Hadamard Inverses, Square Roots and Products of Almost Semidefinite Matrices

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Abstract Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix with all positive entries and just one positive eigenvalue. Bapat proved then that the Hadamard inverse of A, given by $A^{\circ(-1)} = (\frac{1}{a_{ij}})$ is positive semidefinite. We show that if moreover A is invertible then $A^{\circ(-1)}$ is positive definite. We use this result to obtain a simple proof that with the same hypotheses on A, except that all the diagonal entries of A are zero, the Hadamard square root of A, given by $A^{\circ\frac{1}{2}} = (a_{ij}^{\frac{1}{2}})$, has just one positive eigenvalue and is invertible. Finally, we show that if A is any positive semidefinite matrix and B is almost positive definite and invertible then $A \circ B \succeq \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}} A$.

1. Introduction Let $A = (a_{ij})$, $B = (b_{ij})$ be $n \times n$ matrices with real entries, i.e. $A,\ B \in \mathbf{R}^{n\times n}$. The Hadamard product of A and B is defined by $A \circ B = (a_{ij}b_{ij})$ [11]. The Hadamard inverse of A (with $a_{ij} > 0$, $1 \leq i, j \leq n$) is defined by $A^{\circ(-1)} = (\frac{1}{a_{ij}})$, and the Hadamard square root by $A^{\circ\frac{1}{2}} = (a_{ij}^{\frac{1}{2}})$. In Section 2, we extend a result due to Bapat [2], [3], who showed that if A is symmetric, has all positive entries and just one positive eigenvalue, then its Hadamard inverse $A^{\circ(-1)}$ is positive semidefinite. We provide necessary and sufficient conditions on the invertibility of $A^{\circ(-1)}$. A corollary of this theorem will then be used to prove that if A is a symmetric matrix which has all off-diagonal entries positive, all diagonal entries zero, and A has just one positive eigenvalue, then the Hadamard square root of A has just one positive eigenvalue, and is invertible. This was proved for distance matrices (distance matrices are a special case of matrices which satisfy the hypotheses) most recently by Auer [1], and it had previously been proved by Schoenberg [18], Micchelli [17], and Marcus and Smith [16]. See also Blumenthal [4,p.135], Kelly [14], and Critchley and Fichet [5,p.26]. We recall here the Perron-Frobenius Theorem [15], which states that if a matrix $A \in \mathbf{R}^{n\times n}$ has all positive entries then it has a positive eigenvalue $r > |\lambda|$, for all other eigenvalues λ of A. Furthermore, the eigenvector that corresponds to r has positive components. This theorem remains true under more general conditions, including in the case when all off-diagonal entries are positive and the diagonal entries are zero.

Let A and B be symmetric. The Loewner partial order $A \succeq B$ denotes that $A - B$ is positive semidefinite, and $A \succ B$ that $A - B$ is positive definite. Let $\mathbf{e} = (1, 1, ..., 1)^T$, i.e. $\mathbf{e}$ is the $n \times 1$ vector of all ones. A symmetric matrix A is almost positive semidefinite (or conditionally positive semidefinite) if $\mathbf{x}^T A \mathbf{x} \geq 0$, for all $\mathbf{x} \in \mathbf{R}^n$ such that $\mathbf{x}^T \mathbf{e} = 0$, and almost positive definite (or conditionally positive definite) if $\mathbf{x}^T A \mathbf{x} > 0$, for all $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{x}^T \mathbf{e} = 0$. In Section 3, we prove that if A is positive semidefinite and B is almost positive definite and invertible then $A \circ B \succeq \frac{1}{\mathbf{e}^T B^{-1} \mathbf{e}} A$. This extends the validity of Fiedler and Markham's inequality [9], since they required that B is positive definite.

2. Hadamard Inverses and Square Roots

The following five lemmas are essentially well known [3], [7], [13], [17], however for completeness we provide short proofs. Let $\operatorname{diag}(a_{11}, \ldots, a_{nn})$ denote the $n \times n$ diagonal matrix with diagonal entries $a_{11}, \ldots, a_{nn}$, and $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and minimum eigenvalues of $A \in \mathbf{R}^{n \times n}$, respectively.

