On Copositive Matrices and their Spectrum

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Abstract

Let $A \in \mathbf{R}^{n\times n}$. We provide a block characterization of copositive matrices, with the assumption that one of the principal blocks is positive definite. Haynsworth and Hoffman showed that if r is the largest eigenvalue of a copositive matrix then $r \geq |\lambda|$, for all other eigenvalues λ of A. We continue their study of the spectral theory of copositive matrices and show that a copositive matrix must have a positive vector in the subspace spanned by the eigenvectors corresponding to the nonnegative eigenvalues. Moreover, if a symmetric matrix has a positive vector in the subspace spanned by the eigenvectors corresponding to its nonnegative eigenvalues, then it is possible to increase the the nonnegative eigenvalues to form a copositive matrix A', without changing the eigenvectors. We also show that if a copositive matrix has just one positive eigenvalue, and $n-1$ nonpositive eigenvalues then A has a nonnegative eigenvector corresponding to a nonnegative eigenvalue.

Key words: symmetric, eigenvalue, eigenvector, copositive, strictly copositive, Schur complement, positive semidefinite, nonnegative eigenvector
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1 Introduction

Let $e_i \in \mathbf{R}^n$ denote the vector with a 1 in the ith position and all 0's elsewhere. For $x = (x_1, \ldots, x_n)^T \in \mathbf{R}^n$ we will use the notation that $x \geq 0$ when $x_i \geq 0$ for all i, $1 \leq i \leq n$, and $x > 0$ when $x_i > 0$ for all i, $1 \leq i \leq n$. We will say that a matrix is nonnegative (nonpositive) in the event that all of its entries are nonnegative (nonpositive). A symmetric matrix is positive semidefinite if $x^TAx \geq 0$, for all $x \in \mathbf{R}^n$, and positive definite if $x^TAx > 0$, for all $x \in \mathbf{R}^n$,

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$x \neq 0$. A symmetric matrix $A \in \mathbf{R}^{n\times n}$ is said to be copositive when $x^TAx \geq 0$ for all $x \geq 0$, and A is said to be strictly copositive when $x^TAx > 0$ for all $x \geq 0$ and $x \neq 0$. A nonnegative matrix is a copositive matrix, as is a positive semidefinite matrix. Clearly, the sum of two copositive matrices is copositive. It was shown by Alfred Horn (see [2]) that a copositive matrix need not be the sum of a positive semidefinite matrix and a nonnegative matrix.

2 Conditions for copositivity of block matrices

Lemma 1, which is an extension of a similar result for positive semidefinite matrices, appeared in [4],[6], [9]. We include a proof for completeness.

Lemma 1 *Let $A \in \mathbf{R}^{n\times n}$ be copositive. If $x_0 \geq 0$ and $x_0^TAx_0 = 0$, then $Ax_0 \geq 0$.*

PROOF. Let $\epsilon > 0$. Then for any i, $1 \leq i \leq n$, since $x_0 + \epsilon e_i \geq 0$, we have

$$(x_0 + \epsilon e_i)^T A(x_0 + \epsilon e_i) = x_0^TAx_0 + 2\epsilon e_i^TAx_0 + \epsilon^2 e_i^TAe_i \geq 0. \tag{1}$$

This says that $2\epsilon e_i^TAx_0 \geq -\epsilon^2 a_{ii}$, so $e_i^TAx_0 \geq -\frac{\epsilon}{2}a_{ii}$. But this is true for any $\epsilon > 0$, so $Ax_0 \geq 0$. □

A form of Theorem 2, restricted to when $b \geq 0$, or $b \leq 0$, was given in [1].

Theorem 2 *Let $A = \begin{pmatrix} a & b^T \\ b & A' \end{pmatrix} \in \mathbf{R}^{n\times n}$, where $A' \in \mathbf{R}^{(n-1)\times(n-1)}$, $b \in \mathbf{R}^n$, and $a \in \mathbf{R}$. Then A is copositive if and only if $a \geq 0$; A' is copositive; if $a > 0$ then $x'^T(A' - \frac{bb^T}{a})x' \geq 0$, for all $x' \in \mathbf{R}^{n-1}$, such that $x' \geq 0$ and $b^Tx' \leq 0$; if $a = 0$ then $b \geq 0$.*

