FUZZY MAPS AND THEIR APPLICATION IN THE SIMPLIFICATION OF FUZZY SWITCHING FUNCTIONS

by

Abraham Kandel
Computer Science Department
New Mexico Institute of Mining & Technology
Socorro, New Mexico 87801

ABSTRACT

In Boolean logic, a Karnaugh map may be regarded either as a pictorial form of a trugh table, or as an extension of the Venn diagram.

However, when fuzzy logic is concerned another minimization method is required, and therefore an extension of a Karnaugh map is investigated.

In this paper a new minimization algorithm is developed in order to remove the existing disadvantage of simplifying fuzzy forms. The algorithm is based on a new representation of fuzzy forms that assures a very suitable automatic generation of the fuzzy prime implicants and of the essential fuzzy prime implicants.

1. INTRODUCTION

The strictly Boolean approach to the treatment of combinational systems today is not always adequate to describe systems in the so called "real world". The complexity of behavior of a multi-value system derives mainly from the interrelations between the variables and real-world constraints, and therefore the attributes of system variables are often ambiguously defined. Cases with such attributes arise, for example, in pattern recognition, artificial intelligence, natural languages, and optimization.

Ever since the introduction of fuzzy set theory by Zadeh [1], a number of authors have been concerned with the design of fuzzy systems and fuzzy switching circuits [2]-[11]. This paper is concerned with the study of minimization in fuzzy logic by means of a suitable fuzzy map. It has been motivated by a statement from Marinos [7]:

> "Since the fuzzy variables can assume infinitely many values within the interval [0,1], the use of mapping or other graphical techniques which play a very important role in two-valued logic is not possible".

As will be shown, this statement is proved to be premature.

2. FUZZY ALGEBRA

Intuitively, a fuzzy set is a class which admits the possibility of partial membership in it. Let X denote a space of objects. Then a fuzzy set A in X is a set of ordered pairs

$$A=\{(x,\mu_A(x)): x\varepsilon X\}$$

where $\mu_A(x)$ is termed the grade of membership of x in A. We shall assume for simplicity that $\mu_A(x)$ is a number in the interval [0,1], with the grades 1 and 0 representing respectively, full membership and non-membership in a fuzzy set.

<u>Definition 1</u>: A <u>fuzzy algebra</u> is the system

$$\mathcal{Z} = \langle Z,+,*,\bar{\ }\rangle$$

where Z has at least two distinct elements, and system $\mathcal{Z}$ satisfies the following set of axioms $\forall x,y,z\varepsilon Z$:

(1) Idempotency: $x+x=x \quad x*x=x$

(2) Commutativity: $x+y=y+x \quad x*y=y*x$

(3) Associativity: $(x+y)+z=x+(y+z)$
$(x*y)*z=x*(y*z)$

(4) Absorption: $x+(x*y)=x \quad x*(x+y)=x$

(5) Distributivity: $x+(y*z)=(x+y)*(x+z)$
$x*(y+z)=x*y+x*z$

(6) Complement: If $x\varepsilon Z$ then there is a unique complement $\bar{x}$ of x such that $\bar{x}\varepsilon Z$ and $\bar{\bar{x}}=x$.

(7) Identities: $(\exists! e_+)(\forall x)$ such that $x+e_+ = e_++x=x$
$(\exists! e_*)(\forall x)$ such that $x*e_* = e_**x=x$

(8) De-Morgan Laws: $\overline{x+y} = \bar{x}*\bar{y} \quad \overline{x*y} = \bar{x}+\bar{y}$

Clearly the system is a distributive lattice with existence of unique identities under + and *. It is noted that a Boolean algebra is a complemented distributive lattice with existence of unique identities under + and *. However, for every element x in Boolean algebra we have a complement $\bar{x}$ such that $x\bar{x} = 0$ and $x+\bar{x}=1$ which is not true in general in fuzzy algebra. Hence, every Boolean algebra defined by the system

$$\mathcal{Z} = \langle[0,1],+,*,\bar{\ }\rangle,$$

where +,*, and $\bar{\ }$ are interpreted as Max, min, and complement ($\bar{x} = 1-x$, $\forall x\varepsilon[0,1]$), respectively. The unique identities e_+ and e_* are 0 and 1, respectively.