Lemma 2.1: Let $A,\ B \in \mathbf{R}^{n \times n}$ be symmetric. If $A \succeq 0$ then

$$\operatorname{diag}(a_{11}, \ldots, a_{nn}) \lambda_{\max}(B) I \succeq A \circ B \succeq \operatorname{diag}(a_{11}, \ldots, a_{nn}) \lambda_{\min}(B) I.$$

Proof: Let $C \in \mathbf{R}^{n \times n}$ and $C \succeq 0$. We know then that $A \circ C \succeq 0$, since $A \circ C$ is a principal submatrix of $A \otimes C$, the Kronecker product of A and C, which is positive semidefinite. Since $B - \lambda_{\min}(B) I \succeq 0$ and $B - \lambda_{\max}(B) I \preceq 0$, we can re-write this as

$\lambda_{\max}(B) I \succeq B \succeq \lambda_{\min}(B) I$, and then Hadamard multiply all the way across by A.

Lemma 2.2: Let $A,\ B \in \mathbf{R}^{n \times n}$ be symmetric. If $A \succeq 0$, $B \succ 0$ and all the diagonal entries of A are nonzero then $A \circ B$ is positive definite.

Proof: Since $\lambda_{\min}(B) > 0$, Lemma 2.2 follows from Lemma 2.1.

Lemma 2.3: Let $A \in \mathbf{R}^{n \times n}$ be symmetric and positive semidefinite. Then the Hadamard exponential $e^{\circ A} = (e^{a_{ij}})$ is positive semidefinite. Moreover, $e^{\circ A}$ is positive definite if and only if A has distinct rows.

Proof: Evidently, $e^{\circ A} = \mathbf{e}\mathbf{e}^T + A + \frac{1}{2!} A^{\circ 2} + \frac{1}{3!} A^{\circ 3} + \cdots$ is positive semidefinite, and $e^{\circ A}$ positive definite implies that the rows of A must be distinct. Suppose now that for some $\mathbf{y} = (y_1, \ldots, y_n)^T \in \mathbf{R}^n$, $\mathbf{y} \neq 0$, $\mathbf{y}^T e^{\circ A} \mathbf{y} = 0$, then $\mathbf{y}^T A^{\circ k} \mathbf{y} = 0$, and thus $A^{\circ k} \mathbf{y} = 0$, for $k = 0, 1, 2, \ldots$. Write $A = (\mathbf{x}_i \cdot \mathbf{x}_j) = (||\mathbf{x}_i|| ||\mathbf{x}_j|| \cos\, \theta_{ij})$, for some $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbf{R}^n$. Let $||\mathbf{x}_i||$ be maximum among those $||\mathbf{x}_1||, \ldots, ||\mathbf{x}_n||$ such that $y_i \neq 0$. We must have $||\mathbf{x}_i|| \neq 0$, or else for every nonzero y_j we have $||\mathbf{x}_j|| = 0$. In the latter case, if there are two or more nonzero y_j's for which $||\mathbf{x}_j|| = 0$ then A has two rows the same. While if there is just one $y_j \neq 0$ this would imply $e^{\circ A}$ has a zero j^{th} column, which is not possible.

Then, with $||\mathbf{x}_i|| \neq 0$, after dividing all the way across

$$||\mathbf{x}_i||^k(||\mathbf{x}_1||^k\cos^k\theta_{i1}y_1 + ||\mathbf{x}_2||^k\cos^k\theta_{i2}y_2 + \cdots + ||\mathbf{x}_i||^k y_i + \cdots + ||\mathbf{x}_n||^k\cos^k\theta_{in}y_n) = 0,$$

by $||\mathbf{x}_i||^{2k}$ and letting $k \to \infty$, we must have $||\mathbf{x}_i|| = ||\mathbf{x}_j||\cos\theta_{ij}$, for some $i \neq j$. Since $||\mathbf{x}_i|| \geq ||\mathbf{x}_j||$ we also have $\cos\theta_{ij} = 1$ and thus $||\mathbf{x}_i - \mathbf{x}_j||^2 = ||\mathbf{x}_i||^2 + ||\mathbf{x}_i||^2 - 2\mathbf{x}_i.\mathbf{x}_j = 0$. So $\mathbf{x}_i = \mathbf{x}_j$, and A has two rows the same.

Lemma 2.4: Let $A \in \mathbf{R}^{n\times n}$ be symmetric. A is almost positive (semi)definite if and only if $B = (a_{ij} - a_{in} - a_{nj} + a_{nn}) \in \mathbf{R}^{(n-1)\times(n-1)}$ is positive (semi)definite.

