Probabilistic Operations Research Models

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Outline

1. Operations Research Models
2. Axioms of Probability
   - Definition
   - Interpretations of Probability
   - Probability Rules
   - Example: Blood Types
   - Random Variables
3. Markov Chains
   - Markov Property
   - Blood Types II
4. Simulation
   - The Nature of Simulation Modeling
   - An Example of a Discrete-Event Simulation
Operations Research Models

Deterministic OR
- Continuous Variables
  - Linear Functions
  - Nonlinear Functions
- Discrete Variables
  - Linear Functions
  - Nonlinear Functions

Probabilistic OR
- Discrete Time
  - Discrete Space
  - Continuous Space
- Continuous Time
  - Discrete Space
  - Continuous Space
Markov Chains

Operations Research Models
- Axioms of Probability
- Markov Chains
- Simulation

Markov Chains

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Discrete vs. Continuous Models

Discrete
- means “space between”
- countable, e.g., integers, binary numbers
- attributes, variables, time, space

Continuous
- uncountable, e.g., real numbers, intervals of real numbers
- attributes, variables, time, space
Linear vs. Nonlinear Models

**Linear**
- additivity - every function is the sum of the individual contributions of activities
- proportionality - the contribution of an activity to a function is proportional to the level of the activity.


**Nonlinear**
Additivity or proportionality (or both) are violated
Probabilistic vs. Deterministic Models

**Probabilistic**
Probability is used to model behaviors that are uncertain or unknown

**Deterministic**
Randomness is not considered; systems are assumed to be totally determined. Sensitivity analysis can help incorporate uncertainty into models.
What is Probability?

- Simply speaking **probabilities** are numbers between 0 and 1 that reflect the chances of “something” happening.
- Synonymous with **chance, likelihood, odds**.
- Has different interpretations.
Cards and Dice

- What is the probability
  - that I draw a black card?
  - that I roll a 7?
  - that I roll doubles?

  These are called events

- Are you sure? What assumptions did you make?
- Were they correct?
- How can I correct these assumptions?
- How can I determine a more accurate probability?
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Interpretations of Probability

Classical/Analytical

- Theoretically determined probabilities
  - Probability of rolling a 3 on a fair (normally marked) die: 1/6
  - Probability of drawing a black card in a standard deck: 1/2

- Advantages
  - Probabilities are accurate
  - No experimentation required
  - Objective

- Disadvantage: only possible to compute under the best of circumstances (e.g., we know that the die is fair)
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Relative Frequency/Empirical

- Observed proportion of successful events
  - 10 cards selected, 6 of them black $\rightarrow$ probability of selecting a black card is 0.6
  - Minnesota has had snowfall of at least 60 inches in 95 of the last 100 years $\rightarrow$ probability of having at least 60 inches of snow this year is 0.95.

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  - can collect empirical data to estimate probabilities when they can’t be determined analytically
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- Disadvantage: situation must be replicable
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Personal/Subjective

- “What do you think are the odds?”
  - What’s the chance of Florida repeating as NCAA basketball champion?
  - What’s the probability that a nuclear bomb will be deployed in your lifetime?

- Relies on expert information (definition of “expert” is fluid).

- Advantage:
  - always applicable - *everybody* has an opinion
  - useful in risk analysis

- Disadvantage: difficult (sometimes impossible?) to determine accuracy.
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### Probability Rules

1. Probability is always between 0 and 1.
   - probability of an event $E$ is written $P(E)$

2. If event $E$ cannot occur then $P(E) = 0$.
   - $E = \text{“Jill will grow to be 6 feet tall”}$, $P(E) = 0$

3. If an event is certain, then $P(E) = 1$.
   - $E = \text{“Class will end before midnight.”}$, $P(E) = 1$

4. The sum of the probabilities of all possible outcomes is 1.
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5. For two events $A$ and $B$:

- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- $P(A \text{ and } B) = P(A) + P(B) - P(A \text{ or } B)$
6. For two events $A$ and $B$

- $P(B \text{ occurs given that } A \text{ occurs})$
  
  $= P(B|A) = P(A \text{ and } B)/P(A)$
  
  $\Rightarrow P(A \text{ and } B) = P(B|A)P(A)$

- If $P(B|A) = P(B)$ and $P(A|B) = P(A)$ then $A$ and $B$ are said to be independent.
  
  That is, knowing that one event will occur doesn’t give us any information about the other.
  
- If $A$ and $B$ are independent, then $P(A \text{ and } B) = P(A)P(B)$. 

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- If $A$ and $B$ are independent, then $P(A \text{ and } B) = P(A)P(B)$.
Definition

Mutually exclusive events are non-overlapping—that is, they cannot happen at the same time.

- $A =$ randomly chosen person is male
  $B =$ randomly chosen person if female
  These events are mutually exclusive.