This system is the fuzzy algebra used in [1]. In this fuzzy algebra the elements 0,1 satisfy for every fuzzy element x

$$x + 0 = x \qquad x*0 = 0$$
$$x + 1 = 1 \qquad x*1 = x.$$

Let X be a space of objects as defined above. Let A and B be two fuzzy sets in X. The following terminology parallels closely the presentation of fuzzy sets in [1]. Equality (A=B) is defined by

$$A = B \leftrightarrow \mu_A(x) = \mu_B(x),\ \forall x \varepsilon X.$$

Fuzzy set A is contained in B($A \subset B$) iff $\mu_A(x) \leq \mu_B(x)$, $\forall x \varepsilon X$. A fuzzy set $\bar{A}$ is the complement of a fuzzy set A iff $\mu_{\bar{A}}(x) = 1-\mu_A(x)$, $\forall x \varepsilon X$. The union of two fuzzy sets A and B in X is defined by the membership function of A+B given by

$$\mu_{A+B}(x) = \min[\mu_A(x), \mu_B(x)].$$

The intersection of A and B in X, denoted by A*B is defined similarly by

$$\mu_{A*B}(x) = \min[\mu_A(x), \mu_B(x)].$$

In the sequel, the term "fuzzy variable" will replace the term membership grade of a variable in a set. Conventionally, we shall drop the * symbol, i.e., x*y will be written xy.

We can now define fuzzy forms, generated by $x_1, \ldots, x_n$, recursively as follows:

a) The numbers 0 and 1 are fuzzy forms.
b) A fuzzy variable x_i is a fuzzy form.
c) If A is a fuzzy form, then $\bar{A}$ is a fuzzy form.
d) If A and B are fuzzy forms, then A+B and AB are fuzzy forms.
e) The only fuzzy forms are those given by rules (a)-(d).

Since we are interested mainly in fuzzy algebra, we can drop the word "fuzzy" without any confusion. It should be noted that the grade membership $\mu(S)$ of a form S is uniquely determined through the following rules:

1. $\mu(S) = 0$ if $S = 0$;
2. $\mu(S) = 1$ if $S = 1$;
3. $\mu(S) = \mu(x_i)$ if $S = x_i$;
4. $\mu(S) = 1-\mu(A)$ if $S = \bar{A}$;
5. $\mu(S) = \min[\mu(A), \mu(B)]$ if $S=AB$; and
6. $\mu(S) = \mathrm{Max}[\mu(A), \mu(B)]$ if $S=A+B$.

Example 1: Let

$$f(x_1, x_2, x_3) = x_1(\bar{x}_2 + \bar{x}_1 x_3) + x_2(x_1 + \bar{x}_1 \bar{x}_3)$$

and let the grade memberships of x_1, x_2, and x_3 be $\mu(x_1)=0.4$, $\mu(x_2)=0.6$, and $\mu(x_3)=0.8$. Then

$$\mu[f(x_1,x_2,x_3)] = \mathrm{Max}\{\mu[x_1(\bar{x}_2+\bar{x}_1x_3)],\ \mu[x_2(x_1+\bar{x}_1\bar{x}_3)]\} =$$
$$= \mathrm{Max}\{\min[\mu(x_1), \mu(\bar{x}_2+\bar{x}_1x_3)], \min[\mu(x_2),\ \mu(x_1+\bar{x}_1\bar{x}_3)]\} =$$
$$= \mathrm{Max}\{\min[\mu(x_1), \mathrm{Max}(\mu(\bar{x}_2), \mu(\bar{x}_1x_3))],\ \min[\mu(x_2), \mathrm{Max}(\mu(x_1), \mu(\bar{x}_1\bar{x}_3))]\} =$$
$$= \mathrm{Max}\{\min[\mu(x_1), \mathrm{Max}(\mu(\bar{x}_2), \min(\mu(\bar{x}_1), \mu(x_3)))],\ \min[\mu(x_2), \mathrm{Max}(\mu(x_1), \min(\mu(\bar{x}_1), \mu(\bar{x}_3)))]\} =$$
$$= \mathrm{Max}\{\min[0.4, \mathrm{Max}(0.4, \min(0.6,\ 0.8))],\ \min[0.6, \mathrm{Max}(0.4, \min(0.6,\ 0.2))]\} =$$
$$= \mathrm{Max}\{\min[0.4, \mathrm{Max}(0.4,\ 0.6)],\ \min[0.6, \mathrm{Max}(0.4,\ 0.2)]\} =$$
$$= \mathrm{Max}[\min(0.4,\ 0.6),\ \min(0.6,\ 0.4)] = 0.4.$$

Evidently, among the infinite number of distinct assignments of grade membership to the variables, there are a finite number of binary assignments (binary assignments of 0 or 1 to every variable).

