2} & 0 \end{pmatrix}}_{B},$$

so $CA = B$. Then it can be checked that

$$A^{-1}CA = A^{-1}B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{s_2}{5} & 0 & 1 & 0 & 0 \\ \frac{s_3}{5} & \frac{s_2}{20} & 0 & 1 & 0 \\ \frac{4s_4 - s_2^2}{20} & \frac{s_3}{20} & \frac{s_2}{12} & 0 & 1 \\ \frac{12s_5 - 7s_2s_3}{60} & \frac{3s_4 - s_2^2}{60} & \frac{s_3}{12} & \frac{s_2}{6} & 0 \end{pmatrix}.$$

As in the proof of Theorem 3 we perform a conveniently chosen similarity

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{4s_4 - s_2^2}{20} & 0 & \frac{s_2}{12} & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{s_2}{5} & 0 & 1 & 0 & 0 \\ \frac{s_3}{5} & \frac{s_2}{20} & 0 & 1 & 0 \\ \frac{4s_4 - s_2^2}{20} & \frac{s_3}{20} & \frac{s_2}{12} & 0 & 1 \\ \frac{12s_5 - 7s_2s_3}{60} & \frac{3s_4 - s_2^2}{60} & \frac{s_3}{12} & \frac{s_2}{6} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{-4s_4 + s_2^2}{20} & 0 & \frac{-s_2}{12} & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{s_2}{5} & 0 & 1 & 0 & 0 \\ \frac{s_3}{5} & \frac{s_2}{20} & 0 & 1 & 0 \\ 0 & \frac{s_3}{20} & 0 & 0 & 1 \\ \frac{2s_5 - s_2s_3}{10} & \frac{4s_4 - s_2^2}{16} & \frac{s_3}{12} & \frac{s_2}{4} & 0 \end{pmatrix}, \qquad (**)$$

proving the theorem.

Continuing with our investigation of solutions to the nniep for 5×5 matrices, note that the matrix at $(**)$ improves on a trace zero companion matrix (in the sense that if the entries of a companion matrix, for a given σ, are nonnegative then the matrix at $(**)$ is nonnegative also, but not conversely) which has the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{6s_5-5s_2s_3}{30} & \frac{2s_4-s_2^2}{8} & \frac{s_3}{3} & \frac{s_2}{2} & 0 \end{pmatrix}.$$

To illustrate that we have improved on a companion matrix consider $\sigma = \{6, 1, 1, -4, -4\}$.

Concerning the (5,1) and (5,2) entries of the matrix at $(**)$, it is worth mentioning that the inequality $2s_5 - s_2s_3 \geq 0$ is not true for all (even trace zero) nonnegative matrices, as can be seen from the following nonnegative matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

for which $s_5 = 0$, whilst $s_2 = 2$ and $s_3 = 3$. Although, for a trace zero 5×5 nonnegative matrix in lower Hessenberg form we claim that $4s_4 - s_2^2 \geq 0$, as we now show. In fact we show something slightly stronger than this.

Let

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & 0 & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & 0 & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & 0 \end{pmatrix}.$$

Then with the usual notation $s_k = \text{trace}(A^k)$, $s_1 = 0$,

$$\frac{s_2}{2} = a_{12}a_{21} + a_{13}a_{31} + a_{14}a_{41} + a_{15}a_{51} + a_{23}a_{32} + a_{24}a_{42} + a_{25}a_{52} + a_{34}a_{43} + a_{35}a_{53} + a_{45}a_{54},$$

and

$$-\frac{2s_4 - s_2^2}{8} = \begin{vmatrix} 0 & a_{23} & a_{24} & a_{25} \\ a_{32} & 0 & a_{34} & a_{35} \\ a_{42} & a_{43} & 0 & a_{45} \\ a_{52} & a_{53} & a_{54} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{13} & a_{14} & a_{15} \\ a_{31} & 0 & a_{34} & a_{35} \\ a_{41} & a_{43} & 0 & a_{45} \\ a_{51} & a_{53} & a_{54} & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & a_{12} & a_{14} & a_{15} \\ a_{21} & 0 & a_{24} & a_{25} \\ a_{41} & a_{42} & 0 & a_{45} \\ a_{51} & a_{52} & a_{54} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} & a_{15} \\ a_{21} & 0 & a_{23} & a_{25} \\ a_{31} & a_{32} & 0 & a_{35} \\ a_{51} & a_{52} & a_{53} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{vmatrix},$$