Proof: If $\mathbf{x}^T\mathbf{e} = 0$ then $x_n = -\sum_{i=1}^{n-1} x_i$, and substituting we have

$$\begin{aligned}\sum_{i,j=1}^{n} a_{ij}x_ix_j &= \sum_{i,j=1}^{n-1} a_{ij}x_ix_j + x_n\sum_{i=1}^{n-1} a_{in}x_i + x_n\sum_{j=1}^{n-1} a_{nj}x_j + a_{nn}x_n^2,\\ &= \sum_{i,j=1}^{n-1} a_{ij}x_ix_j - \sum_{j=1}^{n-1} x_j\sum_{i=1}^{n-1} a_{in}x_i - \sum_{i=1}^{n-1} x_i\sum_{j=1}^{n-1} a_{nj}x_j + a_{nn}\sum_{i,j=1}^{n-1} x_ix_j,\\ &= \sum_{i,j=1}^{n-1} (a_{ij} - a_{in} - a_{nj} + a_{nn})x_ix_j.\end{aligned}$$

Remark: If $i = n$ or $j = n$ then $a_{ij} - a_{in} - a_{nj} + a_{nn} = 0$.

Lemma 2.5: Let $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ be almost positive semidefinite then $e^{\circ A}$ is positive semidefinite. Moreover, $e^{\circ A}$ is positive definite if and only if $a_{ii} + a_{jj} > 2a_{ij}$, for all $i \neq j$.

Proof: Write $\alpha_i = a_{in} - (a_{nn}/2)$, for $1 \leq i \leq n$. From Lemma 2.4, since $A = (a_{ij})$ is almost positive semidefinite we can write, for $1 \leq i, j \leq n$,

$$a_{ij} = b_{ij} + a_{in} + a_{nj} - a_{nn} = b_{ij} + \alpha_i + \alpha_j,$$

where $B = (b_{ij}) = (a_{ij} - a_{in} - a_{nj} + a_{nn}) \in \mathbf{R}^{n\times n}$ is positive semidefinite. Then $e^{\circ B} = (e^{b_{ij}})$ is positive semidefinite also. It follows that $e^{\circ A} = (e^{a_{ij}}) = (e^{b_{ij}+\alpha_i+\alpha_j}) = (e^{\alpha_i}e^{b_{ij}}e^{\alpha_j}) = De^{\circ B}D$ is positive semidefinite, where $D = \mathrm{diag}(e^{\alpha_1}, \ldots, e^{\alpha_n})$.

Finally, $e^{\circ A}$ is positive definite iff $e^{\circ B}$ is positive definite iff the rows of $B = (\mathbf{x}_i \cdot \mathbf{x_j})$ are distinct iff $0 < ||\mathbf{x}_i - \mathbf{x}_j||^2 = b_{ii} + b_{jj} - 2b_{ij} = a_{ii} + a_{jj} - 2a_{ij}$, for all $i \neq j$.

Corollary 2.6: Let $A \in \mathbf{R}^{n\times n}$ be almost positive definite then $e^{\circ A}$ is positive definite.

Proof: $(\mathbf{e}_i - \mathbf{e}_j)^TA(\mathbf{e}_i - \mathbf{e}_j) = a_{ii} + a_{jj} - 2a_{ij} > 0$, for all $i \neq j$.

Remarks: For (symmetric) positive semidefinite matrices the condition $a_{ii} + a_{jj} > 2a_{ij}$ for all $i \neq j$ is equivalent to saying A has distinct rows. This is not true for almost positive semidefinite matrices however, since for example consider $A = \begin{bmatrix} 1 & 2 & 2\\ 2 & 3 & 3\\ 2 & 3 & 5\end{bmatrix}$. This matrix satisfies $\mathbf{x}^TA\mathbf{x} \geq 0$, for any $\mathbf{x} = (x_1, x_2, -x_1 - x_2)^T$, A has distinct rows, but $a_{ii} + a_{jj} = 2a_{ij}$, when $i = 1$ and $j = 2$. From Lemma 2.5 we can also see that for A almost positive semidefinite, $e^{\circ A}$ is positive definite if and only if all principal 2×2 submatrices of $e^{\circ A}$ are positive definite.

Theorem 2.7: Let $A \in \mathbf{R}^{n \times n}$ be symmetric, have positive entries and just one positive eigenvalue, then the Hadamard inverse $A^{\circ(-1)} = (\frac{1}{a_{ij}})$ is positive semidefinite.

Moreover, $A^{\circ(-1)}$ is positive definite if and only if $\dfrac{a_{ii}}{v_i^2} + \dfrac{a_{jj}}{v_j^2} < 2\dfrac{a_{ij}}{v_i v_j}$, for all $i \neq j$, where $\mathbf{v} = (v_1, \dots, v_n)^T \in \mathbf{R}^n$ is the Perron eigenvector for A.