In two-valued logic every form can be expressed in disjunctive and conjunctive normal forms, due to the existence of the distributive laws, De-Morgan's laws, and the following definitions:

A literal is a variable x_i, or $\bar{x}_i$, the complement of x_i.

A clause is a disjunction of one or more literals.

A phrase is a conjunction of one or more literals.

A form S is said to be in disjunctive normal form if $S=P_1+P_2+\ldots+P_m, m \geq 1$ and every P_i, $1 \leq i \leq m$, is a phrase.

A form S is said to be in conjunctive normal form if $S=C_1C_2\ldots C_k$, $k \geq 1$ and every C_j, $1 \leq j \leq k$, is a clause.

Following [9] we can easily see that forms in fuzzy logic can be expressed in disjunctive and conjunctive normal forms, in a similar way to two-valued logic.

We will merge the concepts of fuzzy functions and fuzzy forms, to an extent, by representing the mapping by fuzzy forms. This is similar, again, to procedures in two-valued logic. Thus, the minimization of a fuzzy function (mapping) is really finding a minimal form for the function.

3. MAP REPRESENTATION OF FUZZY FORMS

In a disjunctive normal form, each phrase corresponds to a logic gate and each literal to an input line. The ratio between the cost of a logic gate and the cost of an input line will depend on the type of gates used in the realization. However, practically, the cost of an additional input line on an already existing gate, will be several times less than the cost of an additional logic gate. On this basis, the elimination of gates will be the primary objective of the minimization process, leading to the following definition of a minimal expression.

Definition: A disjunctive normal form is regarded as a minimal complexity form if there exists:

(1) no other equivalent form involving fewer number of phrases, and

(2) no other equivalent form involving the same number of phrases but a smaller total number of literals.

Definition: A phrase f_j subsumes another phrase f_ℓ iff f_j contains all the variables of f_ℓ, and thus $f_j \leq f_\ell$. A function(denoting also the corresponding phrase) f_k is said to be a fuzzy implicant of function f iff $f_k \leq f$. A fuzzy implicant f_j is said to be a fuzzy prime implicant (F.P.I.) if it subsumes no other fuzzy implicant of f, (i.e., $f_j \leq f_k \leq f \leftrightarrow k=j$).

It has been claimed that the fuzzy consensus as defined in [12], can be used to produce the sum of all fuzzy prime implicants of a fuzzy switching function via the following theorem.

Theorem 1 [12]: A sum of products expression $F=P_1+P_2+\ldots+P_r$ for the function $f(x_1,x_2,\ldots,x_n)$ is the sum of all the fuzzy prime implicants of $f(x_1,x_2,\ldots,x_n)$ iff:

1) No phrase includes any other phrase, $P_j \nleq P_i$ for any i and j, $i \neq j$, $i,j \in \{1,2,\ldots,r\}$.

2) The fuzzy consensus of any two phrases $P_i \psi P_j$ either does not exist ($P_i \psi P_j = 0$) or every phrase that belongs to the set describing $P_i \psi P_j$ is included in some other phrase from $\{P_k\}_{k=1}^{r}$.

Unfortunately the original definition of fuzzy consensus, when applied in the above theorem, will not yield the entire set of fuzzy prime implicants of $f(x_1,x_2,\ldots,x_n)$, but a set consisting of all essential fuzzy prime implicants plus some fuzzy prime implicants which are not essential.

Definition: Let R and Q be two phrases over the set of fuzzy variables $x_1,x_2,\ldots,x_n$. The fuzzy consensus of R and Q, written $R\psi Q$, is defined to be the set of phrases $\{R_iQ_i\}$, where $R=x_iR_i$ and $Q=\bar{x}_iQ_i$ (or $R=\bar{x}_iR_i$ and $Q=x_iQ_i$) and $x_i \in \{x_1,x_2,\ldots,x_n\}$, and when the phrase R_iQ_i includes the conjunction $x_j\bar{x}_j$ for at least one j, $j \in \{1,2,\ldots,n\}$. If the phrase R_iQ_i does not include $x_j\bar{x}_j$ for any j, $j \in \{1,2,\ldots,n\}$, then

$$R\psi Q=\{R_iQ_ix_j\bar{x}_j \mid j=1,2,\ldots,n\},\ x_i \in \{x_1,x_2,\ldots,x_n\}.$$

If none of the above occurs then we say that $R\psi Q = 0$. Theorem 1 is valid with the proof as given in [12] utilizing the above definition of fuzzy consensus.

