LEAST WASSERSTEIN DISTANCE BETWEEN DISJOINT SHAPES WITH PERIMETER REGULARIZATION

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Abstract. We prove the existence of global minimizers to the double minimization problem

$$\inf \Big\{ P(E) + \lambda W_p(\mathcal{L}^n \llcorner E, \mathcal{L}^n \llcorner F) \colon |E \cap F| = 0,\ |E| = |F| = 1 \Big\},$$

where $P(E)$ denotes the perimeter of the set E, W_p is the p-Wasserstein distance between Borel probability measures, and $\lambda > 0$ is arbitrary. The result holds in all space dimensions, for all $p \in [1, \infty)$, and for all positive λ. This answers a question of Buttazzo, Carlier, and Laborde.

1. Introduction

Comparing equal volume *shapes*, i.e., Lebesgue measurable sets E and F in $\mathbb{R}^n$ of equal volume, is a ubiquitous task in numerous applications. From a mathematical standpoint, by identifying shapes with probability measures via their normalized characteristic functions, optimal transportation theory provides natural choices of metrics, the p-Wasserstein distances, which metrize the weak convergence of probability measures on compact spaces [San15]. Indeed, length-minimizing Wasserstein geodesics between equal volume sets, known as displacement interpolants, offer a (length minimizing) path joining the shapes being compared [McC97].

In this note we investigate the role of perimeter regularization in variational problems involving the Wasserstein distance between equal volume sets. As we subsequently discuss, examples of this type of problem arise in different applications. Our principal goal in this paper is to show that such perimeter-regularized variational problems, even when posed on all of space, *do not suffer a loss of compactness of minimizing sequences*. In order to focus on the technical essence in the simplest possible setting while capturing the main difficulties, we consider the following problem:

$$\inf \Big\{ P(E) + \lambda W_p(\mathcal{L}^n \llcorner E, \mathcal{L}^n \llcorner F) \colon E, F \subset \mathbb{R}^n,\ |E \cap F| = 0,\ |E| = |F| = 1 \Big\}. \tag{1.1}$$

Here $P(E)$ denotes the perimeter of $E \subset \mathbb{R}^n$, W_p denotes the p-Wasserstein distance on the space of probability measures, and $\lambda > 0$ is a constant. The parameter p belongs to the interval $[1, \infty)$, and λ represents the strength of the Wasserstein term relative to perimeter.

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The problem (1.1) was recently analyzed by Buttazzo, Carlier and Laborde in [BCL20], in addition to more general minimization problems involving the minimal Wasserstein distance between a measure μ and measures singular with respect to μ. In the context of (1.1), the authors in [BCL20] show, for any $\lambda > 0$, the existence of minimizers when admissible sets E and F are required to be subsets of a bounded domain Ω. In two dimensions, they prove the existence of minimizers for the problem (1.1) on all of $\mathbb{R}^2$, and conjectured that it should hold in all dimensions. This whole-space result was extended by Xia and Zhou [XZ21] to higher dimensions but under the additional assumptions that λ is sufficiently small and that $p < n/(n-2)$. In our result we lift all these restrictions and obtain that minimizers exist in any dimension *and* for all values of $\lambda > 0$ and $p \in [1, \infty)$, thereby completely answering the conjecture of Buttazzo, Carlier and Laborde. Precisely, we prove the following theorem:

Theorem 1.1 (Existence)**.** *For any $\lambda > 0$ and $p \in [1, \infty)$, there exists a minimizing pair (E, F) to the problem (1.1).*

The proof of Theorem 1.1 is based on tools developed in the context of constrained geometric variational problems on all of space for which symmetrization principles cannot rule out loss of volume at infinity for a minimizing sequence. First, for a minimizing sequence $\{E_m, F_m\}$, the nucleation lemma of Almgren [Alm76, VI.13] yields a finite number of bounded "chunks" which contain most of the volume. Then, classical density arguments for constrained perimeter minimizers allow one to argue that the minimizing sequence is essentially confined to finitely many (potentially diverging) balls on which there is no volume loss, at which point lower-semicontinuity of the energy yields the existence of a minimizing pair.

Nonlocal isoperimetric problems are well-studied and consist of minimizing the perimeter functional with some additional nonlocal term that precludes coalescence of sets. The problem (1.1) has several interesting mathematical features and exhibits both similarities and differences to other nonlocal isoperimetric models. The behavior of (1.1) is driven by the competition between the perimeter term and the Wasserstein term. There is an inherent frustration between the two, due the fact that while there exists sequences $\{(E_m, F_m)\}$ of admissible sets to (1.1) such that $W_p(E_m, F_m) \to 0$, any such sequence necessarily has perimeters approaching infinity, cf. Lemma 2.8. However, a crucial feature is that the construction of such a sequence can be achieved within a *bounded* set. This is one reason why we are able to prove the existence of minimizers in all parameter regimes, which does not hold for some other examples of perimeter energies perturbed by a nonlocal term. As we recall presently, the celebrated liquid drop model of Gamow displays non-existence phenomena in certain parameter regimes.

A classical example of a nonlocal isoperimetric problem is the liquid drop model of Gamow (see [Gam30]),

$$\inf \left\{ P(E) + \int_E \int_E |x-y|^{-\alpha}\, dxdy \colon |E| = M \right\},$$

where the nonlocal term is given by Riesz-type interactions. Here the two terms present in the energy functional (perimeter and nonlocal interactions) are in direct competition, as in (1.1). The surface energy is minimized by a ball whereas the repulsive term prefers to disperse the mass into vanishing components diverging infinitely apart. The parameter of the problem, that is M, sets a length scale between these competing forces (see [CMT17] for a review).

There are two major differences between the problem (1.1) and the aforementioned one:

- The nonlocal isoperimetric problems considered in the literature involve the minimization of functionals over *single* sets of finite perimeter with a volume constraint. The energy functional in (1.1), on the other hand, is minimized over a pair of disjoint sets of finite perimeter of equal volume. A similar phenomenon appears in ternary systems (involving both interfacial energy and nonlocal pairwise interactions) with three different phases, where two of which interact via long-range Riesz-type potentials (see [BK16]).
- Perhaps the most important distinction between (1.1) and the nonlocal isoperimetric problems studied in the literature is that in our case a minimizing sequence for the repulsive nonlocal term (the Wasserstein distance) does not necessarily consist of vanishing components that are diverging away to infinity (cf. [ABCT19, KMN16]). Rather, in some sense, it prefers oscillations reminiscent of nonexistence of minimizers in shape optimization problems via nonlocal attractive-repulsive interactions in models of swarming [BCT18, FL18].

An important manifestation of these differences is that, as shown by Knüpfer and Muratov [KM14], in Gamow's model minimizers *fail to exist* for values of the mass constraint that are larger than a critical value of M and for $\alpha \in (0, 2)$ (see also [LO14, FKN16] for the physically relevant case of $n = 3$, $\alpha = 1$, and [FN21] for a newer proof in the general case). This is in striking contrast with our main result for (1.1).

Let us briefly discuss some of the mathematical literature related to (1.1), as it arises in various applications. First, geometric variational problems with a Wasserstein term are useful in the modelling of bilayer membranes. In [PR09], Peletier and Röger derived the energy

$$P(E) + \varepsilon^{-2} W_1(\mathcal{L}^n \llcorner E, \mathcal{L}^n \llcorner F) \quad \text{for } E, F \subset \mathbb{R}^n,\ |E \cap F| = 0,\ |E| = |F| = \varepsilon, \tag{1.2}$$

as a simplified model for lipid bilayer membranes. Here the sets E and F represent the densities of the hydrophobic tails and hydrophilic heads, respectively, of the two part lipid molecules. The perimeter term signifies an interfacial energy arising from hydrophobic effects, while the Wasserstein term is a weak remainder of the bonding between the head and tail particles. The authors in [LPR14, PR09] considered the asymptotic expansion as $\varepsilon \to 0$ of the energy in $\mathbb{R}^2$ and $\mathbb{R}^3$ and identified a limiting energy concentrated on a codimension one set. The competition described earlier between the two terms in the energy drives the system toward *partially localized* structures that are thin in one direction ($\sim \varepsilon$) and extended in the remaining directions. Since (1.2) is equivalent to (1.1) up to rescaling and choosing the correct $\lambda = \lambda(\varepsilon)$, our existence theorem applies to (1.2). Nonlocal isoperimetric problems (mostly related to models of diblock copolymers) where the perimeter functional is perturbed by a nonlocal term involving the 2-Wasserstein distance have also appeared elsewhere in the literature (cf. [BPR14, PV10]).

