TAKING MATH TO THE HEART: Pulse Propagation in a Discrete Ring

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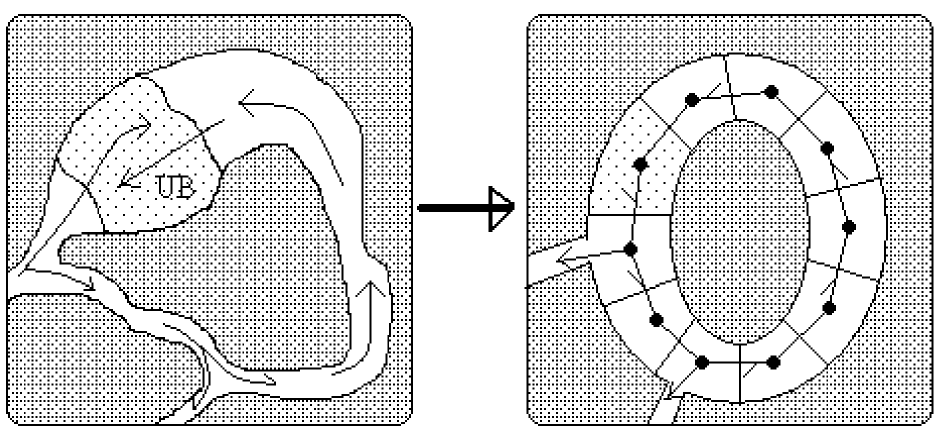
The Discrete Ring or Loop

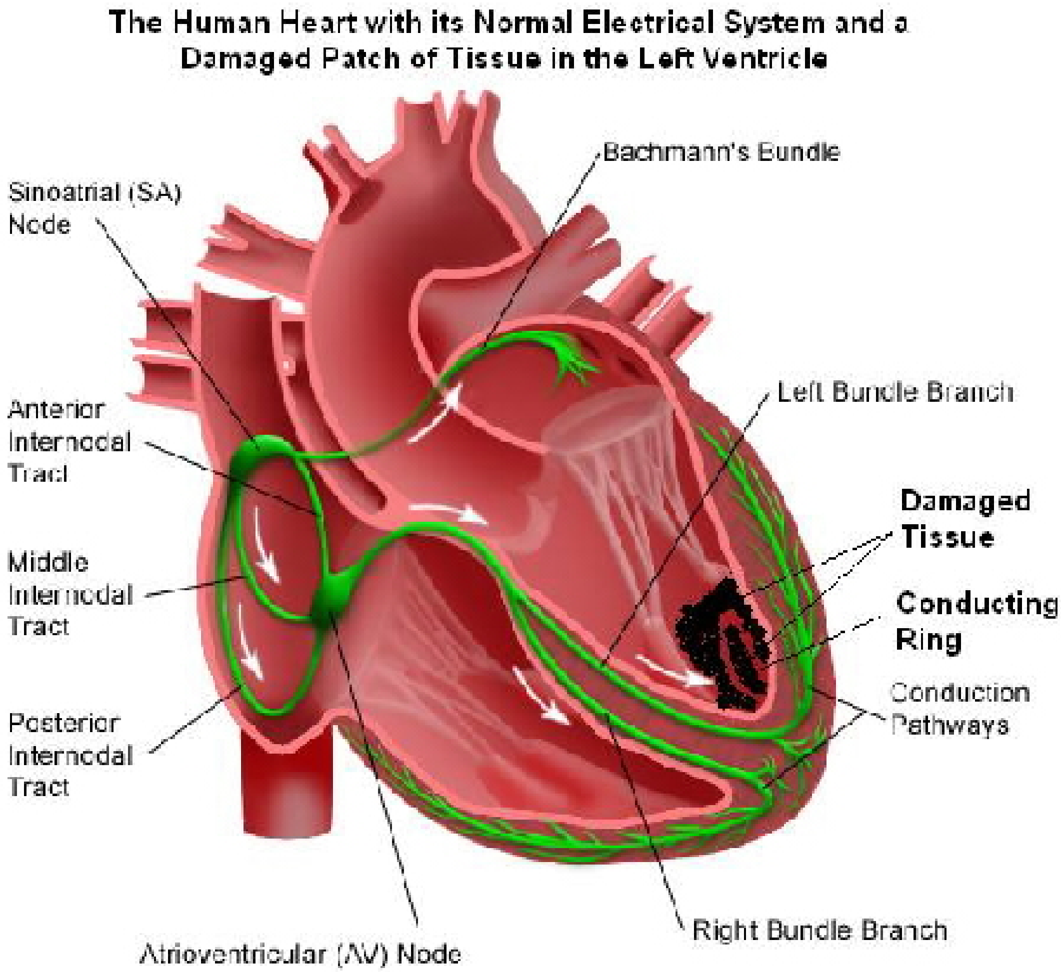
A (discrete) ring of "excitable media" is a closed, finite network of cells capable of generating and conducting electro-chemical signals, or pulses called *action potentials*. Prominent examples of such networks include circuits of neurons in the brain and the nervous system, and also loops of cardiac cells within the heart (we use the heart for the scientific context of this note).