The phrases added whenever $R_iQ_i \ngeq x_j\bar{x}_j$ are not essential [12] fuzzy prime implicants. This can be seen from the following proposition.

Proposition 1: Let $R=x_iR_i$ and $Q=\bar{x}_iQ_i$ (or $R=\bar{x}_iR_i$ and $Q=x_iQ_i$) and R_iQ_i does not include $x_j\bar{x}_j$ for any j, $j \in \{1,2,\ldots,n\}$. Then $R+Q+(R\psi Q)= =R+Q$.

Proposition 2: Let $f(x_1,x_2,\ldots,x_n)=\prod_k(x_k+\bar{x}_k+\sigma_k)$ where σ_k is an arbitrary function in $x_1,x_2,\ldots,x_n$ and $k \in \{1,2,\ldots,n\}$. Then $f(x_1,x_2,\ldots,x_n)=\prod_k(x_k+\bar{x}_k+\sigma_k)+\sum_j x_j\bar{x}_j\gamma_j$ where γ_j is also some arbitrary function in $x_1,x_2,\ldots,x_n$ and $j \in \{1,2,\ldots,n\}$.

Theorem 2 [12]: The maximum number of fuzzy implicants in a disjunctive normal form representation of a fuzzy function $f(x_1,x_2,\ldots,x_n)$ is 4^n+1.

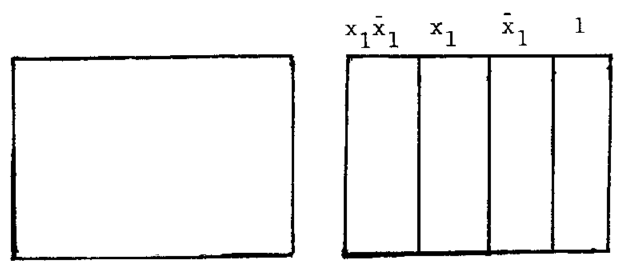
Thus excluding the implicant 0, which does not have any effect when appended to the function, the number is reduced to 4^n.

Based on these two theorems an algorithm to minimize $f(x_1,x_2,\ldots,x_n)$ is derived.

The power of the Karnaugh map [13] used in Boolean logic lies in its utilization of the ability of the human mind to perceive complex patterns in pictorial representations of data. The Karnaugh map may be regarded either as a pictorial form of a truth table, or as an extension of the Venn diagram.

The fuzzy map may be regarded as an extension of the Veitch diagram [14], which generated the basis for the Karnaugh map. It pictorially describes the set of all fuzzy implicants which represent the fuzzy function f. We interpret the universal set as the set of all 4^n combinations of values of 2n variables, divide this set into 4^n equal areas, and then place 1's in the areas corresponding to those combinations for which a fuzzy implicant appears in the description of the fuzzy function $f(x_1,x_2,\ldots,x_n)$.

We start with the universal set represented by a single area (Fig. 1a), and divide it to 4 areas, corresponding to input combinations of $1,\bar{x}_1,x_1$, and $x_1\bar{x}_1$ (Fig. 1b). We then divide it into 4 again, corresponding to $1,\bar{x}_2,x_2$, and $x_2\bar{x}_2$ (Fig. 1c).



Universal Set
(a)

(b)

	$x_1\bar{x}_1$	x_1	$\bar{x}_1$	1
$x_2\bar{x}_2$				
x_2				
$\bar{x}_2$				
1				

(c)

FIGURE 1. DEVELOPMENT OF FUZZY MAPS

With this notation, the interpretation of OR as the union of sets and AND as the intersection of sets make it particularly simple to determine in which area we should place 1's (Fig. 2).