- $A =$ randomly chosen person has blue eyes
  $B =$ randomly chosen person has brown hair
  These events are not mutually exclusive.
Bayes’ Rule

- Suppose $A_1, A_2, \ldots, A_k$ are mutually exclusive events so that
  - $P(A_i) > 0, \ i = 1, \ldots, k$
  - $P(A_1) + P(A_2) + \cdots + P(A_k) = 1$ (i.e., they are exhaustive).
- Let $B$ be another event with $P(B) > 0$. Then

$$P(A_i | B) = \frac{P(A_i \text{ and } B)}{P(B)} = \frac{P(B | A_i) P(A_i)}{\sum_{i=1}^{k} P(B | A_i) P(A_i)}$$
Bayes’ Rule

\[ P(A_i | B) = \frac{P(A_i \text{ and } B)}{P(B)} = \frac{P(B | A_i)P(A_i)}{\sum_{i=1}^{k} P(B | A_i)P(A_i)} \]
Bayes’ Rule: Example

Suppose that 5% of all athletes use performance-enhancing drugs. Suppose further that for the drug test in use, the false positive rate is 3% and the false negative rate is 7%.

An athlete is tested, and her results are positive. What is the probability that she uses drugs?

- 5%?
- 97%
- something else?
Bayes’ Rule: Example

- We want to find \( P(\text{drug use}|\text{positive test}) \). What we have are
  - \( P(\text{positive test}|\text{no drug use}) = P(\text{false positive}) \)
  - \( P(\text{negative test}|\text{drug use}) = P(\text{false negative}) \)

- Bayes’ Rule says that

\[
P(\text{drugs}|\text{positive}) = \frac{P(\text{positive}|\text{drugs})P(\text{drugs})}{P(\text{positive}|\text{drugs})P(\text{drugs}) + P(\text{positive}|\text{no drugs})P(\text{no drugs})}
\]
Bayes’ Rule: Example

\[ P(\text{drugs}|\text{positive}) = \frac{P(\text{positive}|\text{drugs})P(\text{drugs})}{P(\text{positive}|\text{drugs})P(\text{drugs}) + P(\text{positive}|\text{no drugs})P(\text{no drugs})} \]

\[ = \frac{(0.93)(0.05)}{(0.93)(0.05) + (0.03)(0.95)} \]

\[ = 0.62 \]

When we first met the athlete, we thought the chance of her being a drug user was 5%. We were able to use Bayes’ Rule, along with the test results, to update our “expert” information.
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Example: Blood Types

Suppose the distribution of blood types (and genotypes) in a population is as follows:

<table>
<thead>
<tr>
<th>Blood Type</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>40%</td>
</tr>
<tr>
<td>AA</td>
<td>20%</td>
</tr>
<tr>
<td>AO</td>
<td>20%</td>
</tr>
<tr>
<td>B</td>
<td>12%</td>
</tr>
<tr>
<td>BB</td>
<td>6%</td>
</tr>
<tr>
<td>BO</td>
<td>6%</td>
</tr>
<tr>
<td>AB</td>
<td>5%</td>
</tr>
<tr>
<td>AB</td>
<td>5%</td>
</tr>
<tr>
<td>O</td>
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What is the probability of producing a child with type O blood if your genotype is
Example: Blood Types

What is the probability of producing a child with type O blood if your genotype is AO?

\[ P(OO_{\text{child}}) = P(\text{mate with AO or BO, you contribute O, mate contributes O}) + P(\text{mate with OO, you contribute O}) \]

\[ = (0.26)(0.5)(0.5) + (0.43)(0.5) \]

\[ = 0.28 \]
What is the probability of producing a child with type O blood if your genotype is OO?

\[
P(OO_{child}) = P(\text{mate with AO or BO, and mate contributes O}) + P(\text{mate with OO})
\]

\[
= (0.26)(0.5) + 0.43
\]

\[
= 0.56
\]
What is the probability of producing a child with type O blood if your genotype is BB?

$P(OO_{\text{child}})$ is zero! (impossible event)
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Random Variables

- Sometimes we’re more interested in some function of an outcome, rather than the outcome itself.
  - If I flip a coin 5 times, how many are heads? This is a function of the outcomes on five separate flips.
  - How long will it be before Jill and Paul stop talking?

- This function of the outcome is called a random variable.

- Observed value is determined by chance.
Random Variables

- Need to know what values are possible— discrete or continuous?
- Need to know what values are probable— how likely are each of these values?
- Probabilities defined by the probability density function (or probability mass function)
pmf’s and pdf’s

Probability Mass Function

For a discrete random variable $X$, $f(x) = P(X = x)$ for each possible value of $x$.