In a completely different line of research, in the recent article [LPS19], Liu, Pego, and Slepčev study incompressible flows between equal volume shapes, as critical points for action, given by kinetic energy along transport paths that are constrained to be characteristic function densities. Formally, viewing the space of equal volume shapes as an infinite dimensional "manifold", the critical points for action are geodesics – they verify incompressible Euler equations for an inviscid potential flow with zero pressure, and *zero surface tension along free boundaries.* The authors in [LPS19] find that, in particular, locally minimizing action exhibits an instability associated with microdroplet formation. They show that any two shapes of equal volume can be approximately connected by what they refer to as an "Euler spray", a countable superposition of ellipsoidal geodesics. Furthermore, associated with the aforementioned instability, the infimum of action, which is equal to the squared 2-Wasserstein distance, is not attained.

Unlike [LPS19], we do not focus on *paths* joining shapes – investigating the role of surface tension in alleviating the microdroplet instability alluded to above is an interesting research direction that we hope to pursue elsewhere. For now we simply note that in the absence of the perimeter regularization in (1.1), minimizing sequences disintegrate into tiny "microdroplets", driving the minimum energy to zero, a form of microdroplet instability (see Lemma 2.8). We believe our technical contributions precluding the loss of compactness via microdroplet formation will be useful in studying the effect of including surface tension in [LPS19].

Finally, we mention some future directions and questions that remain regarding (1.1). While the one-dimensional calculations in [BCL20, Example 4.4] determine the minimizers depending on λ explicitly, the characterization of minimizers for any $\lambda > 0$ in higher dimensions remains an open problem.

Shortly after submission of the present article, Candau-Tilh and Goldman uploaded a preprint on arXiv which studies the same minimization problem (see [CTG22]). They obtain the existence of minimizers via an alternative argument. They also characterize global minimizers in the small λ regime, partially answering a question left open in our paper.

2. Notation and Preliminaries

We introduce some notation that we will use throughout the paper. Let $B(x,r)$ denote the open ball in $\mathbb{R}^n$ centered at x with radius r, and let $\omega_n := |B(0,1)|$. For any Lebesgue measurable set $E \subset \mathbb{R}^n$, $|E|$ is the Lebesgue measure of E. Finally, we use uppercase C_n, C_p, and $C_{n,p}$ to refer to constants that depend on one or both of the spatial dimension n and $p \in [1,\infty)$. The values of these constants may change from line to line. An exception to this convention is Lemma 2.1, so we denote the dimensional constant appearing there by the lowercase $c(n)$.

We work within the setting of sets of finite perimeter in $\mathbb{R}^n$ (see e.g. [Mag12]). Given a Lebesgue measurable set $E \subset \mathbb{R}^n$ we use the perimeter functional in the sense of De Giorgi, defined by

$$P(E) := \sup\left\{ \int_E \operatorname{div} V(x)\, dx \colon V \in C^1_c(\mathbb{R}^n;\mathbb{R}^n), |V| \leqslant 1 \right\}.$$

This notion of perimeter possesses properties such as lower-semicontinuity under L^1_{loc}-convergence, which is immediate from the definition, and compactness.

In the sequel, we will need the following nucleation lemma, due to Almgren [Alm76, VI.13] and quoted from [Mag12, Lemma 29.10].

Lemma 2.1 (Nucleation)**.** *For every $n \geqslant 2$, there exists a positive constant $c(n)$ with the following property. If E is of finite perimeter, $0 < |E| < \infty$, and*

$$\varepsilon \leqslant \min\left\{|E|, \frac{P(E)}{2nc(n)}\right\},$$

then there exists a finite family of points $x_i \subset \mathbb{R}^n$, $1 \leqslant i \leqslant I$ such that

$$\left|E \setminus \bigcup_{1\leqslant i\leqslant I} B(x_i, 2)\right| < \varepsilon,$$

$$|E \cap B(x_i, 1)| \geqslant \left(c(n)\frac{\varepsilon}{P(E)}\right)^n. \tag{2.1}$$

Moreover, $|x_i - x_{i'}| > 2$ for every $i \neq i'$, and

$$I < |E|\left(\frac{P(E)}{c(n)\varepsilon}\right)^n. \tag{2.2}$$

Remark 2.2 (Nucleation/compactness)**.** We will often employ the nucleation lemma, in particular the conclusion (2.1), in conjunction with the compactness theorem for sets of finite perimeter (cf. for example [Mag12, Corollary 12.27]) to obtain a positive measure subsequential L^1_{loc}-limit of a suitable sequence $\{E_m\}$. Precisely, if $\{E_m\}$ is a sequence of sets of finite perimeter satisfying

$$0 < \varepsilon := \inf_m \min\left\{|E_m|, \frac{P(E_m)}{2nc(n)}\right\}, \qquad \sup_m P(E_m) < \infty,$$

then, up to extraction of a non-relabeled subsequence, there exists a non-empty set E and sequence $\{x_m\} \subset \mathbb{R}^n$ such that $(E_m - x_m) \overset{\mathrm{loc}}{\to} E$ and

$$|E \cap B(0, 1)| \geqslant \left(c(n)\frac{\varepsilon}{\sup P(E_m)}\right)^n > 0. \tag{2.3}$$

Here the local convergence for sets is the strong L^1_{loc} convergence of the corresponding characteristic functions. We remark that this compactness property has also been obtained by Frank and Lieb in [FL15] using different arguments.

The next lemma is an amalgamation of several standard arguments [Mag12, Lemmas 17.21 and 17.9]. It allows for comparison of the energies of a minimizing sequence against local variations which do not necessarily preserve the volume constraint, cf. (3.6), which is useful in the derivation of density estimates for example. For convenience we include the proof of this lemma in the appendix.

Lemma 2.3 (Volume-fixing variations along a sequence)**.** *Let E be a set of finite perimeter and A be an open set such that $\mathcal{H}^{n-1}(\partial^* E \cap A) > 0$. Suppose also that $\{E_m\}$ satisfy*

$$\sup P(E_m; A) \leqslant M < \infty$$

and $E_m \overset{\text{loc}}{\to} E$ *in* $\mathbb{R}^n$. *Then there exists* $\sigma_0 = \sigma_0(E, A, M) > 0$ *and* $C_0 = C_0(E, A, M) < \infty$ *such that for every* $\sigma \in (-\sigma_0, \sigma_0)$ *and large enough* m *there exist sets of finite perimeter* G_m *with* $G_m \triangle E_m \subset\subset A$ *and*

$$|G_m \cap A| = |E_m \cap A| + \sigma, \tag{2.4}$$

$$|G_m \triangle E_m| \leqslant C_0|\sigma|, \quad \text{and} \quad |P(G_m; A) - P(E_m; A)| \leqslant C_0|\sigma|. \tag{2.5}$$

We turn to recalling notions from optimal transport that we use throughout the paper (see [Amb03, San15, Vil03, Vil09] for further details). The family of finite, positive Borel measures on $\mathbb{R}^n$ is denoted by $\mathcal{M}_+(\mathbb{R}^n)$. We work with this class instead of the usual space of Borel probability measures since it will sometimes be useful to have a notion of transport between measures of equal mass other than 1; this of course entails no significant change in the theory. Given μ, $\nu \in \mathcal{M}_+(\mathbb{R}^n)$ with $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n)$, we let $\Pi(\mu, \nu)$ be the set of all couplings between μ and ν:

$$\Pi(\mu, \nu) := \{\gamma \in \mathcal{M}_+(\mathbb{R}^n \times \mathbb{R}^n) \colon (\pi_1)_{\#}\gamma = \mu,\ (\pi_2)_{\#}\gamma = \nu\},$$

where # is the push-forward operation, and π_1, π_2 respectively denote projections onto the first and second copies of $\mathbb{R}^n$. A transport map from μ to ν is a map $T \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $T_{\#}\mu = \nu$. Any such T induces a coupling γ via the relation $\gamma = (\mathrm{Id} \times T)_{\#}\mu$. When $\mu = \mathcal{L}^n \llcorner E$ and $\nu = \mathcal{L}^n \llcorner F$, we will refer to T as transporting E to F.