	$x_1\bar{x}_1$	x_1	$\bar{x}_1$	1
$x_2\bar{x}_2$				
x_2				1
$\bar{x}_2$				
1		1		

x_1+x_2

	$x_1\bar{x}_1$	x_1	$\bar{x}_1$	1
$x_2\bar{x}_2$				
x_2		1		
$\bar{x}_2$				
1				

x_1x_2

FIGURE 2. OR and AND Representations (n=2)

Using the rules of fuzzy algebra, the fuzzy function f of n variables can be expressed in the form

$$f(x_1,x_2,\ldots,x_n) = \sum_{k\varepsilon\Gamma} (\prod_{j\varepsilon S(k)} x_j)$$

where Γ and $S(k)$ are index sets and $\bar{x}_j=x_{j+n}$. To illustrate the meaning of the above expression, let

$$f(x_1,x_2,x_3)=x_1\bar{x}_1x_2+x_1x_2\bar{x}_3+x_1x_3\bar{x}_3+x_1x_2x_3=$$

$$=\prod_{j\varepsilon S(1)} x_j + \prod_{j\varepsilon S(2)} x_j + \prod_{j\varepsilon S(3)} x_j + \prod_{j\varepsilon S(4)} x_j = \sum_{k\varepsilon\Gamma} (\prod_{j\varepsilon S(k)} x_j)$$

where $n=3$, $\Gamma=\{1,2,3,4\}$ and $S(1)=\{1,2,4\}$, $S(2)=\{1,2,6\}$, $S(3)=\{1,3,6\}$, $S(4)=\{1,2,3\}$.

Thus, every implicant besides the number 0, can be represented uniquely as a binary string of length 2n, where 1 in the ith place represents the variable x_i in the ith place and 0 in the ith place means that the implicant is vacuous in x_i.

Because of the commutative law we can change the order of the literals in each phrase without affecting the function. Thus the phrases will be represented as conjunctions of the form

$$x_1x_{n+1}x_2x_{n+2}\ldots x_jx_{j+n}\ldots x_nx_{2n}$$

and not as

$$x_1x_2\ldots x_j\ldots x_nx_{n+1}\ldots x_{j+n}\ldots x_{2n-1}x_{2n}.$$

We can replace the column's and row's heading with decimal numbers representing the binary headings. Thus, there are four combinations for each variable x_i, $1\le i\le n$, to be presented in the heading of a row or a column:

1) The heading is vacuous in x_i. The pair $x_i\bar{x}_i$ is denoted by 00 and represented by 0.
2) The heading includes $\bar{x}_i$ but not x_i. The pair $x_i\bar{x}_i$ is denoted by 01 and represented by 1.
3) The heading includes x_i but not $\bar{x}_i$. The pair $x_i\bar{x}_i$ is denoted by 10 and represented by 2.
4) The heading includes x_i and $\bar{x}_i$. The pair $x_i\bar{x}_i$ is denoted by 11 and represented by 3.

Thus, every square on the fuzzy map is represented uniquely as a decimal number representing the binary sequence of length 2n. The following illustration is given for a fuzzy map where n=2, where the order is $x_1\bar{x}_1x_2\bar{x}_2(x_1x_3x_2x_4)$.

	$x_1\bar{x}_1$	x_1	$\bar{x}_1$	1
$x_2\bar{x}_2$	15	11	7	3
x_2	14	10	6	2
$\bar{x}_2$	13	9	5	1
1	12	8	4	0

FIGURE 3. DECIMAL REPRESENTATION OF FUZZY IMPLICANTS, n=2.

The fuzzy map has a row dimension of 4^s and a column dimension of 4^{n-s} squares, where s and n-s are the number of row and column variables. respectively.

For simplicity the headings of the rows and columns will be expressed as decimal numbers. Thus, Fig. 3 will be represented as

x_2 \ x_1	3	2	1	0
3	15	11	7	3
2	14	10	6	2
1	13	9	5	1
0	12	8	4	0

The fuzzy map can be considered as being a graphical representation of a table of combinations and hence a graphical representation of a fuzzy function. Figure 4 illustrates the fuzzy map of the following three variables function:

$$f(x_1,x_2,x_3)=x_1\bar{x}_1x_2x_3+x_1x_2\bar{x}_3+\bar{x}_2x_3+x_1\bar{x}_1\bar{x}_2\bar{x}_3.$$

Using our notation

$$f(x_1,x_2,x_3)=x_1x_2x_3x_4+x_1x_2x_6+x_5x_3+x_1x_4x_5x_6=$$

$$=x_1x_4x_2x_3+x_1x_2x_6+x_5x_3+x_1x_4x_5x_6=$$

$=\Sigma(111010,101001,000110,110101)=$
$=\Sigma(58,41,6,53)=\Sigma(6,41,53,58)$.