Proof: Let the eigenvalues of A be $\lambda_1 \leq \cdots \leq \lambda_{n-1} \leq r$ with $A\mathbf{v} = r\mathbf{v}$ and $A\mathbf{u}_i = \lambda_i \mathbf{u}_i$, for $1 \leq i \leq n-1$. The Perron eigenvalue is r, and $\mathbf{v} = (v_1, v_2, ..., v_n)^T$ the Perron eigenvector has positive entries, from the Perron-Frobenius Theorem. If we now write A in the form

$$A = r\mathbf{v}\mathbf{v}^T + \lambda_{n-1}\mathbf{u}_{n-1}\mathbf{u}_{n-1}^T + \cdots + \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T,$$

and let $V = \text{diag}(\frac{1}{v_1}, \frac{1}{v_2}, ..., \frac{1}{v_n})$, we can also write

$$VAV = r\mathbf{e}\mathbf{e}^T + \lambda_{n-1}(V\mathbf{u}_{n-1})(V\mathbf{u}_{n-1})^T + \cdots + \lambda_1(V\mathbf{u}_1)(V\mathbf{u}_1)^T.$$

If $\mathbf{x}^T\mathbf{e} = 0$ then $\mathbf{x}^T VAV\mathbf{x} \leq 0$, i.e. $VAV = B = (b_{ij})$ is almost negative semidefinite.

Next, recall that for $t > 0$

$$\frac{1}{t} = \int_0^\infty e^{-ts} ds, \qquad \text{so} \qquad \mathbf{x}^T(\frac{1}{b_{ij}})\mathbf{x} = \int_0^\infty \mathbf{x}^T(e^{-b_{ij}s})\mathbf{x}\, ds,$$

and since $(-b_{ij}s)$, for $s > 0$ is almost positive semidefinite, from Lemma 2.5 $(e^{-b_{ij}s})$ is positive semidefinite, so $(\frac{1}{b_{ij}}) = (VAV)^{\circ(-1)} = V^{-1}A^{\circ(-1)}V^{-1}$ is positive semidefinite. We conclude that $A^{\circ(-1)}$ is positive semidefinite.

Finally, $A^{\circ(-1)}$ is positive definite iff $V^{-1}A^{\circ(-1)}V^{-1} = (\frac{1}{b_{ij}})$ is positive definite iff $(e^{-b_{ij}s})$ is positive definite iff $b_{ii} + b_{jj} < 2b_{ij}$, for all $i \neq j$ iff $\dfrac{a_{ii}}{v_i^2} + \dfrac{a_{jj}}{v_j^2} < 2\dfrac{a_{ij}}{v_i v_j}$, for all $i \neq j$.

Corollary 2.8: Let $A \in \mathbf{R}^{n \times n}$ be symmetric, have positive entries and just one positive eigenvalue. If A is invertible then $A^{\circ(-1)}$ is positive definite.

Proof: A invertible implies $B = VAV$ is almost negative definite, so $(e^{-b_{ij}s})$ is positive definite (Corollary 2.6), which implies $A^{\circ(-1)}$ is positive definite.

We now use Corollary 2.8 to give a simple proof of a well-known result for distance matrices (distance matrices are almost negative semidefinite matrices with positive off-diagonal entries, and zeroes on the diagonal [10],[19]). Recall that a real symmetric $n \times n$ matrix has at least k nonnegative (positive) eigenvalues, including multiplicities, if and only if A is positive semidefinite (positive definite) on a subspace of dimension k [12,p.192].

Theorem 2.9: Let $A \in \mathbf{R}^{n \times n}$ be symmetric, with positive off-diagonal entries, all diagonal entries equal to zero, and just one positive eigenvalue. Then the Hadamard square root $A^{\circ\frac{1}{2}} = (a_{ij}^{\frac{1}{2}})$ has just one positive eigenvalue and is invertible.

Proof: We use induction on n. Clearly the result is true for $n = 2$. We shall assume the result is true for $n-1$. As in the proof of Theorem 2.7, there is a diagonal matrix V, with positive diagonal entries, such that $VAV = B = (b_{ij}) \in \mathbf{R}^{n\times n}$ is almost negative semidefinite. From Lemma 2.4 we know that $C = (b_{in} + b_{nj} - b_{ij}) \in \mathbf{R}^{(n-1)\times(n-1)}$ is positive semidefinite. We will show that $D = (b_{in}^{\frac{1}{2}} + b_{nj}^{\frac{1}{2}} - b_{ij}^{\frac{1}{2}}) \in \mathbf{R}^{(n-1)\times(n-1)}$ is positive definite.