Kantorovich's problem with cost $c(x, y)$ for measures μ, ν of equal total mass is

$$\mathbf{K}_c(\mu, \nu) = \inf \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y)\, d\gamma(x, y) \colon \gamma \in \Pi(\mu, \nu) \right\}.$$

Since we are interested in the case where μ and ν are the restrictions of Lebesgue measure to two subsets of $\mathbb{R}^n$ and the cost is

$$c_p(x, y) := |x - y|^p,$$

the existence of a solution to Kantorovich's problem in this instance is relevant. For stronger versions of this theorem and more comprehensive discussions of the vast mathematical literature on optimal transport, we refer the reader to the monographs mentioned above and the references therein.

Theorem 2.4 (Existence of an optimal transport map)**.** *Let* $p \in [1, \infty)$*, and suppose* E *and* F *are Lebesgue measurable sets with* $|E| = |F| > 0$. *Then there exists a map* $\Phi \colon \mathbb{R}^n \to \mathbb{R}^n$*, called an optimal transport map, such that* $\Phi_{\#}(\mathcal{L}^n \llcorner E) = \mathcal{L}^n \llcorner F$ *and*

$$\int_E |x - \Phi(x)|^p\, dx = \mathbf{K}_{c_p}(\mathcal{L}^n \llcorner E, \mathcal{L}^n \llcorner F).$$

Using optimal transport theory, one may define a distance between finite Lebesgue measure sets. This notion and more general ones involving mutually singular measures were analyzed in [BCL20].

Definition 1. For positive Lebesgue measure sets E and F with equal measure, let

$$W_p(E, F) := \mathbf{K}_{c_p}(\mathcal{L}^n \llcorner E, \mathcal{L}^n \llcorner F)^{\frac{1}{p}}. \tag{2.6}$$

Also, we set

$$\mathcal{W}_p(E) := \inf\{W_p(E, F) \colon |F| = |E|,\ |F \cap E| = 0\},$$

with the convention that $\mathcal{W}_p(E) = 0$ if $|E| = 0$.

When $\mathcal{L}^n \llcorner E$, $\mathcal{L}^n \llcorner F$ are in the space of Borel probability measures with finite p^{th} moments $\mathcal{P}_p(\mathbb{R}^n)$, the definition (2.6) coincides with the much-studied p-Wasserstein distance between two disjoint sets, hence the duplicate notation. The rest of the preliminaries are dedicated to the properties of $\mathcal{W}_p$ necessary for our analysis.

Lemma 2.5 (Properties of $\mathcal{W}_p$)**.** *Let $E \subset \mathbb{R}^n$ be Lebesgue measurable.*

(i) (Monotonicity) *If $E \subset F$, where F is Lebesgue measurable, then $\mathcal{W}_p(E) \leqslant \mathcal{W}_p(F)$.*
(ii) (Positivity) *If $|E| > 0$, then $\mathcal{W}_p(E) > 0$.*
(iii) (Scaling) *For any $r \geqslant 0$,*

$$\mathcal{W}_p(rE) = r^{1+\frac{n}{p}} \mathcal{W}_p(E). \tag{2.7}$$

(iv) (L^q-bound) *There exists C_n such that*

$$\mathcal{W}_p(E) \leqslant C_n |E|^{\frac{1}{p}+\frac{1}{n}};$$

(cf. [XZ21, Equation 4.2] for the same statement when E is bounded).

Proof. Items (i) and (iii) follow immediately from the definition of $\mathcal{W}_p$. By (i), it suffices to prove (ii) in the case that $|E| > 0$ and E is bounded.

Suppose for a contradiction that F_m is such that $\mathbf{K}_{c_p}(\mathcal{L}^n \llcorner E, \mathcal{L}^n \llcorner F_m) \to 0$. In this case, E and F_m have finite p^{th} moments, so by the properties of the p-Wasserstein distance, $W_p(E, F_m) \to 0$ implies that $\mathcal{L}^n \llcorner F_m \overset{*}{\rightharpoonup} \mathcal{L}^n \llcorner E$ (see for example [San15, Theorem 5.11]). But this is incompatible with $|E \cap F_m| = 0$ and $|E| = |F_m| > 0$, so we have a contradiction.

For (iv), by the scaling (2.7), it is enough to prove the claim when $|E| = 1$. Divide $\mathbb{R}^n$ into disjoint cubes Q_j of volume 2. For each j, since $|Q_j| = 2$ and $|E \cap Q_j| \leqslant 1$, we can find $F_j \subset Q_j$ such that $|F_j| = |Q_j \cap E|$ and $|F_j \cap E| = 0$. Let T_j transport $E \cap Q_j$ onto F_j, and set $F = \bigcup_j F_j$. Then it is easy to see that the map T defined by

$$T(x) = T_j(x) \qquad \text{for } x \in E \cap Q_j$$

transports E onto F and satisfies $|x - T(x)| \leqslant \operatorname{diam}(Q_j)$ for $x \in E \cap Q_j$. Thus

$$\mathcal{W}_p(E) \leqslant \left(\int_E |x - T(x)|^p \, dx \right)^{\frac{1}{p}} \leqslant C_n,$$

since $\operatorname{diam}(Q_j) =: C_n$, independent of j, and $|E| = 1$. The claim follows. $\square$

Proposition 2.6 (Continuity of $\mathcal{W}_p$ with respect to L^1-convergence)**.** *There exists $C_{n,p}$ such that for any $|E|$, $|\tilde{E}|$,*

$$\left| \mathcal{W}_p^p(\tilde{E}) - \mathcal{W}_p^p(E) \right| \leqslant C_{n,p} \max\{|E|^{\frac{p}{n}}, |\tilde{E}|^{\frac{p}{n}}\} |E \triangle \tilde{E}| \tag{2.8}$$

and

$$\left| \mathcal{W}_p(\tilde{E}) - \mathcal{W}_p(E) \right| \leqslant C_{n,p} \max\{\mathcal{W}_p^{1-p}(E), \mathcal{W}_p^{1-p}(\tilde{E})\} \max\{|E|^{\frac{p}{n}}, |\tilde{E}|^{\frac{p}{n}}\} |E \triangle \tilde{E}|. \tag{2.9}$$

Remark 2.7. When E and F are both bounded with unit measure, Proposition 2.6 is contained in [BCL20, Lemma 4.5].

Proof of Proposition 2.6. First we demonstrate how (2.9) follows from (2.8). By applying the mean value theorem to the function $t \mapsto t^{1/p}$, we deduce that

$$\left|\mathcal{W}_p(\tilde{E}) - \mathcal{W}_p(E)\right| \leqslant \frac{1}{p}\max\{\mathcal{W}_p^{1-p}(E), \mathcal{W}_p^{1-p}(\tilde{E})\}\left|\mathcal{W}_p^p(\tilde{E}) - \mathcal{W}_p^p(E)\right|.$$

The bound (2.9) follows immediately from this equation and (2.8).

It remains to prove (2.8). Without loss of generality,

$$\mathcal{W}_p^p(\tilde{E}) \geqslant \mathcal{W}_p^p(E). \tag{2.10}$$

We may also assume that

$$|\tilde{E}| = 1; \tag{2.11}$$

the full case then follows from rescaling. Fix any F with $|F| = |E|$ and $|F \cap E| = 0$. If we can show that

$$\mathcal{W}_p^p(\tilde{E}) - W_p^p(E, F) \leqslant C_{n,p}\max\{|E|^{\frac{p}{n}}, 1\}|E \triangle \tilde{E}|, \tag{2.12}$$

then, in light of (2.10), taking the infimum over F disjoint from E gives (2.8) when $|\tilde{E}| = 1$.