$x_3 \backslash x_1x_2$	33	32	31	23	13	03	30	22	21	12	11	02	01	20	10	00
3																
2		1											1			
1			1					1								
0																

FIGURE 4. THE FUZZY MAP OF $f(x_1,x_2,x_3)=\Sigma(6,41,53,58)$.

4. FUZZY PRIME IMPLICANTS AND FUZZY MAPS

The significance of the fuzzy map lies in the fact that it is possible to determine the implicants of a function from the patterns of 1's appearing on the map.

Define a square of the map with a 1 entry as being a 1-square. The simplification procedure used on the fuzzy map is an exact translation of that used in fuzzy consensus theorem. However, this simplification procedure is quite different from the one used on a Karnaugh map. In Boolean logic, two minterms can be "combined" by means of the theorem $xy+x\bar{y}=x$ if their corresponding binary representations differ in exactly one bit. Namely, two minterms combine only if the corresponding points on an n-cube map are distance one apart, and thus, they are adjacent squares on a Karnaugh map. On the fuzzy map, however, the simplification is performed through subsuming operation, $x+xy=x$, and the fuzzy consensus result,

$$\alpha x_i\bar{x}_i\beta + \bar{\alpha}x_i\bar{x}_i\beta = x_i\bar{x}_i\beta$$

where $x_i \varepsilon \{x_1,x_2,\ldots,x_n\}$ and α and β are strings of literals from the set $\{x_j\}_{j=1}^{n}$. The technique for minimization is straightforward.

To restate the basic principles involved in the derivation of a minimal complexity expression we state the following two steps:

1) Choose as few groupings as possible.
2) Choose each grouping as large as possible.

Step (1) ensures that the number of first-level gates is reduced as much as possible, and step (2) ensures that each of these gates has a minimal number of input lines.

Theorem 3: An expression for a fuzzy function can be obtained by summing a set of phrases represented by rectangular groupings on a fuzzy map such that each subsumed 1-square is contained in at least one of the groupings.

Proof: By the fuzzy consensus operation

$$\alpha x_j\bar{x}_j\beta + \bar{\alpha}x_j\bar{x}_j\beta = x_j\bar{x}_j\beta$$

where $x_j \varepsilon\{x_1,x_2,\ldots,x_n\}$, α and β are strings of literals from the set $\{x_i\}_{i=1}^{n}$. Thus, each rectangular grouping of 1-squares of dimension $2^k \times s^q$ where $k\neq 0$ and $q\neq 0$, represents a phrase which includes a conjunction of the form $x_j\bar{x}_j$ for at least one j, $1\leq j\leq n$. Clearly such a rectangular grouping is performed on a set of columns or rows which have the number 3 in the same digit of their headings.

All other phrases are represented by rectangular groupings of dimension 1 x 1, namely, k=q=0.

In case some rectangular groupings are subsumed, the subsuming groupings are deleted from the map. The remaining groupings represent phrases which imply the fuzzy function and thus, it is sufficient to select the set of groupings such that every subsumed 1-square is included in at least one of these groupings. Hence, the function is properly described by an expression from the fuzzy map. |X|

Definition 2: The right-bottom square on the fuzzy map having 0's as row and column headings, is called the dominant square. This square is represented by the binary 2n-tuple of 0's.

Definition 3: By a fuzzy column-extension (FCE) we mean that any original square on the map, in which a 1 is placed, forms an extension of 1's throughout a set of squares G, where G is the set of squares which possesses a row-heading that subsumes the original square row-heading, and that appears in the same column where the original square appears.

Example 2: Let $f(x_1,x_2) = \bar{x}_1+x_1\bar{x}_1x_2$

	$x_1\bar{x}_1$	x_1	$\bar{x}_1$	1
$x_2\bar{x}_2$				
x_2	1			
$\bar{x}_2$				
1			1	

The fuzzy column-extension of $f(x_1,x_2)$ is given by

	$x_1\bar{x}_1$	x_1	$\bar{x}_1$	1
$x_2\bar{x}_2$	1		1	
x_2	1		1	
$\bar{x}_2$			1	
1			1	

Similarly, a fuzzy row-extension (FRE) can be defined. The reverse operations are called fuzzy column-condensation (FCC) and fuzzy row-condensation (FRC), respectively.