Write

$$b_{in} + b_{nj} - b_{ij} = (b_{in}^{\frac{1}{2}} + b_{nj}^{\frac{1}{2}} - b_{ij}^{\frac{1}{2}})(b_{in}^{\frac{1}{2}} + b_{nj}^{\frac{1}{2}} + b_{ij}^{\frac{1}{2}}) - 2b_{in}^{\frac{1}{2}}b_{nj}^{\frac{1}{2}},$$

then

$$C + 2\mathbf{c}\mathbf{c}^T = (b_{in}^{\frac{1}{2}} + b_{nj}^{\frac{1}{2}} - b_{ij}^{\frac{1}{2}}) \circ (b_{in}^{\frac{1}{2}} + b_{nj}^{\frac{1}{2}} + b_{ij}^{\frac{1}{2}}), \qquad (*)$$

where $\mathbf{c} = (b_{in}^{\frac{1}{2}}) \in \mathbf{R}^{n-1}$. We will use the fact that $C + 2\mathbf{c}\mathbf{c}^T$ is positive semidefinite, and has all diagonal entries nonzero. Write $(b_{in}^{\frac{1}{2}} + b_{nj}^{\frac{1}{2}} + b_{ij}^{\frac{1}{2}}) = \mathbf{c}\mathbf{e}^T + \mathbf{e}\mathbf{c}^T + \tilde{B}$, where $\tilde{B} = (b_{ij}^{\frac{1}{2}}) \in \mathbf{R}^{(n-1)\times(n-1)}$. $\tilde{B}^{\circ 2}$ is almost negative semidefinite, since it is a principal submatrix of B, and by induction $\tilde{B}$ is almost negative definite, so also $\mathbf{c}\mathbf{e}^T + \mathbf{e}\mathbf{c}^T + \tilde{B}$ is almost negative definite and hence invertible (one eigenvalue is positive, from the Perron-Frobenius Theorem). But then $(\mathbf{c}\mathbf{e}^T + \mathbf{e}\mathbf{c}^T + \tilde{B})^{\circ(-1)}$ is positive definite, and Hadamard multiplying on both sides of $(*)$ by this Hadamard inverse we conclude, using Lemma 2.2, that $D = (b_{in}^{\frac{1}{2}} + b_{nj}^{\frac{1}{2}} - b_{ij}^{\frac{1}{2}})$ is positive definite. So $(VAV)^{\circ\frac{1}{2}} = V^{\circ\frac{1}{2}}A^{\circ\frac{1}{2}}V^{\circ\frac{1}{2}} = B^{\circ\frac{1}{2}}$ is almost negative definite. Then since $V^{\circ\frac{1}{2}}A^{\circ\frac{1}{2}}V^{\circ\frac{1}{2}}$ is negative definite on a subspace of dimension $n-1$, $A^{\circ\frac{1}{2}}$ is negative definite on a subspace of dimension $n-1$, so $A^{\circ\frac{1}{2}}$ has at least $n-1$ negative eigenvalues, and one positive eigenvalue by the Perron-Frobenius Theorem.

Remark: Along the way we have shown that if $B = (b_{ij}) \in \mathbf{R}^{n\times n}$ is almost negative semidefinite, has positive off-diagonal entries, and zeroes on the diagonal, then $B^{\circ\frac{1}{2}} = (b_{ij}^{\frac{1}{2}})$ is almost negative definite and is invertible (this is the result for distance matrices).

3. Hadamard Products

The following theorem gives a Loewner partial order lower bound for the Hadamard product of two symmetric matrices under some fairly restrictive conditions. More theory on almost semidefinite matrices may be found in [6], [8], [20].

Theorem 3.1: Let $A,\ B \in \mathbf{R}^{n\times n}$ be symmetric. If $A \succeq 0$ and B is positive definite or is almost positive definite and invertible then $A \circ B \succeq \dfrac{1}{\mathbf{e}^T B^{-1}\mathbf{e}} A$.