To show (2.12) under the assumptions (2.10) and (2.11), first consider the case that

$$|E| \geqslant 2.$$

Then since $|\tilde{E}| = 1$, we have $|E \triangle \tilde{E}| \geqslant 1$, so that

$$\max\{|E|^{\frac{p}{n}}, 1\}|E \triangle \tilde{E}| \geqslant 2^{\frac{p}{n}}. \tag{2.13}$$

In addition,

$$\mathcal{W}_p^p(\tilde{E}) - W_p^p(E, F) \leqslant \mathcal{W}_p^p(\tilde{E}) \leqslant C_{n,p},$$

which together with (2.13) gives (2.12) after suitably modifying $C_{n,p}$. For the rest of the proof of (2.12), we therefore assume that

$$|E| \leqslant 2. \tag{2.14}$$

Let Φ be an optimal transport map from E to F, which exists by Theorem 2.4. The idea is to modify Φ to create a transport map $\tilde{\Phi}$ for $\tilde{E}$ (to a set of the appropriate measure), which allows for comparison between $\mathcal{W}_p^p(\tilde{E})$ and $W_p^p(E, F)$. When $x \in E \cap \tilde{E}$ and $\Phi(x) \notin \tilde{E}$, we can define $\tilde{\Phi}$ simply by using Φ:

$$\tilde{\Phi}(x) = \Phi(x) \quad \text{if } x \in \tilde{E} \cap E \cap \Phi^{-1}(F \cap \tilde{E}^c). \tag{2.15}$$

For the rest of the points in $\tilde{E}$, we must make a new definition. We partition $\mathbb{R}^n$ into cubes Q_j of volume 4. Since $|Q_j \setminus (\tilde{E} \cup F)| \geqslant 1$ and $|\tilde{E}| = 1$, there exist measurable sets $D_j \subset Q_j$ such that

$$D_j \cap \tilde{E} = \emptyset = D_j \cap F \quad \text{and} \quad |D_j| = |Q_j \cap \tilde{E} \cap (E^c \cup \Phi^{-1}(F \cap \tilde{E}))| \leqslant 1.$$

We may obtain optimal transport maps Φ_j from $Q_j \cap \tilde{E} \cap (E^c \cup \Phi^{-1}(F \cap \tilde{E}))$ to D_j and define

$$\tilde{\Phi}(x) = \Phi_j(x) \quad \text{if } x \in Q_j \cap \tilde{E} \cap (E^c \cup \Phi^{-1}(F \cap \tilde{E})).$$

Before estimating the energy difference, we note that since Φ is a transport map and $F \subset E^c$,

$$|\tilde{E} \cap \Phi^{-1}(F \cap \tilde{E})| \leqslant |\Phi^{-1}(\tilde{E} \cap F)| = |\tilde{E} \cap F| \leqslant |\tilde{E} \cap E^c|. \tag{2.16}$$

Then

$$\begin{aligned}
\mathcal{W}_p^p(\tilde{E}) - W_p^p(E,F) &\leqslant \int_{\tilde{E}} |x-\tilde{\Phi}(x)|^p\,dx - \int_E |x-\Phi(x)|^p\,dx \\
&\leqslant \sum_j \int_{Q_j\cap\tilde{E}\cap(E^c\cup\Phi^{-1}(F\cap\tilde{E}))} |x-\Phi_j(x)|^p\,dx \\
&\leqslant \sum_j \operatorname{diam}(Q_j)^p(|Q_j\cap\tilde{E}\cap E^c| + |Q_j\cap\tilde{E}\cap\Phi^{-1}(F\cap\tilde{E})|) \\
&= \operatorname{diam}(Q_j)^p(|\tilde{E}\cap E^c| + |\tilde{E}\cap\Phi^{-1}(F\cap\tilde{E})|) \\
&\overset{(2.16)}{\leqslant} 2\operatorname{diam}(Q_j)^p|\tilde{E}\triangle E|. \qquad (2.17)
\end{aligned}$$

Since $1 \leqslant \max\{|E|^{\frac{p}{n}}, 1\} \leqslant 2^{\frac{p}{n}}$, (2.17) implies (2.12). The proof is complete. □

Lemma 2.8 (Non-existence of minimizers for $\mathcal{W}_p$)**.** *There exists a sequence $\{(E_m, F_m)\}$ such that $|E_m\cap F_m| = 0$, $|E_m| = |F_m| = 1$, and*

$$W_p(E_m, F_m) \to 0.$$

Furthermore, for any sequence satisfying those three properties,

$$P(E_m),\ P(F_m) \to \infty. \qquad (2.18)$$

Proof. We omit a full proof of the construction of such a sequence, which is straightforward. There are many ansatzes one could use; for example, E_m could be a single thin, arbitrarily long cylinder, and F_m a suitable tubular neighborhood. Alternatively, E_m and F_m could be suitably many disjoint arbitrarily small balls and corresponding annuli around them. The latter example may be viewed as an analogue of the microdroplet instability discovered in [LPS19] in our static setting.

To prove (2.18), assume for contradiction that $W_p(E_m, F_m) \to 0$ but $\limsup P(E_m) < \infty$. By Remark 2.2, the uniform perimeter bound implies that, up to translations which we ignore, there exists a set E with $|E\cap B(0,1)| > 0$ and $E_m \overset{\text{loc}}{\to} E$. Therefore, $E_m\cap B(0,1) \to E\cap B(0,1)$, and so by the L^1-continuity of $\mathcal{W}_p$,

$$0 < \mathcal{W}_p(E\cap B(0,1)) = \liminf_{m\to\infty} \mathcal{W}_p(E_m\cap B(0,1)) \leqslant \liminf_{m\to\infty} \mathcal{W}_p(E_m, F_m) = 0.$$

We have thus arrived at a contradiction. The proof that $P(F_m)$ diverges is the same. □

Remark 2.9. In their paper [CTG22], Candau-Tilh and Goldman obtain the following interpolation inequality

$$W_p(E,F)P(E) \geqslant C(n)|E|^{1+\frac{1}{p}}. \qquad (2.19)$$

As a consequence of this inequality one can obtain (2.18) in Lemma 2.8. Here we provide an alternative proof of this interpolation inequality which effectively quantifies our proof of (2.18).

Proof of (2.19)*.* We first observe that there exists $C_n > 0$ such that if $|E\cap B_r| \geqslant 3\omega_n r^n/4$, then

$$\mathcal{W}_p(E\cap B_r) \geqslant C_n r^{1+\frac{n}{p}}. \qquad (2.20)$$

This is due to the fact that at least $\omega_n r^n/4$ of the mass of $E\cap B_r$ is contained in $B_{3r/4}$ and must be transported outside B_r. Let $|E| = |F|$ and $|E\cap F| = 0$ and consider any

Lebesgue point $x \in E^{(1)}$. By the continuity of $r \to |E \cap B_r(x)|$, the fact that $x \in E^{(1)}$, and the intermediate value theorem, there exists

$$r_x \leqslant \left(\frac{4|E|}{3\omega_n}\right)^{\frac{1}{n}} \tag{2.21}$$

such that $|E \cap B_{r_x}(x)| = 3\omega_n r_x^n/4$. By (2.21), we can apply the Besicovitch covering theorem to the family of closed balls $\mathcal{F} = \{\overline{B}_{r_x} : x \in E^{(1)}\}$, to obtain subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_{\xi(n)}$, each of which consists of disjoint balls, such that

$$E^{(1)} \subset \bigcup_{i=1}^{\xi(n)} \bigcup_{\overline{B}_{r_x} \in \mathcal{F}_i} \overline{B}_{r_x} .$$

By the relative isoperimetric inequality, since $|E \cap B_{r_x}| = 3\omega_n r_x^n/4$, we have for some c_n

$$P(E; B_{r_x}) \geqslant c_n r_x^{n-1} \quad \forall x \in E^{(1)}. \tag{2.22}$$

Also, with Φ denoting the optimal transport map from E to F, we may use the observation (2.20) to see that

$$W_p(E \cap B_{r_x}, \Phi(E \cap B_{r_x})) \geqslant \mathcal{W}_p(E \cap B_{r_x}(x)) \geqslant C_n r^{1+\frac{n}{p}} . \tag{2.23}$$