Theorem 4: The set of all needed F.P.I.'s of a fuzzy function can be obtained from the fuzzy map.

Proof: The proof is by construction. Let

$f(x_1,x_2,\dots,x_n)$ be a fuzzy function, over n variables, represented in disjunctive normal form.

Given an n-variable fuzzy map, if there is a 1-entry on the dominant square, the function is identically equal to 1. This 1 is the only F.P.I. of the function.

If the dominant square does not have a 1-entry, perform all possible FCE's. Determine all rectangular groupings of 1-squares with maximal dimensions ($2^k \times 2^q = 2^{n-1}$), using the fuzzy consensus theorem,

$$\alpha x_j \bar{x}_j \beta + \bar{\alpha} x_j \bar{x}_j \beta = x_j \bar{x}_j \beta.$$

Next, form all rectangular groupings of 1-squares with dimensions $2^k \times 2^q = 2^{n-j}$, for j=2, such that no grouping is totally within a single previously formed grouping.

Repeat this process for $j=3,4,\dots,n$. It should be pointed out that since in Boolean logic all squares in the groupings on the Karnaugh map are 1-squares, they must describe prime implicants. However, on the fuzzy map one grouping may subsume another grouping, and thus, not all the groupings are F.P.I.'s. In order to find the set of all F.P.I.'s perform all possible FRC's between complete rectangular groupings.

Consider the implicant in row i and column j, where $1 \le i \le 4^s$, $1 \le j \le 4^{n-s}$. Every implicant in row i and in column k, such that column k possesses a heading that subsumes the heading of column j, will be deleted by the occurrence of 1 in row i and column j. This is due to the fact that entry ij on the map is subsumed by entry ik and is therefore deleted. However, this deletion can take place only on the basis of rectangular groupings. Namely, a complete grouping is deleted from the map if and only if it subsumes another complete grouping and not only part of it.

The next step is to perform all possible FCC's. This operation is also performed on complete groupings only. Practically, the above procedure is just an application of the fuzzy consensus theorem. Therefore, the set of all phrases formed is the set of all needed F.P.I.'s of the function. |X|

Definition 4: A fuzzy implicant is called a fundamental phrase if it has either one of the following two attributes:

(i) for every literal x_j in the phrase, the complement of x_j does not appear in the phrase.

(ii) the phrase contains as many variables (complemented, uncomplemented, or both) as the function.

Usually it is the minimal complexity form (minimal sum) rather than the sum of all F.P.I.'s of a fuzzy function that is desired. It has been shown [12] that this minimal complexity form must consist of a sum of phrases representing F.P.I.'s. A general approach for determining a minimal complexity form can now be stated using Theorem 4 and the fact that in the minimal representation each fundamental phrase of the function must be covered.

Definition 5: A F.P.I. which is subsumed by a fundamental phrase which does not subsume any other F.P.I. is called an essential F.P.I. (E.F.P.I.).

The fundamental phrase which is subsumed by an E.F.P.I. is represented on the map by an essential 1-square.

Theorem 5: Every E.F.P.I. of a fuzzy function must appear in all the irredundant disjunctive normal forms of the functions and, hence, in the minimal complexity form.

Proof: Associated with an E.F.P.I. is a 1-square that is associated with no other F.P.I. |X|

A general approach for determining a minimal complexity form can now be stated.

The Algorithm:

Input: The set of phrases representing $f(x_1,x_2,\dots,x_n)$.

Output: Minimal complexity form of $f(x_1,x_2,\dots,x_n)$.

Step 1: Expand $f(x_1,x_2,\dots,x_n)$ into sum of fundamental phrases.

Step 2: Draw a fuzzy map of the expanded function.

Step 3: If there is a 1-entry on the dominant square, then f=1 and go to step 8. Otherwise go to step 4.

Step 4: Perform all possible FCE's.

Step 5: Determine all possible groupings with respect to Theorem 4.

Step 6: Perform all possible FRC's. At this point, a grouping will be deleted if it subsumes a set of 1-squares, which do not necessarily have to be in a single grouping as required in Theorem 4.

Step 7: Perform all possible FCC's under the same conditions in Step 6.

Step 8: Halt. The set of the remaining groupings represent the phrases of the minimal complexity form.

