Math 307: Multivariable Calculus

SOLUTIONS FOR THE SAMPLE FINAL EXAM

(The questions may be downloaded and printed separately)

1. Start with the first-order partial derivatives:

$$f_x = -6xz\sin(x^2 - y), \quad f_y = 3z\sin(x^2 - y), \quad f_z = 3\cos(x^2 - y)$$

Now proceed to the second partial derivatives:

$$\begin{aligned} f_{xx} &= -6z\sin(x^2 - y) - 12x^2z\cos(x^2 - y) \\ f_{xy} &= f_{yx} = 6xz\cos(x^2 - y), \quad f_{yy} = -3z\cos(x^2 - y) \\ f_{yz} &= f_{zy} = 3\sin(x^2 - y), \quad f_{zz} = 0. \end{aligned}$$

2. The given surface is a level surface of the function $F(z, y, z) = xy+yz+zx$. Take partial derivatives:

$$F_x = y + z, \quad F_y = x + z, \quad F_z = y + x$$

The equation of the tangent plane is:

$$\begin{aligned} F_x(1, 2, 1/3)(x - 1) + F_y(1, 2, 1/3)(y - 2) + F_z(1, 2, 1/3)(z - \frac{1}{3}) &= 0 \\ \frac{7}{3}(x - 1) + \frac{4}{3}(y - 2) + 3(z - \frac{1}{3}) &= 0 \\ 7x + 4y + 9z &= 18. \end{aligned}$$

3. Using the chain rule:

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = -y\sin(xy) - \cos y + 2u[-x\sin(xy) + x\sin y] \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -(2v)[y\sin(xy) + \cos y] - x\sin(xy) + x\sin y. \end{aligned}$$

4. The Lagrange system for this problem is:

$$4 = 2x\lambda, \quad -2 = 2y\lambda, \quad -2 = 2z\lambda, \quad x^2 + y^2 + z^2 = 6.$$

$$x = \frac{2}{\lambda}, \qquad y = \frac{-1}{\lambda}, \qquad z = \frac{-1}{\lambda}, \qquad \left(\frac{2}{\lambda}\right)^2 + \left(\frac{-1}{\lambda}\right)^2 + \left(\frac{-1}{\lambda}\right)^2 = 6$$

The last equation gives $\lambda = \pm 1$. These values give two critical points that can be checked for max/min as follows:

$$\lambda = 1 \to (2,-1,-1) \to f(2,-1,-1) = 15 \quad \text{(Max)}$$
$$\lambda = -1 \to (-2,1,1) \to f(-2,1,1) = -9 \quad \text{(Min)}.$$

5. The region of integration in the given integral is of Type II:

$$D = \{(x,y) : 0 \le y \le 1, \quad y^2 \le x \le 1\}$$

This can be re-written as Type I (you may find it helpful to sketch D):

$$D = \{(x,y) : 0 \le x \le 1, \quad 0 \le y \le \sqrt{x}\}$$

Now we calculate the integral in reverse order:

$$\begin{aligned}
\int_0^1 \int_{y^2}^1 y\cos(x^2)\,dxdy &= \int_0^1 \int_0^{\sqrt{x}} y\cos(x^2)\,dydx \\
&= \int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{x}} \cos(x^2)dx \\
&= \frac{1}{2}\int_0^1 x\cos(x^2)\,dx \quad (u = x^2,\ du = 2xdx) \\
&= \left[\frac{1}{4}\sin(x^2)\right]_{x=0}^{x=1} = \frac{\sin 1}{4}.
\end{aligned}$$

6. (a) The region of integration

$$D = \left\{(x,y) : 0 \le y \le a,\ -\sqrt{a^2-y^2} \le x \le \sqrt{a^2-y^2}\right\}$$

is the upper half of the disk $x^2 + y^2 \le a^2$. In polar coordinates this is

$$D = \{(r,\theta) : 0 \le \theta \le \pi,\ 0 \le r \le a\}$$

So the integral becomes

$$I = \int_0^\pi \int_0^a \frac{2}{\sqrt{4+r^2}}\,rdrd\theta = \int_0^\pi d\theta \int u^{-1/2}du = 2\pi(\sqrt{4+a^2} - 2)$$

(b) Setting $I = \pi a$ gives the equation $2\pi(\sqrt{4+a^2} - 2) = \pi a$ which we can solve for a:

$$\begin{aligned} 2(\sqrt{4+a^2} - 2) &= a \\ \sqrt{4+a^2} &= \frac{a}{2} + 2 \\ 4 + a^2 &= \left(\frac{a}{2} + 2\right)^2 = \frac{a^2}{4} + 2a + 4 \\ \frac{3a^2}{4} &= 2a \to a = 0, \frac{8}{3} \end{aligned}$$

Therefore, a positive solution is $a = 8/3$.