PROOF. For $x = (x_1, x')^T \in \mathbf{R}^n$, where $x_1 \in \mathbf{R}$ and $x' \in \mathbf{R}^{n-1}$, we have

$$\begin{aligned} x^TAx &= ax_1^2 + 2b^Tx'x_1 + x'^TA'x', & (2)\\ &= a[x_1 + \frac{b^Tx'}{a}]^2 + x'^T(A' - \frac{bb^T}{a})x', \qquad (\text{if } a > 0). & (3) \end{aligned}$$

Suppose A is copositive. Then evidently $a \geq 0$ and A' is copositive. If $a > 0$ and $b^Tx' \leq 0$ then taking $x_1 = -\frac{b^Tx'}{a}$ we have that $x'^T(A' - \frac{bb^T}{a})x' \geq 0$. If $a = 0$ then from Lemma 1 with $x_0 = e_1$, we have $b \geq 0$.

For the converse, if $a = 0$, $b \geq 0$, so from equation 2, A is copositive, since A' is copositive. Suppose $a > 0$. If $b^T x' > 0$, then use equation 2 to conclude that $x^T Ax \geq 0$, since A' is copositive. Whereas, if $b^T x' \leq 0$ use equation 3 to conclude A is copositive. $\square$

Corollary 3 *Suppose $a > 0$ in the matrix A in the statement of the theorem. If $A' - \dfrac{bb^T}{a}$ is copositive then A is copositive.*

PROOF. Follows from equation 3. $\square$

For an example to illustrate that if the matrix A in Theorem 2 is copositive it does not necessarily follow that $A' - \dfrac{bb^T}{a}$ is copositive, consider $A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$. A is copositive, since $A = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} (1, -1, 1) + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

We can extend Theorem 2 to provide a Schur complement-like condition, analogous to the well known block characterization of positive definite matrices.

Theorem 4 *Let $A = \begin{pmatrix} A_1 & B^T \\ B & A_2 \end{pmatrix} \in \mathbf{R}^{n \times n}$, where $A_1 \in \mathbf{R}^{l \times l}$, $B \in \mathbf{R}^{m \times l}$, and $A_2 \in \mathbf{R}^{m \times m}$, where $l + m = n$. Let A_1 be positive definite, and suppose that $A_1^{-1} B^T$ is nonpositive. Then A is copositive if and only if $A_2 - BA_1^{-1} B^T$ is copositive.*

PROOF. For $x = (x_1, x_2)^T \in \mathbf{R}^n$, where $x_1 \in \mathbf{R}^l$ and $x_2 \in \mathbf{R}^m$, we have

$$\begin{aligned} x^T Ax &= x_1^T A_1 x_1 + 2x_1^T B^T x_2 + x_2^T A_2 x_2, & (4) \\ &= (x_1 + A_1^{-1} B^T x_2)^T A_1 (x_1 + A_1^{-1} B^T x_2) + x_2^T (A_2 - BA_1^{-1} B^T) x_2. & (5) \end{aligned}$$

Reasoning as in Theorem 2, and using only equation 5, if A is copositive, taking $x_1 = -A_1^{-1} B^T x_2$, for any $x_2 \geq 0$, we see that $A_2 - BA_1^{-1} B^T$ is copositive. Conversely, A_1 being positive definite, and $A_2 - BA_1^{-1} B^T$ being copositive implies the copositivity of A. $\square$

3 Spectral theory of copositive matrices

Haynsworth and Hoffman [3] showed that for a copositive matrix A, its largest eigenvalue r satisfies $r \geq |\lambda_i|$, where the λ_i's are the other eigenvalues of A.

Their proof is: Let $\lambda < 0$ and $Ax = \lambda x$, where $||x|| = 1$. Then write $x = y - z$, where $y \geq 0, z \geq 0$, and $y^T z = 0$, so $||y+z|| = 1$. Then $r \geq (y+z)^T A(y+z) = 2y^T Ay + 2z^T Az - (y-z)^T A(y-z) \geq -\lambda$.

The Perron-Frobenius Theorem [5] states that a nonnegative matrix A has an eigenvalue r, such that $r \geq |\lambda|$ for all other eigenvalues λ of A, and that the eigenvector corresponding to r has nonnegative components. For a copositive matrix, in general, the eigenvector corresponding to r need not be nonnegative, since a positive semidefinite matrix can be constructed as $\sum_{i=1}^{k} u_i u_i^T$, $n \geq k \geq 1$, with orthogonal u_i's where the components of each u_i has mixed signs.

