The Copositive Completion Problem

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Abstract

An $n \times n$ real symmetric matrix A is called (strictly) copositive if $x^T Ax \geq 0$ (> 0) whenever $x \in \mathbf{R}^n$ satisfies $x \geq 0$ ($x \geq 0$ and $x \neq 0$). The (strictly) copositive matrix completion problem asks which partial (strictly) copositive matrices have a completion to a (strictly) copositive matrix. We prove that every partial (strictly) copositive matrix has a (strictly) copositive matrix completion and give a lower bound on the values used in the completion. We answer affirmatively an open question whether an $n \times n$ copositive matrix $A = (a_{ij})$ with all diagonal entries $a_{ii} = 1$ stays copositive if each off-diagonal entry of A is replaced by $\min\{a_{ij}, 1\}$.

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An $n \times n$ real symmetric matrix A is called copositive if $x^T Ax \geq 0$ whenever $x \in \mathbf{R}^n$ satisfies $x \geq 0$ (entry-wise), and is called strictly copositive if $x^T Ax > 0$ whenever $x \geq 0$ and $x \neq 0$. The copositive matrices arise in a number of ways (e.g. they constitute the cone theoretic dual of the completely positive matrices [2]), and have received notable study [1], [3], [4], [5], [7]. Checking a given matrix may be carried out definitively [8], [9], [10], [11] but is generally computationally time-consuming. Since the vector argument for the quadratic form of a principal submatrix may be embedded into an argument for the quadratic form for the full matrix by insertion of 0's, copositivity and strict copositivity, are inherited by principal submatrices. Thus, the diagonal entries of a (strictly) copositive matrix are nonnegative (positive).

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A partial matrix is one in which some entries are specified, while the remaining entries are unspecified and free to be chosen. A completion of a partial matrix is a choice of values for the unspecified entries, resulting in a conventional matrix, and a matrix completion problem asks which matrices have completions with a desired property. The (strictly) copositive matrix completion problem asks which partial symmetric matrices have a (strictly) copositive completion. Our purpose here is to answer these two questions. We assume, without loss of generality, that the diagonal entries are specified. An obvious necessary condition that a symmetric partial matrix B have a (strictly) copositive completion is that every fully specified principal submatrix of B be (strictly) copositive. Such a partial matrix is called partial (strictly) copositive. We show that in each case, the necessary condition is sufficient. Thus, the copositive problems are rather like the combinatorially symmetric P-matrix completion problem [6] and quite different from the positive (semi-)definite completion problem, for which complicated additional conditions are needed when the graph of the specified entries is not chordal.

We first analyse the copositive completion problems in the case of one symmetrically placed pair of unspecified entries. Since the property of (strict) copositivity is permutation similarity invariant, we may assume that the unspecified entry is in the upper right and lower left corners, without loss of generality.

Theorem 1 *Let* $A = \begin{pmatrix} a & b^T & ? \\ b & A' & c \\ ? & c^T & d \end{pmatrix}$ *be a partial copositive matrix. Then* $A = \begin{pmatrix} a & b^T & s \\ b & A' & c \\ s & c^T & d \end{pmatrix}$ *is a copositive matrix for* $s \geq \sqrt{ad}$*. If* A *is partial strictly copositive then* A *with* $s \geq \sqrt{ad}$ *is strictly copositive. Furthermore,* $\sqrt{ad}$ *is best possible in general.*

PROOF. Let $x = (x_1, x'^T, x_n)^T \geq 0$, where $x' \in \mathbf{R}^{n-2}$ and $x_1, x_n \in \mathbf{R}$. Then $x^T A x = a x_1^2 + x'^T A' x' + d x_n^2 + 2 s x_1 x_n + 2 x_1 x'^T b + 2 x_n x'^T c$. A is partial copositive, so if $x_n = 0$, $x^T A x = a x_1^2 + x'^T A' x' + 2 x_1 x'^T b \geq 0$, for any $x' \geq 0$ and $x_1 \geq 0$. Let $f(x_1) = a x_1^2 + 2 x_1 x'^T b + x'^T A' x'$, then $f(x_1) \geq 0$, for any $x_1 \geq 0$.

If $x'^T b < 0$ then by choosing x_1 as large as desired, we must have $a > 0$. Then f has a minimum at $x_1 = -\dfrac{x'^T b}{a}$, and we have $f(-\dfrac{x'^T b}{a}) = x'^T A' x' - \dfrac{(x'^T b)^2}{a} \geq 0$. Similarly, if $x'^T c < 0$ we have $x'^T A' x' - \dfrac{(x'^T c)^2}{d} \geq 0$.

If $x'^T b \geq 0$, then for $s \geq 0$ we have

$$\begin{aligned}
x^T A x &= a x_1^2 + x'^T A' x' + d x_n^2 + 2 s x_1 x_n + 2 x_1 x'^T b + 2 x_n x'^T c, \\
&\geq x'^T A' x' + d x_n^2 + 2 x_n x'^T c, \\
&= (0, x'^T, x_n) A (0, x'^T, x_n)^T \geq 0.
\end{aligned}$$

Similarly, if $x'^T c \geq 0$ and $s \geq 0$ then $x^T A x \geq 0$.

Assume $x'^T b < 0$ and $x'^T c < 0$, and without loss of generality $\dfrac{x'^T b}{\sqrt{a}} \geq \dfrac{x'^T c}{\sqrt{d}}$.