Furthermore, if B is positive definite (so $\mathbf{e}^T B^{-1}\mathbf{e} > 0$), or if B is almost positive definite and invertible, in which case $\mathbf{e}^T B^{-1}\mathbf{e} < 0$, then

$$B - \frac{\mathbf{e}\mathbf{e}^{\mathbf{T}}}{\mathbf{e}^T B^{-1}\mathbf{e}} \succeq 0 \quad \text{and} \quad \frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}} = \sup\{t \in \mathbf{R} | B - t\mathbf{e}\mathbf{e}^T \succeq 0\}.$$

Proof: We show that $\mathbf{e}^T B^{-1}\mathbf{e} \neq 0$. If B is positive definite then certainly $\mathbf{e}^T B^{-1}\mathbf{e} > 0$. Suppose B is invertible and almost positive definite. Let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of B, and $\hat{\lambda}_1 \leq \cdots \leq \hat{\lambda}_{n+1}$ the eigenvalues of $\begin{bmatrix} B & e \\ e^T & 0 \end{bmatrix}$, then from "interlacing"

$$\hat{\lambda}_1 \leq \lambda_1 \leq \hat{\lambda}_2 \leq \lambda_2 \leq \cdots \leq \hat{\lambda}_n \leq \lambda_n \leq \hat{\lambda}_{n+1}.$$

Since B is almost positive definite we must have that $\lambda_2 > 0$. We must also have that $\lambda_1 < 0$, or otherwise B would be positive definite. Let $\mathbf{y} = (\mathbf{x}\ z)^T \in \mathbf{R}^{n+1}$, where $\mathbf{x}^T\mathbf{e} = 0$ and $z \in \mathbf{R}$. Then $\mathbf{y}^T \begin{bmatrix} B & e \\ e^T & 0 \end{bmatrix} \mathbf{y} = \mathbf{x}^T B\mathbf{x} > 0$, i.e. $\begin{bmatrix} B & e \\ e^T & 0 \end{bmatrix}$ is positive definite on a subspace of dimension n, so $\begin{bmatrix} B & e \\ e^T & 0 \end{bmatrix}$ has at least n positive eigenvalues, which implies $\hat{\lambda}_2 > 0$. Using Schur complements and properties of determinants we have

$$\det \begin{bmatrix} B & e \\ e^T & 0 \end{bmatrix} = -\det(B)\ \mathbf{e}^T B^{-1}\mathbf{e} = -(-1)|\lambda_1|\ \lambda_2 \cdots \lambda_n\ \mathbf{e}^T B^{-1}\mathbf{e} = (-1)|\hat{\lambda}_1|\ \hat{\lambda}_2 \cdots \hat{\lambda}_{n+1},$$

so we must also have that $\mathbf{e}^T B^{-1}\mathbf{e} < 0$.

Let $\mathbf{u}$ be any vector in $\mathbf{R}^n$, and $\mathbf{v} = (I - \frac{B^{-1}\mathbf{e}\mathbf{e}^T}{\mathbf{e}^T B^{-1}\mathbf{e}})\mathbf{u}$, then notice that $\mathbf{e}^T\mathbf{v} = 0$. Further, note that

$$(I - \frac{B^{-1}\mathbf{e}\mathbf{e}^T}{\mathbf{e}^T B^{-1}\mathbf{e}})^T B (I - \frac{B^{-1}\mathbf{e}\mathbf{e}^T}{\mathbf{e}^T B^{-1}\mathbf{e}}) = B - \frac{\mathbf{e}\mathbf{e}^\mathbf{T}}{\mathbf{e}^T B^{-1}\mathbf{e}}.$$

So if B is positive definite or almost positive definite (and invertible) then $B - \frac{\mathbf{e}\mathbf{e}^T}{\mathbf{e}^T B^{-1}\mathbf{e}} \succeq 0$ (not strict inequality here since $(I - \frac{B^{-1}\mathbf{e}\mathbf{e}^T}{\mathbf{e}^T B^{-1}\mathbf{e}})B^{-1}\mathbf{e} = \mathbf{0}$). In either case, Hadamard multiplying on both sides of this inequality by $A \succeq 0$ gives the inequality of our theorem.

Finally, we prove the "sup" part of the statement of the theorem. If B is positive definite or B is almost positive definite (and invertible) and $B - t\mathbf{e}\mathbf{e}^T \succeq 0$, taking $\mathbf{x} = B^{-1}\mathbf{e}$ we have that $\mathbf{x}^T(B - t\mathbf{e}\mathbf{e}^T)\mathbf{x} = \mathbf{e}^T B^{-1}\mathbf{e} - t(\mathbf{e}^T B^{-1}\mathbf{e})^2 \geq 0$, and this implies $\frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}} \geq t$.