Finally, combining (2.22)-(2.23) with Hölder's inequality, we may estimate

$$\begin{aligned}
W_p(E,F)P(E) &\geqslant \xi(n)^{-1-\frac{1}{p}} \left(\sum_{i=1}^{\xi(n)} \sum_{\overline{B}_{r_x} \in \mathcal{F}_i} \int_{E \cap \overline{B}_{r_x}} |z - \Phi(z)|^p \, dz\right)^{\frac{1}{p}} \left(\sum_{i=1}^{\xi(n)} \sum_{\overline{B}_{r_x} \in \mathcal{F}_i} P(E; \overline{B}_{r_x})\right) \\
&\geqslant \xi(n)^{-2} \left[\left(\sum_{i=1}^{\xi(n)} \sum_{\overline{B}_{r_x} \in \mathcal{F}_i} C_n^p r_x^{p+n}\right)^{\frac{1}{p+1}} \left(\sum_{i=1}^{\xi(n)} \sum_{\overline{B}_{r_x} \in \mathcal{F}_i} c_n r_x^{n-1}\right)^{\frac{p}{p+1}}\right]^{\frac{p+1}{p}} \\
&\geqslant \frac{C_n c_n}{\xi(n)^2} \left(\sum_{i=1}^{\xi(n)} \sum_{\overline{B}_{r_x} \in \mathcal{F}_i} r_x^{\frac{p+n}{p+1}} r_x^{\frac{p(n-1)}{p+1}}\right)^{\frac{p+1}{p}} \\
&= \frac{C_n c_n}{\xi(n)^2} \left(\sum_{i=1}^{\xi(n)} \sum_{\overline{B}_{r_x} \in \mathcal{F}_i} r_x^n\right)^{\frac{p+1}{p}} \\
&= \frac{C_n c_n}{\xi(n)^2} \left(\sum_{i=1}^{\xi(n)} \sum_{\overline{B}_{r_x} \in \mathcal{F}_i} \frac{4|E \cap B_{r_x}(x)|}{3\omega_n}\right)^{\frac{p+1}{p}} \\
&\geqslant \tilde{C}_n |E|^{1+\frac{1}{p}},
\end{aligned}$$

where in the last equality we have used the fact that $|E \cap B_{r_x}| = 3\omega_n r_x^n/4$. □

The last preliminary result is drawn from [BCL20] and [XZ21].

Theorem 2.10. *Let E be a bounded Lebesgue measurable set.*

(i) [BCL20, Theorem 3.21] *There exists a Lebesgue measurable set F with $|F| = |E|$ and $|E \cap F| = 0$ and an optimal transport map Φ from E to F such that*

$$W_p(E, F) = \mathcal{W}_p(E).$$

(ii) [XZ21, Lemma 4.3] *There exists C_n such that for $\mathcal{L}^n$-a.e. $x \in E$,*

$$|x - \Phi(x)| \leqslant C_n |E|^{\frac{1}{n}}.$$

Remark 2.11 (Additivity of $\mathcal{W}_p^p$)**.** Arguing directly from items (i) and (ii) of the above theorem, it follows that if $E_1, \ldots, E_K$ are bounded sets such that

$$\operatorname{dist}(E_k, E_{k'}) \geqslant 2C_n \max_{1 \leqslant j \leqslant K} |E_j|^{\frac{1}{n}} \quad \text{for } k \neq k',$$

then the sets F_k minimizing $W_p(E_k, F_k)$ are pairwise disjoint and

$$W_p^p\left(\bigcup_k E_k, \bigcup_k F_k\right) = \mathcal{W}_p^p\left(\bigcup_{k=1}^K E_k\right) = \sum_{k=1}^K \mathcal{W}_p^p(E_k) = \sum_{k=1}^K W_p^p(E_k, F_k).$$

3. Proof of Theorem 1.1

We write the main functional as

$$\mathcal{G}(E) := P(E) + \lambda \mathcal{W}_p(E), \tag{3.1}$$

where $\mathcal{W}_p$ is given as in Definition 1.

Proof of Theorem 1.1. We prove this theorem in multiple steps.

Step one: First, we extract a nontrivial set E^0 which is the limit of sets E_m corresponding to a minimizing sequence $\{E_m\}_m$ with

$$P(E_m) + \lambda \mathcal{W}_p(E_m) \leqslant \inf \mathcal{G} + \frac{1}{m}. \tag{3.2}$$

From this inequality we have the immediate upper bound

$$P(E_m) \leqslant 1 + \inf \mathcal{G}. \tag{3.3}$$

on the perimeters. Since in addition $|E_m| = 1$ for all m, we may then apply the nucleation lemma and compactness as in Remark 2.2. Therefore, up to a subsequence which we do not relabel and translations which, without loss of generality, are trivial, there exists a set E^0 with

$$0 \overset{(2.3)}{<} |E^0| \leqslant 1,$$

$$E_m \overset{\text{loc}}{\to} E^0 \quad \text{in } \mathbb{R}^n.$$

Step two: Here we identify δ, $\alpha > 0$ such that if $\tilde{E}_m$ is any set with $|E_m \triangle \tilde{E}_m| \leqslant \delta$, then

$$\left|\mathcal{W}_p(E_m) - \mathcal{W}_p(\tilde{E}_m)\right| \leqslant C_{n,p} \alpha^{1-p} |E_m \triangle \tilde{E}_m|. \tag{3.4}$$

We first observe that by the uniform perimeter bound and Lemma 2.8 we can consider

$$\alpha := \inf_m \mathcal{W}_p(E_m) > 0. \tag{3.5}$$

Due to the continuity of $\mathcal{W}_p$ with respect to L^1-convergence from (2.8), we may choose $0 < \delta \leqslant 1$ small enough so that if $|E_m \triangle \tilde{E}_m| \leqslant \delta$,

$$\begin{aligned}\mathcal{W}_p^p(\tilde{E}_m) &\geqslant \mathcal{W}_p^p(E_m) - C_{n,p}\max\{|E_m|^{\frac{p}{n}}, |\tilde{E}_m|^{\frac{p}{n}}\}|E_m \triangle \tilde{E}_m| \\ &\geqslant \alpha^p - C_{n,p}(1+\delta)^{\frac{p}{n}}\delta \\ &\geqslant \frac{\alpha^p}{2^p}.\end{aligned}$$

Then since $\mathcal{W}_p(E_m)$ and $\mathcal{W}_p(\tilde{E}_m)$ are both bounded from below by $\alpha/2$, (2.9) gives

$$\begin{aligned}\left|\mathcal{W}_p(\tilde{E}_m) - \mathcal{W}_p(E_m)\right| &\leqslant C_{n,p}\max\{\mathcal{W}_p^{1-p}(E_m), \mathcal{W}_p^{1-p}(\tilde{E}_m)\}\max\{|E_m|^{\frac{p}{n}}, |\tilde{E}_m|^{\frac{p}{n}}\}|E_m \triangle \tilde{E}_m| \\ &\leqslant C_{n,p}\alpha^{1-p}2^{p-1}(1+\delta)^{\frac{p}{n}}|E_m \triangle \tilde{E}_m|.\end{aligned}$$

Upon recalling that $\delta \leqslant 1$ and modifying $C_{n,p}$, we have shown (3.4).

Step three: In this step, we utilize (3.4) and Lemma 2.3, the volume-fixing variations lemma, to show that there exists r_0 and $\Lambda > 0$ such that for all m large enough, E_m satisfies the inequality

$$P(E_m) \leqslant P(\tilde{E}_m) + \Lambda|E_m \triangle \tilde{E}_m| + \frac{1}{m} \qquad \text{if } E_m \triangle \tilde{E}_m \subset\subset B(x,r),\ 0 < r < r_0. \tag{3.6}$$

Fix x and consider $\tilde{E}_m$ with $E_m \triangle \tilde{E}_m \subset B(x,r)$, with $r < r_0$ to be determined shortly. Since $|\tilde{E}_m|$ is not necessarily 1, we proceed using Lemma 2.3. Let $y_1,\ y_2 \in \partial^*(E^0)$ and $\eta > 0$ be such that

$$\mathcal{H}^{n-1}(\partial^*(E^0) \cap B(y_i,\eta)) > 0$$

for $i = 1, 2$ and

$$B(y_1,\eta) \cap B(y_2,\eta) = \emptyset.$$

We apply the volume-fixing variations lemma with the choice of $A = B(y_i,\eta)$, yielding σ_0 and C_0 such that for any $|\sigma| < \sigma_0$ and $i = 1, 2$, there exists G_m^i with $G_m^i \triangle E_m \subset\subset B(y_i,\eta)$ and

$$\begin{gathered}|G_m \cap B(y_i,\eta)| = |E_m \cap B(y_i,\eta)| + \sigma, \\ |G_m \triangle E_m| \leqslant C_0|\sigma|, \quad \text{and} \quad |P(G_m; B(y_i,\eta)) - P(E_m; B(y_i,\eta))| \leqslant C_0|\sigma|.\end{gathered} \tag{3.7}$$