$= a_{32}a_{23}(a_{45}a_{54} + a_{15}a_{51} + a_{41}a_{14}) + a_{21}a_{12}(a_{45}a_{54} + a_{35}a_{53} + a_{34}a_{43})$
$+ a_{31}a_{13}(a_{45}a_{54} + a_{52}a_{25} + a_{42}a_{24}) + a_{42}a_{24}a_{53}a_{35} + a_{52}a_{25}a_{34}a_{43}$
$+ a_{14}a_{41}a_{53}a_{35} + a_{51}a_{15}a_{42}a_{24} + a_{51}a_{15}a_{34}a_{43} + a_{41}a_{14}a_{52}a_{25} - x,$ where $x \geq 0$,

(grouping some of the terms and letting $-x$ include all the negative terms)

$= a_{32}a_{23}(\frac{s_2}{2} - a_{12}a_{21} - a_{13}a_{31} - a_{34}a_{43} - a_{35}a_{53} - a_{23}a_{32} - a_{24}a_{42} - a_{25}a_{52})+$
$a_{21}a_{12}(\frac{s_2}{2} - a_{12}a_{21} - a_{13}a_{31} - a_{14}a_{41} - a_{15}a_{51} - a_{23}a_{32} - a_{24}a_{42} - a_{25}a_{52})+$
$a_{31}a_{13}(\frac{s_2}{2} - a_{12}a_{21} - a_{13}a_{31} - a_{14}a_{41} - a_{15}a_{51} - a_{23}a_{32} - a_{34}a_{43} - a_{35}a_{53})+a_{42}a_{24}a_{53}a_{35}$
$+ a_{52}a_{25}a_{34}a_{43} + a_{14}a_{41}a_{35}a_{53} + a_{51}a_{15}a_{42}a_{24} + a_{15}a_{51}a_{34}a_{43} + a_{41}a_{14}a_{52}a_{25} - x,$
$= -a_{32}^2a_{23}^2 - a_{12}^2a_{21}^2 - a_{31}^2a_{13}^2 - 2a_{12}a_{21}a_{23}a_{32} - 2a_{13}a_{31}a_{12}a_{21} - 2a_{31}a_{13}a_{23}a_{32}$
$+ \frac{s_2}{2}(a_{32}a_{23} + a_{12}a_{21} + a_{13}a_{31}) - y + a_{42}a_{24}a_{53}a_{35} + a_{52}a_{25}a_{34}a_{43} + a_{14}a_{41}a_{53}a_{35}+$
$a_{51}a_{15}a_{42}a_{24} + a_{51}a_{15}a_{34}a_{43} + a_{41}a_{14}a_{52}a_{25} - x,$

where $y \geq 0$. Collecting appropriate terms we can write the above equation as

$$(a_{32}a_{23} + a_{21}a_{12} + a_{31}a_{13})^2 - \frac{s_2}{2}(a_{32}a_{23} + a_{21}a_{12} + a_{31}a_{13}) + x + y - \frac{s_4}{4} + \frac{s_2^2}{8} =$$
$$a_{42}a_{24}a_{53}a_{35}+a_{52}a_{25}a_{34}a_{43}+a_{14}a_{41}a_{53}a_{35}+a_{51}a_{15}a_{42}a_{24}+a_{51}a_{15}a_{34}a_{43}+a_{41}a_{14}a_{52}a_{25}.$$

When $a_{14} = a_{24} = a_{15} = a_{25} = 0$, or $a_{15} = a_{25} = a_{35} = 0$, we have a quadratic with real roots and therefore $(\frac{s_2}{2})^2 + 4(\frac{s_4}{4} - \frac{s_2^2}{8}) \geq 4(x+y)$, so $s_4 - \frac{s_2^2}{4} \geq 0$, i.e. $4s_4 - s_2^2 \geq 0$, proving the claim. A is in lower Hessenberg form if $a_{13} = a_{14} = a_{15} = a_{24} = a_{25} = a_{35} = 0$.