Remarks: Useful examples to illustrate the theorem are $A = I_k \oplus O_{n-k}$, for $1 \leq k \leq n$, and $B = I - \epsilon\mathbf{e}\mathbf{e}^T$, where $\epsilon \in \mathbf{R}$, so $\frac{1}{\mathbf{e}^T B^{-1}\mathbf{e}} = \frac{1}{n} - \epsilon$. When $k = n$ and $\epsilon = 2$ notice that $A \circ B$ is negative definite. An example of an almost positive definite matrix which is not invertible is $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Corollary 3.2: If $A \succeq 0$ and B has all positive entries and is almost negative definite, then $A \circ B \preceq \dfrac{1}{\mathbf{e}^T B^{-1}\mathbf{e}} A$.

Proof: From the Perron-Frobenius Theorem B has one positive eigenvalue, thus B is invertible. The corollary then follows from the theorem with $-B$ substituted for B (so $\mathbf{e}^T B^{-1}\mathbf{e} > 0$). Notice that under the present hypotheses we can Hadamard multiply both sides of the inequality of the corollary by the positive definite matrix $B^{\circ(-1)}$, to also obtain the inequality $A \circ B^{\circ(-1)} \succeq (\mathbf{e}^T B^{-1}\mathbf{e})A$.

The special role that $\mathbf{e}$ has in Theorem 3.1 stems from the fact that $\mathbf{e}\mathbf{e}^T$ is the identity matrix for the Hadamard product. A restatement of the theorem without reference to $\mathbf{e}$ is Corollary 3.3. If $\mathbf{x}$ is an arbitrary vector in $\mathbf{R}^n$, we will denote by $D_{\mathbf{x}}$ the diagonal matrix $D_{\mathbf{x}} = \text{diag}(x_1, \ldots, x_n)$.

Corollary 3.3: Let $A,\ B \in \mathbf{R}^{n\times n}$ be symmetric, and let $\mathbf{b} \in \mathbf{R}^n$, $\mathbf{b} \neq \mathbf{0}$. If $A \succeq 0$ and B is positive definite or B is positive definite on the subspace $U = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{x}^T\mathbf{b} = 0\}$ and invertible then $A \circ B \succeq \dfrac{1}{\mathbf{b}^T B^{-1}\mathbf{b}} D_{\mathbf{b}} A D_{\mathbf{b}}$.

Furthermore, if B is positive definite (so $\mathbf{b}^T B^{-1}\mathbf{b} > 0$), or if B is positive definite on U and invertible, in which case $\mathbf{b}^T B^{-1}\mathbf{b} < 0$, then

$$B - \frac{\mathbf{b}\mathbf{b}^{\mathbf{T}}}{\mathbf{b}^T B^{-1}\mathbf{b}} \succeq 0 \quad \text{and} \quad \frac{1}{\mathbf{b}^T B^{-1}\mathbf{b}} = \sup\{t \in \mathbf{R} | B - t\mathbf{b}\mathbf{b}^T \succeq 0\}.$$

Proof: Hadamard multiply A across the inequality $B - \frac{\mathbf{b}\mathbf{b}^{\mathbf{T}}}{\mathbf{b}^T B^{-1}\mathbf{b}} \succeq 0$, and the corollary follows once we use the observation [10,p.104] that for any vector $\mathbf{w} \in R^n$ and any matrix $C \in \mathbf{R}^{n\times n}$ it is true that

$$\mathbf{w}\mathbf{w}^T \circ C = D_{\mathbf{w}} C D_{\mathbf{w}}.$$

A Loewner partial order upper and lower bound based upon the spectral decomposition of B is given in the following proposition.

Proposition 3.4: Let $A \succeq 0$. Let B be symmetric with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Suppose that $\lambda_i < 0$, for $i \in \{1, \ldots, k\}$; $\lambda_i = 0$, for $i \in \{k+1, \ldots, m-1\}$; and $\lambda_i > 0$, for $i \in \{m, \ldots, n\}$ (where any of these index sets can be empty). Let $\mathbf{u}_i$ denote the corresponding unit eigenvectors of B so that $B\mathbf{u}_i = \lambda_i \mathbf{u}_i$, for $1 \leq i \leq n$. Then

$$\lambda_n D_{\mathbf{u}_n} A D_{\mathbf{u}_n} + \cdots + \lambda_m D_{\mathbf{u}_m} A D_{\mathbf{u}_m} \succeq A \circ B \succeq \lambda_k D_{\mathbf{u}_k} A D_{\mathbf{u}_k} + \cdots + \lambda_1 D_{\mathbf{u}_1} A D_{\mathbf{u}_1}.$$