Up to further decreasing σ_0, we may assume that

$$\max\{1, C_0\}\sigma_0 < \delta/2. \tag{3.8}$$

Choose r_0 such that

$$\omega_n r_0^n < \sigma_0 \tag{3.9}$$

and for every $z \in \mathbb{R}^n$, $B(z,r_0)$ is disjoint from at least one of $B(y_i,\eta)$. Therefore, for at least one of $i = 1, 2$,

$$B(x,r) \cap B(y_i,\eta) = \emptyset;$$

let us assume without loss of generality that it is y_1. We introduce the sets

$$\overline{E}_m = (E_m \cap B(x,r)^c \cap B(y_1,\eta)^c) \cup ((G_m^1 \cap B(y_1,\eta)) \cup (\tilde{E}_m \cap B(x,r)), \tag{3.10}$$

where G_m^1 is chosen according to Lemma 2.3 with

$$\sigma_m := |E_m \cap B(x,r)| - |\tilde{E}_m \cap B(x,r)| \in (-\omega_n r^n, \omega_n r^n), \tag{3.11}$$

so that

$$|G_m^1 \cap B(y_1,\eta)| = |E_m \cap B(y_1,\eta)| + |E_m \cap B(x,r)| - |\tilde{E}_m \cap B(x,r)|.$$

This ensures that

$$\begin{aligned}
|\overline{E}_m| &= |E_m| - |E_m \cap B(x,r)| - |E_m \cap B(y_1,\eta)| + |G_m^1 \cap B(y_1,\eta)| + |\tilde{E}_m \cap B(x,r)| \\
&= |E_m| - |E_m \cap B(x,r)| - |E_m \cap B(y_1,\eta)| + |E_m \cap B(y_1,\eta)| + |E_m \cap B(x,r)| \\
&\quad - |\tilde{E}_m \cap B(x,r)| + |\tilde{E}_m \cap B(x,r)| \\
&= |E_m| \\
&= 1.
\end{aligned}$$

By the triangle inequality and the formula $\sigma_m = |E_m \cap B(x,r)| - |\tilde{E}_m \cap B(x,r)|$, the bound

$$|\sigma_m| \leqslant |E_m \triangle \tilde{E}_m| \tag{3.12}$$

holds as well. Furthermore, with the aid of (3.7)–(3.9), we may estimate $|\overline{E}_m \triangle E_m|$ by

$$\begin{aligned}
|\overline{E}_m \triangle E_m| &= |G_m^1 \triangle E_m| + |\tilde{E}_m \triangle E_m| \\
&\leqslant C_0|\sigma_m| + \omega_n r^n \\
&< \delta/2 + \delta/2.
\end{aligned} \tag{3.13}$$

The previous inequality implies that (3.4) holds for E_m and $\overline{E}_m$, in which case

$$\big|\mathcal{W}_p(E_m) - \mathcal{W}_m(\overline{E}_m)\big| \leqslant C_{n,p}\alpha^{1-p}|E_m \triangle \overline{E}_m|.$$

Combining (3.13) and the fact that $|\sigma_m| \leqslant |E_m \triangle \tilde{E}_m|$, we have

$$\big|\mathcal{W}_p(E_m) - \mathcal{W}_m(\overline{E}_m)\big| \leqslant C_{n,p}\alpha^{1-p}(C_0+1)|\tilde{E}_m \triangle E_m|. \tag{3.14}$$

The last preliminary estimate before deriving (3.6) is a consequence of (3.7) and (3.11):

$$|P(E_m; B(y_1,\eta)) - P(\overline{E}_m; B(y_1,\eta))| \leqslant C_0|\sigma_m| \leqslant C_0|\tilde{E}_m \triangle E_m|. \tag{3.15}$$

Finally, since $|\overline{E}_m| = 1$, we may test (3.2) with $\overline{E}_m$ and use (3.10), (3.14), and (3.15) to obtain

$$\begin{aligned}
P(E_m) &\leqslant P(\overline{E}_m) + \lambda\mathcal{W}_p(\overline{E}_m) - \lambda\mathcal{W}_p(E_m) + \frac{1}{m} \\
&= P(\tilde{E}_m; B(y_1,\eta)^c) + P(\overline{E}_m; B(y_1,\eta)) - P(E_m; B(y_1,\eta)) + P(\tilde{E}_m; B(y_1,\eta)) \\
&\qquad + \lambda\mathcal{W}_p(\overline{E}_m) - \lambda\mathcal{W}_p(E_m) + \frac{1}{m} \\
&\leqslant P(\tilde{E}_m) + C_0|\tilde{E}_m \triangle E_m| + \lambda C_{n,p}\alpha^{1-p}(C_0+1)|\tilde{E}_m \triangle E_m| + \frac{1}{m}.
\end{aligned}$$

Taking $\Lambda := C_0 + \lambda C_{n,p}\alpha^{1-p}(C_0+1)$, we have shown (3.6).

Step four: Here we use (3.6) to prove that there exist C_n, $r_1 > 0$ such that *any* positive measure set E which is the L^1_{loc}-limit of translates $E_m - y_m$ for a sequence $\{y_m\}$ satisfies:

$$|E \cap B(x,r)| \geqslant C_n r^n \quad \forall x \in \partial^* E,\ r < r_1. \tag{3.16}$$

Since $|E| \leqslant 1$, such a lower density estimate implies that $\partial^* E$ and E are bounded. For the proof of (3.16), to simplify the notation, assume that $y_m = 0$ for all m.

We set

$$u_m(r) = |E_m \cap B(x,r)|, \quad u(r) = |E \cap B(x,r)|.$$

The coarea formula implies that for almost every r,

$$u_m'(r) = \mathcal{H}^{n-1}(E_m \cap \partial B(x,r)) \quad \text{and} \quad u'(r) = \mathcal{H}^{n-1}(E \cap \partial B(x,r)),$$

while the L^1_{loc}-convergence of E_m to E permits us to extract a subsequence such that

$$u_m'(r) \to u'(r) \quad \text{for almost every } r. \tag{3.17}$$

Furthermore, except for a measure zero set of r values which can be made independent of m, we have the identities

$$\begin{aligned} P(E_m) &= P(E_m; B(x,r)) + P(E_m; \overline{B(x,r)}^c), \qquad (3.18)\\ P(E_m \cap B(x,r)) &= P(E_m; B(x,r)) + \mathcal{H}^{n-1}(E_m \cap \partial B(x,r)),\\ P(E_m \setminus B(x,r)) &= P(E_m; \overline{B(x,r)}^c) + \mathcal{H}^{n-1}(E_m \cap \partial B(x,r)), \end{aligned}$$

and similarly for E. Therefore, for almost every $r < r_1$, with $r_1 \in (0, r_0)$ to be fixed shortly, testing (3.6) with $\tilde{E}_m = E_m \setminus B(x,r)$ yields

$$\begin{aligned} P(E_m;&B(x,r)) + P(E_m; \overline{B(x,r)}^c)\\ &= P(E_m)\\ &\leqslant P(E_m \setminus B(x,r)) + \Lambda |E_m \cap B(x,r)| + \frac{1}{m}\\ &= P(E_m; \overline{B(x,r)}^c) + \mathcal{H}^{n-1}(E_m \cap \partial B(x,r)) + \Lambda |E_m \cap B(x,r)| + \frac{1}{m}. \end{aligned}$$

We add $\mathcal{H}^{n-1}(E_m \cap \partial B(x,r)) - P(E_m; \overline{B(x,r)}^c)$ to both sides, arriving at

$$\begin{aligned} P(E_m; B(x,r)) &+ \mathcal{H}^{n-1}(E_m \cap \partial B(x,r))\\ &\leqslant 2\mathcal{H}^{n-1}(E_m \cap \partial B(x,r)) + \Lambda |E_m \cap B(x,r)| + \frac{1}{m}. \end{aligned}$$