More Restricted Cases of the NNIEP

Notice from the 5×5 companion matrix given earlier that when A has trace zero then $\det(A) = \frac{6s_5 - 5s_3s_2}{30}$, so if $\det(A) \geq 0$, then $2s_5 - s_2s_3 \geq 0$, thus solving the nniep in a restricted case, namely trace zero, determinant nonnegative and the matrix A restricted to being in lower Hessenberg form (the latter implying that $4s_4 - s_2^2 \geq 0$, while $s_2 \geq 0$ and $s_3 \geq 0$ are also necessary conditions).

We consider now the nniep restricted to nonnegative matrices having their diagonal entries equal. It is easy to see that this problem reduces to the trace zero case, since if A has diagonal entries equal, these diagonal entries must be each $\frac{s_1}{n}$ and so $A - \frac{s_1}{n}I$ (where I is the identity matrix) has trace zero, and having solved the trace zero case we can add back on the scalar matrix $\frac{s_1}{n}I$ to obtain a matrix with the eigenvalues of A.

Looking at this in more detail, we solve the nniep for 4×4 matrices where we only consider solving the problem when the diagonal entries are equal. The nniep was solved by

Loewy and London in [L-L] for 3×3 matrices. It is not difficult to show that restricting the diagonal entries to be equal in the 2×2 and 3×3 nniep does not prevent us from solving the general nniep (not counting reducible cases, we will concentrate on the irreducible case). If a 4×4 matrix A has eigenvalues $\sigma=\{r,\lambda,\lambda_3,\lambda_4\}$, then $A-\frac{s_1}{4}I$ has trace zero, and a matrix with the eigenvalues of $A-\frac{s_1}{4}I$ is as given in Theorem 3, where $s_1'=\operatorname{trace}(A-\frac{s_1}{4}I)=0$,

$$s_2'=\operatorname{trace}([A-\frac{s_1}{4}I]^2)=\operatorname{trace}(A^2-2\frac{s_1}{4}A+\frac{s_1^2}{4^2}I)=s_2-\frac{s_1^2}{4}=\frac{4s_2-s_1^2}{4},$$

which is greater than or equal to zero even in the general nniep. Also, $s_3'=\operatorname{trace}([A-\frac{s_1}{4}I]^3)=\frac{8s_3-6s_1s_2+s_1^3}{8}$, must be greater than or equal to zero for a matrix with equal diagonal entries, and likewise we must have $4s_4'-s_2'^2\geq0$. Thus we have two new necessary conditions when the diagonal entries are equal. It is perhaps worth noting that

$$8s_3-6s_1s_2+s_1^3=3(r+\lambda-\lambda_3-\lambda_4)(r-\lambda+\lambda_3-\lambda_4)(r-\lambda-\lambda_3+\lambda_4),$$

for which, in the case of four real eigenvalues, we must have each factor greater than or equal to zero, and this fact could have been deduced from Theorem 1 (with the λ of Theorem 1 taken as λ, λ_3 and λ_4 successively) where here $\max_{1\leq i\leq4}a_{ii}=\frac{(r+\lambda+\lambda_3+\lambda_4)}{4}$. Considering matrices with diagonal entries equal does not prevent us from solving the 4×4 problem if you only consider the nniep with real eigenvalues (again, excluding reducible cases), see the solution in [L-L] of this case to see this (they did not need Suleimanova's result quoted in their proof). Diagonal entries equal does however prevent us from solving the 4×4 general nniep in the remaining case of $\sigma=\{r,\lambda,a+ib,a-ib\}$ (with $b\neq0$), for which Theorem 1 (or $8s_3-6s_1s_2+s_1^3\geq0$) implies $r+\lambda-2a\geq0$. To see that we don't always have $r+\lambda-2a\geq0$, consider the matrix with spectrum $\sigma=\{6,-1,3+i,3-i\}$

$$\begin{pmatrix}0 & 6 & 0 & 0\\ 1 & \frac{11}{3} & \frac{2}{3}+\frac{1}{\sqrt{3}} & \frac{2}{3}-\frac{1}{\sqrt{3}}\\ 1 & \frac{2}{3}-\frac{1}{\sqrt{3}} & \frac{11}{3} & \frac{2}{3}+\frac{1}{\sqrt{3}}\\ 1 & \frac{2}{3}+\frac{1}{\sqrt{3}} & \frac{2}{3}-\frac{1}{\sqrt{3}} & \frac{11}{3}\end{pmatrix}$$

which is nonnegative but $r+\lambda-2a<0$.

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