We will use Gordan's and Stiemke's versions of the Theorem of the Alternative (see for instance [8]) for what follows.

Theorem 5 *(Gordan) Let $B \in \mathbf{R}^{m \times n}$. Then either statement I or II occurs, but not both.*

I $y^T B > 0$, for some $y \in \mathbf{R}^m$;

II $Bx = 0$, for some $x \in \mathbf{R}^n$, $x \geq 0$ $(x \neq 0)$.

Theorem 6 *(Stiemke) Let $B \in \mathbf{R}^{m \times n}$. Then either statement I or II occurs, but not both.*

I $y^T B \geq 0$, for some $y \in \mathbf{R}^m$;

II $Bx = 0$, for some $x \in \mathbf{R}^n$, $x > 0$.

Kaplan [7] proved, among other things, that a copositive matrix cannot have a positive eigenvector corresponding to a negative eigenvalue. We now extend this result as follows.

Lemma 7 *Let A be a copositive matrix, with at least one negative eigenvalue. Then A cannot have a nonnegative vector in the subspace spanned by the eigenvectors corresponding to the negative eigenvalues.*

PROOF. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of A. Suppose also that they are ordered as $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$ and $0 > \lambda_{k+1} \geq \cdots \geq \lambda_n$, with corresponding orthonormal eigenvectors $v_1, ..., v_k, v_{k+1}, ..., v_n$, respectively. If there was a vector $w \geq 0$ in the subspace spanned by $v_{k+1}, ..., v_n$, then writing $w = \sum_{i=k+1}^{n} \mu_i v_i$, we would have $w^T A w = (\sum_{i=k+1}^{n} \mu_i v_i)^T A (\sum_{i=k+1}^{n} \mu_i v_i) = \sum_{i=k+1}^{n} \lambda_i \mu_i^2 < 0$, which is not possible, since A is copositive. $\square$

We now use Theorem 6 and Lemma 7 to prove a weaker result than that A has a nonnegative or positive eigenvector. We also prove a partial converse.

Theorem 8 *Let $A \in \mathbf{R}^{n\times n}$ be symmetric. If A is copositive then there is a positive vector in the subspace spanned by the eigenvectors corresponding to the nonnegative eigenvalues. Moreover, if A has a positive vector in the subspace spanned by the eigenvectors corresponding to the nonnegative eigenvalues, then the nonnegative eigenvalues may be made sufficiently large, without changing any eigenvectors so that the resulting matrix A' is copositive.*

PROOF. If all the eigenvalues of A are nonnegative the result is clear. So, suppose A has at least one negative eigenvalue. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of A, and label the eigenvalues and eigenvectors with the same notation as the lemma. Clearly, $\lambda_1 > 0$. Let B be the $(n-k) \times n$ matrix whose rows are $v_{k+1}, ..., v_n$. Then from Lemma 7 there is no $y \in \mathbf{R}^{n-k}$, such that $y^T B \geq 0$. From Stiemke's Theorem of the Alternative, we must have $Bx = 0$, for some $x \in \mathbf{R}^n$, $x > 0$. But then x must be in the subspace spanned by the eigenvectors $v_1, ..., v_k$.

To prove the "Moreover" part, let $Z = \{z = b_1 v_1 + \cdots + b_n v_n | z \geq 0, ||z|| = 1\}$. Clearly, Z is both closed and bounded. Note also that $\sum_{i=1}^n b_i^2 = 1$, since $v_1, ..., v_n$ are orthonormal. Let $b = \min_{z\in Z} \sum_{i=1}^k b_i^2$, $c = \max_{z\in Z} \sum_{i=k+1}^n b_i^2$, and let λ_m be the smallest positive eigenvalue among $\lambda_1, ..., \lambda_k$. Then $z^T A z = b_1^2\lambda_1 + \cdots + b_k^2\lambda_k + b_{k+1}^2\lambda_{k+1} + \cdots + b_n^2\lambda_n \geq (b_1^2 + \cdots + b_m^2)\lambda_m + (b_{k+1}^2 + \cdots + b_n^2)\lambda_n \geq b\lambda_m + c\lambda_n$. Now if $b = 0$ then this would imply that there is a vector $z \in Z$ which is a linear combination of $v_{k+1}, ..., v_n$. but then if B is the $(n-k) \times n$ matrix whose rows are $v_{k+1}, ..., v_n$, as in Theorem 8, we would have $Bx \geq 0$. But this is not possible, since we already have that there is a positive vector in the subspace spanned by $v_1, ..., v_k$, and so this positive vector is orthogonal to $v_{k+1}, ..., v_n$, that is to say, there is a $y \in \mathbf{R}^n$ such that $y^T B > 0$, which is the other alternative of Gordan's Theorem of the Alternative. So, with $b > 0$ we can increase λ_m, to λ'_m, so that $b\lambda'_m + c\lambda_n \geq 0$. Then $A' = \lambda_1 v_1 v_1^T + \cdots + \lambda'_m v_m v_m^T + \cdots + \lambda_n v_n v_n^T$, is copositive, since b is smallest and c is largest for the given eigenvectors. Thereafter, increasing any of the other nonnegative eigenvalues will retain copositivity, since this is effectively only adding a positive semidefinite matrix to A', to form a new A'. □