Probability Density Function

- $f(x) \geq 0$ for all $x$
- $P(a \leq X \leq b) =$ area under $f(x)$ between $a$ and $b$
- Total area under $f$ is 1
- $P(X = x) = 0$
Probability Density Functions

- Uniform
- Normal
- Triangular
- Exponential
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$$P(X_{t+1} = i_{t+1} | X_0 = i_0, X_1 = i_1, \ldots, X_t = i_t) = P(X_{t+1} = i_{t+1} | X_t = i_t)$$

Translation: “The probabilities that a stochastic process moves to a new state depends only on the current state; the probabilities are independent of all past events.”
Definition

A stochastic process with state variable $X_t$ is said to possess the **Markov Property** if

$$P(X_{t+1} = i_{t+1} | X_0 = i_0, X_1 = i_1, \ldots, X_t = i_t) = P(X_{t+1} = i_{t+1} | X_t = i_t)$$

Translation: “The probabilities that a stochastic process moves to a new state depends only on the current state; the probabilities are independent of all past events.”
Illustrations of the Markov Property

Let $X_t =$ current position on the board after $t$ rolls
Let $X_t =$ location of a unit of ingested lead at time $t$

Illustrations of the Markov Property


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Illustrations of the Markov Property

Let $X_t =$ location of a unit of ingested lead at time $t$


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Illustrations of the Markov Property

Let $X_t =$ political party of the U.S. Representative from Virginia’s 3rd district after election $t$
Outline

1. Operations Research Models
2. Axioms of Probability
   - Definition
   - Interpretations of Probability
   - Probability Rules
   - Example: Blood Types
   - Random Variables
3. Markov Chains
   - Markov Property
   - Blood Types II
4. Simulation
   - The Nature of Simulation Modeling
   - An Example of a Discrete-Event Simulation
## Transition Matrix

<table>
<thead>
<tr>
<th>Parent</th>
<th>AA</th>
<th>AB</th>
<th>AO</th>
<th>BB</th>
<th>BO</th>
<th>OO</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA</td>
<td>0.33</td>
<td>0.12</td>
<td>0.55</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>AB</td>
<td>0.16</td>
<td>0.22</td>
<td>0.28</td>
<td>0.06</td>
<td>0.28</td>
<td>0</td>
</tr>
<tr>
<td>AO</td>
<td>0.16</td>
<td>0.06</td>
<td>0.44</td>
<td>0</td>
<td>0.06</td>
<td>0.28</td>
</tr>
<tr>
<td>BB</td>
<td>0</td>
<td>0.33</td>
<td>0</td>
<td>0.12</td>
<td>0.55</td>
<td>0</td>
</tr>
<tr>
<td>BO</td>
<td>0</td>
<td>0.16</td>
<td>0.16</td>
<td>0.06</td>
<td>0.34</td>
<td>0.28</td>
</tr>
<tr>
<td>OO</td>
<td>0</td>
<td>0</td>
<td>0.33</td>
<td>0</td>
<td>0.12</td>
<td>0.55</td>
</tr>
</tbody>
</table>
Let $X_t$ = the blood type of a person in generation $t$
Blood Types

Let $X_t$ = the blood type of a person in generation $t$
Theorem

Let $P$ be the transition matrix of a Markov Chain. Then $P(X_t = j | X_0 = i)$ is the $ij$th entry of $P^t$.

Corollary

The probability that a grandchild has genotype BB given that the grandparent has genotype AA is given in the appropriate entry of $P^2$ (in our model).
Steady-State Behavior

Theorem

If all states are accessible from one another, then

$$\lim_{t \to \infty} P(X_t = j | X_0 = i) = \pi_j$$

where $\pi_j$ is the $j^{th}$ element of the vector $\pi$ such that $\pi = \pi P$ and $\sum_j \pi_j = 1$

This theorem gives us a means to calculate the steady state distribution of genotypes. The quantity $\pi_j$ represents the probability that, after a long time, a descendant has genotype $j$. It also represents the proportion of descendants that have genotype $j$. 

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A Flowchart for Simulation Modeling

State objective and design study

Collect Data

Construct/Program a Model

Model is Valid?

Yes

Design Experiments

Analyze Output Data

No

## Probabilistic Modeling: Simulation vs. Analysis

### Analysis
- **Advantage:** Produces exact values for the characteristics of a model given varied input. These values can easily be compared for determining optimal input values.

- **Disadvantage:** Often requires strict assumptions about the nature of the model for any hope of solving for exact values.

### Simulation
- **Advantage:** Flexible in terms of assumptions required for model.

- **Disadvantage:** Produces estimates of characteristics of a model; simulations need to be run for a wide variety of inputs and for many replications in order to derive reasonable estimates.
Types of Simulation

**Definition**

A *discrete-event simulation* is a continuous-time, discrete-space simulation.

**Definition**

A *Monte Carlo simulation* is a static simulation.
Pitfalls of Simulation

- Failure to have a well-defined set of objectives at the beginning of the simulation study
- Inappropriate level of model detail
- Failure to collect good system data
- Obliviously using simulation software whose complex macro statements may not be well documented and may not implement the desired modeling logic

- Belief that easy-to-use simulation packages require a lower level of technical competence
- Failure to account correctly for sources of randomness in the actual system
- Using arbitrary distributions as input
- Making a single replication of a particular system design
- Comparing alternative system designs on the basis of one replication
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