The Euclidean isoperimetric inequality and (3.18) imply that for almost every $r < r_1$,

$$\begin{aligned} n\omega_n^{\frac{1}{n}} u_m(r)^{\frac{n-1}{n}} &= n\omega_n^{\frac{1}{n}} |E_m \cap B(x,r)|^{\frac{n-1}{n}}\\ &\leqslant P(E_m \cap B(x,r))\\ &= P(E_m; B(x,r)) + \mathcal{H}^{n-1}(E_m \cap \partial B(x,r))\\ &\leqslant 2\mathcal{H}^{n-1}(E_m \cap \partial B(x,r)) + \Lambda |E_m \cap B(x,r)| + \frac{1}{m}\\ &= 2u_m'(r) + \Lambda u_m + \frac{1}{m}. \end{aligned} \tag{3.19}$$

With the goal of absorbing Λu_m into the left hand side, we note that

$$\Lambda u_m \leqslant \frac{n\omega_n^{\frac{1}{n}} u_m(r)^{\frac{n-1}{n}}}{2} \iff u_m \leqslant \left(\frac{n\omega_n^{1/n}}{2\Lambda} \right)^n.$$

Therefore, choosing $r_1 \in (0, r_0)$ small enough so that

$$u_m \leqslant \omega_n r_1^n \leqslant \left(\frac{n\omega_n^{1/n}}{2\Lambda} \right)^n,$$

we have for almost every $r < r_1$

$$\Lambda u_m \leqslant \frac{n\omega_n^{\frac{1}{n}} u_m(r)^{\frac{n-1}{n}}}{2}.$$

Plugging this into the differential inequality (3.19) and passing to the limit $m \to \infty$ using (3.17), we may write

$$\begin{aligned}\frac{n\omega_n^{\frac{1}{n}} u(r)^{\frac{n-1}{n}}}{2} &= \lim_{m\to\infty} \frac{n\omega_n^{\frac{1}{n}} u_m(r)^{\frac{n-1}{n}}}{2} \\ &\leqslant \lim_{m\to\infty} 2u_m'(r) + \frac{1}{m} \\ &= 2u'(r)\end{aligned}$$

for almost every $r < r_1$. The lower density estimate (3.16) is achieved by dividing by $u^{\frac{n-1}{n}}$ integrating this inequality.

Step five: In this step, we obtain finitely many, bounded, limiting sets E^k and sequences $\{x_m^k\}$ such that E^k are L^1_{loc}-limits of translates $E_m - x_m^k$ and satisfy

$$\sum_k |E^k| = 1. \tag{3.20}$$

To this end, apply the nucleation lemma again to E_m with

$$\varepsilon_0 = \min\left\{1, \frac{1+\inf\mathcal{G}}{2nc(n)}, C_n r_1^n\right\},$$

where C_n is the dimensional constant from the previous step, to locate points x_m^i, $1 \leqslant i \leqslant I(m)$ satisfying the conclusions of Lemma 2.1. Here we include $C_n r_1^n$ in the definition of the constant ε_0 as we would like to control the size of what is not contained in the balls obtained from the nucleation lemma. If the remainder is non-empty, its smallness will then lead to a contradiction with the lower density estimates.

The uniform bound (2.2) on $I(m)$ in terms of $P(E_m)$, $|E_m|$, and ε_0 implies that by restricting to a further subsequence, we can find $I \in \mathbb{N}$ such that $I(m) = I$ for each m. After passing to a further subsequence, we may safely assume that

$$\lim_{m\to\infty} |x_m^i - x_m^j| =: d_{ij}$$

exists for each pair $(i, j) \in I \times I$, with infinity as a possible limit, too. Next, we define equivalence classes of $\{1, \dots, I\}$ based on the relation

$$i \equiv j \iff d_{ij} < \infty.$$

Let $K \leqslant I$ be the number of these equivalence classes, which partition $\{1, \dots, I\}$. For each $1 \leqslant k \leqslant K$ and $m \in \mathbb{N}$, let $x_m^k := x_m^{i(k,m)}$ be a point from the family of points corresponding to E_m such that $i(k, m)$ is a representative of the k-th equivalence class. Recall that due to (2.1) and (3.3), $E_m - x_m^k$ satisfies

$$|(E_m - x_m^k) \cap B(0, 1)| \geqslant \left(c(n)\frac{\varepsilon_0}{P(E_m)}\right)^n \geqslant \left(c(n)\frac{\varepsilon_0}{1+\inf\mathcal{G}}\right)^n.$$

We can therefore find non-trivial sets of finite perimeter E^k such that

$$E_m - x_m^k \overset{\text{loc}}{\to} E^k. \tag{3.21}$$

Since the previous step implies that each E^k is bounded, there exists R_0 such that

$$E^k \subset\subset B(0, R_0) \tag{3.22}$$

for each $1 \leqslant k \leqslant K$. We may also take R_0 to be large enough that

$$\bigcup_{i\in\{1,\dots,I\}:\, i\equiv k} B(x^i_m, 2) \subset B(x^k_m, R_0) \tag{3.23}$$

for all m; in other words $B(x^k_m, R_0)$ contains all the balls at the m-th stage with indices in the same equivalence class as k.

It remains to show that

$$\sum_{k=1}^{K} |E^k| = 1.$$

We first show that $\sum_{k=1}^{K} |E^k| \leqslant 1$. If this were not the case, then

$$\sum_{k=1}^{K} |E^k \cap B(0, R_0)| > 1. \tag{3.24}$$

Now for large m, the sets

$$E_m \cap B(x^k_m, R_0)$$

are pairwise disjoint since $|x^k_m - x^{k'}_m| \to \infty$ if $k \neq k'$. By (3.21) and (3.24), it follows that

$$\sum_{k=1}^{K} |E_m \cap B(x^k_m, R_0)| > \frac{1}{2}\left(1 + \sum_{k=1}^{K} |E^k \cap B(0, R_0)|\right)$$

for large m, which is impossible since $|E_m| = 1$. So

$$\sum_{k=1}^{K} |E^k| \leqslant 1.$$

Assume now for a contradiction that

$$\sum_{k=1}^{K} |E^k| = 1 - \delta$$

for some $\delta > 0$. Since $E^k \subset\subset B(0, R_0 + 2)$ and $E_m \cap B(x^k_m, R_0 + 2)$ are disjoint for large enough m, it must then be the case that

$$\left| E_m \setminus \left(\bigcup_k B(x^k_m, R_0 + 2) \right) \right| \geqslant \frac{\delta}{2}$$

for large enough m. At the same time, the nucleation lemma at the beginning of this step with $\varepsilon_0 \leqslant C_n r_1^n$ gave

$$\left| E_m \setminus \bigcup_{1\leqslant i\leqslant I} B(x^i_m, 2) \right| < \varepsilon_0 \leqslant C_n r_1^n.$$

Together with the assumption (3.23) that $\bigcup_i B(x^i_m, 2) \subset \bigcup_k B(x^k_m, R_0)$, this yields

$$\frac{\delta}{2} \leqslant \left| E_m \setminus \left(\bigcup_k B(x^k_m, R_0 + 2) \right) \right| < C_n r_1^n. \tag{3.25}$$

Applying the nucleation lemma a final time to the sets $E_m \setminus \left(\bigcup_k B(x_m^k, R_0+2)\right)$, we obtain finitely many points y_m^j fulfilling the conclusions of Lemma 2.1. We claim that it must be the case that

$$|y_m^j - x_m^k| \to \infty. \tag{3.26}$$

If $\limsup_{m\to\infty} |y_m^j - x_m^k| < \infty$, then the uniform bound from below on $|B(y_m^j, 1) \cap E_m|$ and the fact that $y_m^j \notin B(x_m^k, R_0+1)$ would imply that $E^k \cap B(0,R_0)^c \neq \emptyset$. However, this contradicts (3.22). Next, by the compactness for sets of finite perimeter and the fourth step, we may find a measurable set E and $R_1 > 0$ such that $E \subset\subset B(0,R_1)$, $E_m - y_m^1 \overset{\text{loc}}{\to} E$, and

$$|E| \geqslant C_n r_1^n. \tag{3.27}$$

Since E is compactly supported, $|E_m \cap B(y_m^1, R_1)| \to |E|$. But (3.26) implies that $B(y_m^1, R_1) \subset \left(\bigcup_k B(x_m^k, R_0+2)\right)^c$ for large m, and hence

$$\begin{aligned} |E| &= \lim_{m\to\infty} |E_m \cap B(y_m^1, R_1)| \\ &\leqslant \limsup_{m\to\infty} \left| E_m \setminus \left(\bigcup_k B(x_m^k, R_0+2) \right) \right| \\ &\overset{(3.25)}{<} C_n r_1^n. \end{aligned}$$

This upper bound is at odds with the lower bound (3.27), so we have derived a contradiction. Thus $\sum_{k=1}^K |E^k| = 1$.

