Sophistication in Simplicity: Unravelling the Secrets of a Simple Recursion

Ever since Fibonacci's time, if not before it, recursions have intrigued people. The sequence of Fibonacci's numbers 1, 1, 2, 3, 5, 8, 13,... is just one solution of the following simple linear recursion:

$$x_{n+1} = x_n + x_{n-1}$$

with initial values $x_0 = 1$, $x_1 = 1$. We discuss the following slightly different though equally simple-looking equation:

$$x_{n+1} = |x_n - x_{n-1}|.$$

This absolute value version holds many surprises. It generates two types of sequences (its solutions): Each pair of initial values can either generate a sequence of period 3 or a sequence that converges to zero. The set of all initial values that lead to period-3 sequences is countable and dense (they contain all rational numbers). Irrational initial values in the complement of this set generate solutions that converge to zero. Hence there is extreme sensitivity to initial values. For solutions that converge to zero, the sequences of ratios of consecutive terms are surprisingly complex. These sequences can be periodic with all possible periods – except 3. Fibonacci numbers determine the periodic solutions. Aperiodic ratio sequences also exist and some of them are chaotic. By contrast, in the Fibonacci recursion the ratio sequences all converge to the number $(1+\sqrt{5})/2$, namely, the "golden ratio". Our discussion will be informal and use only elementary ideas from the theory of nonlinear difference equations.