Proof: Write $B = \lambda_n \mathbf{u}_n \mathbf{u}_n^T + \cdots + \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T$, and notice that

$$B - \lambda_k \mathbf{u}_k \mathbf{u}_k^T - \cdots - \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T \succeq 0 \qquad \text{and} \qquad B - \lambda_n \mathbf{u}_n \mathbf{u}_n^T - \cdots - \lambda_m \mathbf{u}_m \mathbf{u}_m^T \preceq 0.$$

Then Hadamard multiplying A all the way across the inequalities

$$\lambda_n \mathbf{u}_n \mathbf{u}_n^T + \cdots + \lambda_m \mathbf{u}_m \mathbf{u}_m^T \succeq B \succeq \lambda_k \mathbf{u}_k \mathbf{u}_k^T + \cdots + \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T,$$

and using the observation in the proof of the previous corollary, we're done.

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References

[1] J. W. Auer, An Elementary Proof of the Invertibility of Distance Matrices, *Linear and Multilinear Algebra,* 40:119–124 (1995).

[2] R. B. Bapat, Multinomial probabilities, permanents and a conjecture of Karlin and Rinott, *Proc. Amer. Math. Soc.,* 102(3): 467–472 (1988).
[3] R. B. Bapat and T. E. S. Raghavan, *Nonnegative Matrices and Applications,* Encyclopedia of Mathematics and its Applications, No. 64, Cambridge Uni. Press, 1997.
[4] L. M. Blumenthal, *Theory and Applications of Distance Geometry,* Oxford University Press, Oxford, 1953. Reprinted by Chelsea Publishing Co., New York, 1970.
[5] F. Critchley and B. Fichet, The partial order by inclusion of the principal classes of dissimilarity on a finite set, *In* B. Van Cutsem (Ed.) *Classification and Dissimilarity Analysis,* Lect. Notes in Statist., Springer-Verlag, New York, pp. 5–65, 1994.
[6] J. P. Crouzeix and J. Ferland, Criteria for Quasiconvexity and Pseudoconvexity: Relationships and Comparisons, *Mathematical Programming* 23(2):193–205 (1982).
[7] W. F. Donoghue, *Monotone matrix functions and analytic continuation,* Springer-Verlag, New York, 1974.
[8] J. A. Ferland, Matrix-Theoretic Criteria for the Quasiconvexity of Twice Continuously Differentiable Functions, *Linear Algebra and its Applications,* 38:51–63 (1981).
[9] M. Fiedler and T. L. Markham, An Observation on the Hadamard Product of Hermitian Matrices, *Linear Algebra and its Applications,* 215:179–182 (1995).
[10] J. C. Gower, Euclidean Distance Geometry, *Math. Scientist,* 7:1–14 (1982).
[11] R. A. Horn, The Hadamard Product, *In* C. R. Johnson (Ed.) *Proc. Appl. Math.,* Vol. 40, American Mathematical Society, Providence, Rhode Island, pp. 87–169, 1990.
[12] R. A. Horn and C. R. Johnson, *Matrix Analysis,* Cambridge Uni. Press, 1985.
[13] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis,* Cambridge Uni. Press, 1991.
[14] J. B. Kelly, Hypermetric spaces and metric transforms, *In* O. Shisha (Ed.) *Inequalities III*, Academic Press, New York, pp. 149–158, 1972.
[15] P. Lancaster and M. Tismenetsky, *The Theory of Matrices, with applications* 2$^{\text{nd}}$ Ed. Academic Press, San Diego, 1985.
[16] M. Marcus and T. R. Smith, A Note on the Determinants and Eigenvalues of Distance Matrices, *Linear and Multilinear Algebra,* 25:219–230 (1989).
[17] C. A. Micchelli, Interpolation of Scattered Data: Distance Matrices and Conditionally Positive Definite Functions, *Constr. Approx.,* 2:11–22 (1986).
[18] I. J. Schoenberg, On certain metric spaces arising from Euclidean space by a change of metric and their imbedding in Hilbert space, *Ann. of Math.,* 38(4):787–793 (1937).
[19] I. J. Schoenberg, Metric spaces and positive definite functions, *Trans. Amer. Math. Soc.,* 44:522–536 (1938).
[20] P. Tarazaga, T. L. Hayden and J. Wells, Circum-Euclidean Distance Matrices and Faces, *Linear Algebra and its Applications,* 232:77–96 (1996).