Note: It should be noted that there is a difference in performing all possible FRC's and FCC's as described in Steps 6 and 7 of the algorithm and as described in Theorem 4. The difference is clear once it is noted that Theorem 4 describes the construction of the set of all F.P.I.'s and the above algorithm obtains the minimal complexity form of the function, and thus applies the coverage technique to the set obtained by Theorem 4.

Example 3: Let $f(x_1,x_2)=x_1\bar{x}_2+\bar{x}_1x_2+x_1\bar{x}_1+x_2\bar{x}_2$. The phrases $x_1\bar{x}_1$ and $x_2\bar{x}_2$ can be expressed as sum of fundamental phrases, and thus

$$f(x_1,x_2)=x_1\bar{x}_2+\bar{x}_1x_2+x_1\bar{x}_1x_2+x_1\bar{x}_1\bar{x}_2+x_1x_2\bar{x}_2+\bar{x}_1x_2\bar{x}_2.$$

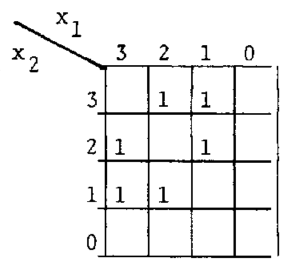


FIGURE 5. FUZZY MAP FOR EXAMPLE 3.

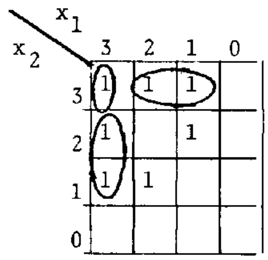


FIGURE 6. EXTENDED FUZZY MAP FOR EXAMPLE 3.

The circled terms are those that can now be deleted by FRC and FCC. The rest is represented by

$$f_{min}(x_1,x_2)=\bar{x}_1x_2+x_1\bar{x}_2.$$

5. CONCLUSION

In this paper we have discussed a new minimization technique for fuzzy forms. The method, based on the practical use of the concept of fuzzy consensus and the fuzzy consensus theorem, is implemented through the use of the fuzzy map. This fuzzy map is a natural extension of the Karnaugh map.

It is hoped that this paper will contribute to many other extensions, from Boolean algebra to fuzzy algebra, and to applications with non-binary minimization methods in general.

REFERENCES

[1] Zadeh, L.A., "Fuzzy sets", Information and Control, Vol.8, pp.338-353, 1965.

[2] _____, "Fuzzy algorithms", Information and Control, Vol.12, pp.94-102, 1968.

[3] _____, Fuzzy sets and systems", 1965 Proc. Symp. on Systems Theory, Polytechnic Institute of Brooklyn, Brooklyn, N.Y. 1965.

[4] _____, "Quantitative fuzzy semantics", Memo. No. ERL-M281, Electronics Research Laboratory, Univ. of California, Berkeley, Aug. 1970.

[5] _____, "Similarity relations and fuzzy ordering", Memo. No. ERL-M277, Electronics Research Lab., University of Calif., Berkeley, July 1970.

[6] _____, "Toward a theory of fuzzy systems", Report No. ERL-69-2, Electronics Research Laboratory, University of Calif. Berkeley, June 1969.

[7] Marinos, P.N., "Fuzzy logic and its application to switching systems", IEEE Trans. on Computers, Vol. C-18, No.4, pp. 343-348, April 1969.

[8] Goguen, J.A. , "L-fuzzy sets, J. Math. Anal. Appl., Vol.18, pp.145-174, 1967.

[9] Lee, R.C.T. and Chang, C.L., "Some properties of fuzzy logic", Information and Control, Vol.19, pp.417-431, 1971.

[10] Siy, P. and Chen, C.S., "Minimization of fuzzy functions", IEEE Trans. on Comp., Vol. C-21, No.1, pp.100-102, Jan. 1972.

[11] Chang, C.L., "Fuzzy algebras, Fuzzy Functions and their application to function approximation", Div. of Comp. Research and Techn., National Inst. of Health, Bethesda, Md., 1971.

[12] Kandel, A., "On minimization of fuzzy functions", IEEE Trans. on Computers, Vol.C-22, No.9, pp.826-832, Sept. 1973.

[13] Karnaugh, M., "The map method for synthesis of combinational logic circuits", Trans. AIEE, Vol.72, Pt.I, pp.593-598, 1953.

[14] Veitch, E.W., "A chart method for simplifying truth functions", Proc. ACM, Pittsburg, Pa., pp.127-133, May 2-3, 1952.