7. The density in this problem is $\rho(x, y) = ky$ where k is a positive constant. Hence

$$m = \iint_D ky\, dA = \int_{-1}^{1} \int_0^{1-x^2} ky\, dydx = \frac{k}{2} \int_{-1}^{1} (1-x^2)^2 dx = \frac{8k}{15}$$

where we used the expansion $(1-x^2)^2 = 1 - 2x^2 + x^4$ to do the last integral. Now we compute the coordinates of the center of mass. First, $\bar{x} = 0$ since the density $\rho = ky$ does not depend on the x-coordinate and also since the region D under the parabola is symmetric with respect to the y-axis (drawing D may be helpful to you). Next,

$$\bar{y} = \frac{1}{m} \iint_D ky^2\, dA = \frac{15}{8} \int_{-1}^{1} \int_0^{1-x^2} y^2\, dydx = \frac{15}{24} \int_{-1}^{1} (1-x^2)^3 dx = \frac{4}{7}$$

where we used the expansion $(1-x^2)^3 = 1 - 3x^2 + 3x^4 - x^6$ to do the last integral.

8. The volume over the rectangle $0 \le x \le 2$ and $0 \le y \le 1$ is given by the double integral

$$V = \iint_D z\, dA = \int_0^2 \int_0^1 (6 - x^2 - 2y^2) dydx = \int_0^2 \left(\frac{16}{3} - x^2\right) dx = 8.$$

9. The triple integral is calculated as follows:

$$\iiint_E 12y\,dV = \int_0^1 \int_0^1 \int_0^{1+x-y^2} 12y\,dzdydx$$
$$= \int_0^1 \int_0^1 12y(1+x-y^2)dydx$$
$$= \int_0^1 (3+6x)dx = 6.$$

10. (a) $\mathbf{F}(\mathbf{r}(t)) = \langle \cos(\pi t), -\sin(\pi t), -2t\rangle$ and $\mathbf{r}'(t) = \langle \pi\cos(\pi t), -\pi\sin(\pi t), 1\rangle$ so work is calculated as

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle \cos(\pi t), -\sin(\pi t), -2t\rangle \cdot \langle \pi\cos(\pi t), -\pi\sin(\pi t), 1\rangle\,dt$$
$$= \int_0^1 (\pi - 2t)dt = \pi - 1.$$

(b) $\mathbf{F}(\mathbf{r}(t)) = \langle t-2, t-1, -2t\rangle$ and $\mathbf{r}'(t) = \langle -1, 1, 1\rangle$ so work is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle t-2, t-1, -2t\rangle \cdot \langle -1, 1, 1\rangle\,dt$$
$$= \int_0^1 (1-2t)dt = 0.$$

11. (a) $\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{i} + (t+1)^{-2}\mathbf{j}$ so

$$\mathbf{v}(t) = \mathbf{r}'(t) = \int [\mathbf{i} + (t+1)^{-2}\mathbf{j}]dt = t\mathbf{i} - (t+1)^{-1}\mathbf{j} + \vec{C}_1$$

Since $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$ we calculate $\vec{C}_1 = \mathbf{i} + \mathbf{k}$ and

$$\mathbf{r}'(t) = (t+1)\mathbf{i} - (t+1)^{-1}\mathbf{j} + \mathbf{k}.$$

Thus

$$\mathbf{r}(t) = \int [(t+1)\mathbf{i} - (t+1)^{-1}\mathbf{j} + \mathbf{k}]dt = \frac{(t+1)^2}{2}\mathbf{i} - \ln|t+1|\,\mathbf{j} + t\mathbf{k} + \vec{C}_2$$

Using $\mathbf{r}(0) = \mathbf{0}$, we calculate $\vec{C}_2 = -(1/2)\mathbf{i}$. Therefore,

$$\mathbf{r}(t) = \frac{t}{2}(t+2)\mathbf{i} - \ln|t+1|\,\mathbf{j} + t\mathbf{k}.$$

(b) At $t = 1$, velocity is $\mathbf{v}(1) = 2\mathbf{i} - (1/2)\mathbf{j} + \mathbf{k}$, speed is $|\mathbf{v}(1)| = \sqrt{21}/2$ and the location of the particle is $\mathbf{r}(1) = (3/2)\mathbf{i} - (\ln 2)\mathbf{j} + \mathbf{k}$. The curvature at $t = 1$ is determined as follows

$$\kappa(1) = \frac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \frac{8\,|(1/4)\mathbf{i} + \mathbf{j} + \mathbf{k}|}{21\sqrt{21}} = \frac{2\sqrt{33}}{21\sqrt{21}}.$$

12. (a) The temperature gradient at the point $(1, 2, 0)$ is computed as

$$\nabla T = \left\langle 2e^{2x-y+z}, -e^{2x-y+z}, e^{2x-y+z} \right\rangle \to \nabla T(1, 2, 0) = \langle 2, -1, 1 \rangle$$

The vector $\mathbf{v} = 3\mathbf{i} - 4\mathbf{k}$ has length $|\mathbf{v}| = 5$ so the unit vector along $\mathbf{v}$ is $\mathbf{u} = \langle 3/5, 0, -4/5 \rangle$. Now the directional derivative, or the rate of change of temperature is

$$D_{\mathbf{u}}T(1, 2, 0) = \langle 2, -1, 1 \rangle \cdot \left\langle \frac{3}{5}, 0, -\frac{4}{5} \right\rangle = \frac{2}{5} = 0.4.$$

(b) The direction of maximum increase in temperature is the same as the direction of the gradient $\nabla T(1, 2, 0)$. Therefore, the maximum possible rate of increase in temperature is $|\nabla T(1, 2, 0)| = \sqrt{6} \approx 2.45$.