One consequence of the construction in the proof of Theorem 8 is that we can see that a copositive matrix might or might not have a nonnegative eigenvector corresponding to one of its eigenvalues (although, of course, we know from Lemma 7, a nonnegative eigenvector can't correspond to a negative eigenvalue), since the only requirement for the construction of A' was that there be a positive vector in the span of the eigenvectors corresponding to the nonnegative eigenvalues.

Theorem 9 *Let $A \in \mathbf{R}^{n\times n}$ be strictly copositive. Then there is a (nonzero)* *nonnegative vector in the subspace spanned by the eigenvectors corresponding to the positive eigenvalues.*

PROOF. Replacing strict inequalities with nonstrict inequalities, or vice versa, in appropriate places, and using Gordan's Theorem of the Alternative [8], proves the result. $\square$

However, with some conditions on the eigenvalues, we can guarantee that a copositive matrix has a nonnegative or positive eigenvector. Consider the copositive matrices $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. A shows that we can have a positive eigenvector going with a zero eigenvalue. B shows that we can have a nonnegative eigenvector going with a positive eigenvalue. C shows that with at least one negative eigenvalue we can have a nonnegative eigenvector going with a positive eigenvalue. To prove the theorem, guaranteeing a nonnegative or positive eigenvector, we will need a lemma.

Lemma 10 *Let $u \in \mathbf{R}^n$ be a vector with components of mixed signs (i.e. at least one positive component and at least one negative component), and let $v \in \mathbf{R}$ be any vector linearly independent with u. Then there is a positive vector w, which is both orthogonal to u and not orthogonal to v.*

PROOF. Since u has mixed signs, there are positive vectors orthogonal to u. Pick any vector $x > 0$ which is orthogonal to u. If x is not orthogonal to v, take $w = x$ and we're done. If x is orthogonal to v, take $w = x + \epsilon v$, where ϵ is small enough so that $w > 0$. Then $w^T v = \epsilon v^T v > 0$, and we're done. $\square$

Theorem 11 *Let $A \in \mathbf{R}^{n \times n}$ be copositive, with one positive eigenvalue r, and $n-1$ nonpositive eigenvalues. Then A has a nonnegative eigenvector corresponding to a nonnegative eigenvalue. Moreover, if A has no negative eigenvalues, then either r has a nonnegative eigenvector, or zero has a positive eigenvector. Further, if A has any negative eigenvalues, then r has a nonnegative eigenvector, and if A has $n-1$ negative eigenvalues, then r has a positive eigenvector.*

PROOF. Using the notation of the previous lemmas and theorems (except that $\lambda_1 = r$), let v_1 be the eigenvector corresponding with r. If A has no negative eigenvalues, we must have $A = r v_1 v_1^T$. If v_1 is nonnegative then we're done, since v_1 is the desired eigenvector. If v_1 has mixed signs then we can find $w > 0$ orthogonal to v_1, and this w is an eigenvector corresponding to zero. Let A now have at least one negative eigenvalue. Suppose v_1 has mixed signs.

Then from Lemma 10 there is a positive vector w which is orthogonal to v_1 and not orthogonal to v_n. Then $w^T A w = \sum_{i=2}^{n} \lambda_i (w^T v_i)^2 < 0$, which is not possible since A is copositive. So, v_1 must be nonnegative. Finally, let A have $n-1$ negative eigenvalues, then A has a positive eigenvector corresponding with r from Theorem 8, since the subspace spanned by the eigenvectors corresponding to the nonnegative eigenvalues is one dimensional. $\square$

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