If $s \geq \sqrt{ad}$ then

$$\begin{aligned}
x^T A x &\geq a x_1^2 + x'^T A' x' + d x_n^2 + 2\sqrt{ad} x_1 x_n + 2 x_1 x'^T b + 2 x_n x'^T c, \\
&= (\sqrt{a} x_1 + \sqrt{d} x_n)^2 + x'^T A' x' + 2\sqrt{a} x_1 \frac{x'^T b}{\sqrt{a}} + 2\sqrt{d} x_n \frac{x'^T c}{\sqrt{d}}, \\
&\geq (\sqrt{a} x_1 + \sqrt{d} x_n)^2 + 2(\sqrt{a} x_1 + \sqrt{d} x_n) \frac{x'^T c}{\sqrt{d}} + x'^T A' x', \\
&= [\sqrt{a} x_1 + \sqrt{d} x_n + \frac{x'^T c}{\sqrt{d}}]^2 - (\frac{x'^T c}{\sqrt{d}})^2 + x'^T A x' \geq 0.
\end{aligned}$$

Now consider when A is partial strictly copositive. Then $a > 0$ and $d > 0$.

With A partial strictly copositive, and $s \geq 0$, then $x'^T b \geq 0$ implies $x^T A x \geq (0, x'^T, x_n) A (0, x'^T, x_n)^T = (x'^T, x_n) \begin{pmatrix} A' & c \\ c^T & d \end{pmatrix} (x'^T, x_n)^T > 0$, if $(x'^T, x_n)^T \neq 0$, since $\begin{pmatrix} A' & c \\ c^T & d \end{pmatrix}$ is strictly copositive.

If $(x'^T, x_n)^T = 0$ then for $x = (x_1, 0, 0)^T$, where $x_1 \neq 0$, we have in this case $x^T A x = a x_1^2 > 0$, also.

If A is partial strictly copositive and $s \geq 0$, $x'^T c \geq 0$ implies $x^T A x > 0$, for $x \neq 0$, by a similar argument to that given in the previous paragraph, but instead using $\begin{pmatrix} a & b^T \\ b & A' \end{pmatrix}$.

If $x'^T b < 0$, then $(x_1, x') \neq 0$, so $f(x_1) > 0$, for any $x_1 \geq 0$, and so $x'^T A' x' - \dfrac{(x'^T b)^2}{a} > 0$, and if $x'^T c < 0$ then $x'^T A' x' - \dfrac{(x'^T c)^2}{d} > 0$. Then, as in the copositive case, for $s \geq \sqrt{ad}$ if $x'^T b < 0$ and $x'^T c < 0$, we have $x^T A x > 0$.

To show that no general improvement in the bound is possible take $A =$

$\begin{pmatrix} 1 & -1 & s \\ -1 & 1 & -1 \\ s & -1 & 1 \end{pmatrix}$, and consider $x^T A x$, where $x = (1, 1, x_3)$ with x_3 small. In the strictly copositive case, take $A + \epsilon I$, $\epsilon > 0$, and argue by continuity. $\square$

Remark: Note that if $b, c \geq 0$, then $s \geq -\sqrt{ad}$ suffices, as $\begin{pmatrix} 0 & b^T \\ b & A' \end{pmatrix}$ and $\begin{pmatrix} A' & c \\ c^T & 0 \end{pmatrix}$ are copositive and $\begin{pmatrix} a & -\sqrt{ad} \\ -\sqrt{ad} & d \end{pmatrix}$ is copositive. We have $s \geq -\sqrt{ad}$, necessarily. It is an interesting question to determine the minimum s that yields copositivity, in terms of the specific data.

Now consider a general partial (strictly) copositive $n \times n$ matrix $B = (b_{ij})$, with fully specified diagonal and focus upon a particular symmetrically placed pair of unspecified entries, say b_{pq}, b_{qp}. We shall apply the rule of Theorem 1 repeatedly, namely, "fill in s in place of b_{pq} and b_{qp}, where $s \geq \sqrt{b_{pp} b_{qq}}$" $(*)$. Thus, the proof could proceed as follows: We choose an arbitrary pair of unspecified entries and apply the rule $(*)$, say with the equals sign, to obtain a value for those two entries. Then we consider an arbitrary principal submatrix, two of whose entries are those just specified and whose other entries are already specified. Theorem 1 asserts that this principal submatrix, possibly after a permutation similarity to put s in the upper right and lower left corners, is (strictly) copositive. We conclude that the matrix B, with the newly specified entries, is partially (strictly) copositive. Repeating the process as needed, one obtains a (strictly) copositive completion of the given matrix. Notice that the completion can be done in one step by applying $(*)$ to all unspecified entries.

Theorem 2 *Let B be a partial (strictly) copositive $n \times n$ matrix. Then, there is a completion A of B that is (strictly) copositive.*

Corollary 3 *If in an $n \times n$ (strictly) copositive matrix each entry in a symmetric set of entries is replaced by application of the rule $(*)$, then the matrix remains (strictly) copositive.*

Thus, from Corollary 3, we can deduce the useful rule: If a matrix A with nonnegative diagonal is the completion of a partially (strictly) copositive matrix whose off-diagonal entries are negative and the other entries of A satisfy $(*)$, then A is (strictly) copositive.

Let A be a copositive matrix with all diagonal entries $a_{ii} = 1$. Now apply rule $(*)$, with the equals sign, to all off-diagonal entries $a_{ij} > 1$, in effect, replacing all such entries with 1. Then Corollary 3 answers affirmatively a question raised by Kaplan in [8] (bottom of page 245, and open question 3 on page 250), whether a copositive matrix $A = (a_{ij})$ with unit diagonal is converted to a copositive matrix by replacing each off-diagonal entry a_{ij} by $\min\{a_{ij}, 1\}$.

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