Step six: At last we can prove Theorem 1.1. Let us choose any K points $z_1, \ldots, z_K \in \mathbb{R}^n$ such that $B(z_k, R_0 + C_n)$ are pairwise disjoint, where C_n is the dimensional constant from Theorem 2.10(ii). We claim that

$$\bigcup_{k=1}^K E^k + z_k$$

is a minimizer. The choice of radius $R_0 + C_n$ and Remark 2.11 ensure that the sets F^k defined by

$$\mathcal{W}_p(E^k + z_k) = W_p(E^k + z_k, F^k)$$

are pairwise disjoint and

$$\mathcal{W}_p^p\left(\bigcup_{k=1}^K E^k + z_k\right) = \sum_{k=1}^K \mathcal{W}_p^p(E^k).$$

Appealing to the continuity result Proposition 2.6 gives

$$\begin{aligned} \lambda \mathcal{W}_p^p\left(\bigcup_{k=1}^K E^k + z_k\right) &= \lambda \sum_{k=1}^K \mathcal{W}_p^p(E^k) \\ &= \lambda \lim_{m\to\infty} \sum_{k=1}^K \mathcal{W}_p^p((E_m - x_m^k) \cap B(0, R_0 + C_n)) \\ &\leqslant \lambda \liminf_{m\to\infty} \mathcal{W}_p^p(E_m), \end{aligned} \tag{3.28}$$

where the last inequality depends on Remark 2.11, the additivity of $\mathcal{W}_p^p$ (which applies since the distance between the x_m^k's goes to infinity as $m \to \infty$). Next, the inequality

$$\begin{aligned}\sum_{k=1}^{K} P(E^k + z_k) &= \sum_{k=1}^{K} P(E^k; B(0, R_0)) \\ &\leqslant \liminf_{m\to\infty} \sum_{k=1}^{K} P(E_m - x_m^k; B(0, R_0)) \\ &\leqslant \liminf_{m\to\infty} P(E_m) \end{aligned} \tag{3.29}$$

is immediate from the lower-semicontinuity of perimeter under L^1-convergence and the pairwise disjointness again. Summing (3.28) and (3.29) finishes the proof, since E_m is a minimizing sequence and $\left|\bigcup_{k=1}^K E^k + z_k\right| = 1$. □

As a byproduct of our existence proof we obtain that the set E in a minimizing pair (E, F) is a quasiminimizer of the perimeter in the following sense; hence, enjoys some regularity properties.

Corollary 3.1. *For any minimizing pair (E, F) to (1.1), the set E is a (Λ, r_0)-perimeter minimizer in $\mathbb{R}^n$. That is, there exists $0 \leqslant \Lambda < \infty$ and $r_0 > 0$ such that*

$$P(E) \leqslant P(\tilde{E}) + \Lambda|E \triangle \tilde{E}| \qquad \text{if } E \triangle \tilde{E} \subset\subset B(x, r),\ 0 < r < r_0.$$

Proof. The analogous inequality for the elements of the minimizing sequence $\{E_m\}$ was derived in (3.6) with an added factor of $1/m$, and the same proof applies to the minimizer E. □

Remark 3.2 (Regularity of minimizers)**.** The classical theory of (Λ, r_0)-perimeter minimality implies that $\partial^* E \in C^{1,\gamma}$ for any $\gamma \in (0, 1/2)$ and the Hausdorff dimension of $\partial E \setminus \partial^* E$ is at most $n - 8$ (see e.g. [Mag12, Theorem 26.3]). This regularity was also observed in [BCL20, Theorem 4.6]. Also, by [BCL20, Theorem 3.13], F is a set of finite perimeter.

Remark 3.3. Alternatively, one could attempt to demonstrate the existence of minimizers using the framework developed in [FL15]. This would require proving that the binding inequality

$$e(M) < e(M') + e(M - M')$$

holds for all $0 < M' < M$, where $e(M) = \inf\left\{P(E) + \mathcal{W}_p(E) \colon |E| = M\right\}$.

Appendix A. Proof of Lemma 2.3

The argument is a straightforward modification of the case where there is one set E [Mag12, Lemma 17.21], as opposed to a sequence.

Proof of Lemma 2.3. Since $\mathcal{H}^{n-1}(\partial^* E \cap A) > 0$, we can find $T \in C_c^\infty(A;\mathbb{R}^n)$ with

$$\gamma := \int_E \operatorname{div} T\, dx = \int_{\partial^* E} T \cdot \nu_E \, d\mathcal{H}^{n-1} > 0.$$

By the L^1_{loc}-convergence of E_m to E, for m large enough, we have

$$\frac{\gamma}{2} < \int_{E_m} \operatorname{div} T\, dx = \int_{\partial^* E_m} T \cdot \nu_{E_m} \, d\mathcal{H}^{n-1} < 2\gamma. \tag{A.1}$$

Let $\varphi_t(x)\colon \mathbb{R}^n \times (-\delta,\delta) \to \mathbb{R}^n$ be a one parameter family of diffeomorphisms with initial velocity T. By the first variation formulae for perimeter and volume (see e.g. [Mag12, Chapter 17]), there exists $\delta_0 > 0$ such that for all $|t| \leqslant \delta_0$,

$$|P(\varphi_t(E_m);A) - P(E_m;A)| \leqslant 2|t|P(E_m;A)\|\nabla T\|_{L^\infty}, \tag{A.2}$$

$$|\varphi_t(E_m) \cap A| = |E_m \cap A| + t\int_{\partial^* E_m} T \cdot \nu_{E_m}\, d\mathcal{H}^{n-1} + O(t^2), \tag{A.3}$$

where the decay rate in t in the second equality depends on T and is thus uniform in m. Also, by (A.1) and (A.3), $|\varphi_t(E_m) \cap A|$ is strictly increasing on $[-\delta_0, \delta_0]$ with

$$\big|\, |\varphi_t(E_m) \cap A| - |\varphi_{t'}(E_m) \cap A| \,\big| \geqslant \frac{\gamma}{4}|t - t'| \tag{A.4}$$

(after decreasing δ_0 if necessary). Therefore, we have the inclusion

$$\left(-\frac{\delta_0\gamma}{4}, \frac{\delta_0\gamma}{4}\right) \subset \big\{|\varphi_t(E_m) \cap A| - |E_m \cap A| \colon |t| \leqslant \delta_0\big\}.$$

So for all $|\sigma| < \sigma_0 := \delta_0\gamma/4$, there exists $t_m = t_m(\sigma) \in (-\delta_0, \delta_0)$ such that

$$|\varphi_t(E_m) \cap A| = |E_m \cap A| + \sigma. \tag{A.5}$$

By (A.4), it must be the case that

$$|t_m| < \frac{4\sigma}{\gamma}. \tag{A.6}$$

Then defining $G_m = \varphi_{t_m}(E_m)$, it follows from (A.5) and (A.2), (A.6) that

$$|G_m \cap A| = |E_m \cap A| + \sigma, \quad |P(G_m;A) - P(E_m;A)| \leqslant C_0|\sigma|,$$

where C_0 depends on $M = \sup P(E_m;A)$, A, and E. The estimate

$$|G_m \triangle E_m| \leqslant C_0|\sigma|$$

can be found in [Mag12, Lemma 17.9] in the form

$$|\varphi_{t_m}(E_m) \triangle E_m| \leqslant C(T)\, |t_m|\, P(E_m;A).$$

Hence, the result is established. □

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