A ring of conducting tissue within a patch of damaged cardiac cells

The schematized ring shown as an annulus, and its abstraction shown as a cyclic graph or network

The next picture shows what such a ring might look like in the heart (black patch at the bottom of the left ventricle):



Reentrant Arrhythmias

The existence of a conducting ring of tissue may cause a form of *tachyarrhythmia*, i.e., abnormal fast rhythm in heart beats. Certain types of tachyarrhythmias may lead to cardiac arrest and sudden death if not immediately stopped.

A form of potentially life-threatening tachyarrhythmia that occurs in a conducting loop is called *reentrant*. The loop itself is called a *reentrant circuit*.

The term reentrant refers to the reentry of an action potential pulse in the loop throught a region of *unidirectional block (UB)* - shaded lightly in the figure. The UB region stops propagation in one direction, thus permitting a pulse to reenter the loop without being stopped by an opposing pulse.

The Ring as a Discrete Dynamical System

The propagation of a reentrant pulse in a loop can be modeled mathematically using difference equations on a discrete space, namely, the abstracted closed network that represents the loop.

The loop is divided into m cell aggregates, or units, that constitute a finite space. An m-dimensional dynamical system is defined on this discrete space where each time unit corresponds to one complete cycle, i.e., once around the loop.

Each vector in the state space has the form

$$\mathbf{DI}_n = (DI_{1,n}, \ldots, DI_{m,n})$$

where non-negative real numbers $DI_{i,n}$ give the "diastolic intervals" or the rest periods for each unit $i = 1, \ldots, m$.

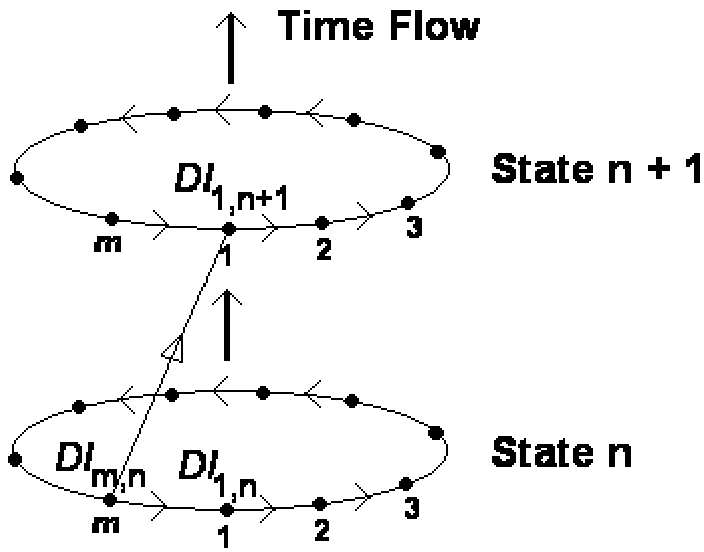
Here is a schematic view of the evolution of the states; in the case of the heart, each state corresponds to one beat. Reentry of the action potential pulse in the loop at unit #1 completes a cycle which is then repeated.

During each beat, the pulse moves outwards, away from the loop and neutralizes the normal action of heart's pacemakers (the self-oscillatory "Sinoatrial or SA" node and the "atrioventricular or AV" node).

If a reentrant pulse takes over, the normal SA beat rate of about 70 bpm is replaced by a fast rate that often exceeds 200 bpm.

The existence of an anomalous ring is not necessary for the initiation of reentry and the occurrence of tachyarrhythmia. A UB region is also required. The imbalance of *ionic current generated* by a unit in the UB region and the *activation current threshold* of the adjacent unit is one of the possible causes for the occurrence of the UB region.

Even if the UB region is active, other factors such as the *concduction velocity CV* of the pulse (also called the "speed of the wavefront") or the *action potential duration, APD* may inhibit the initiation of reentry.



There may also be several rings in multiple locations in the heart whose actions may cancel each other if reentry does initiate in some of them.

If reentry is initated in a ring, it may terminate spontaneously because of several factors, including the inactivation of the UB feature.

Sustained Reentry

If reentry initiates and does not terminate internally, then we have *sustained reentry*. This can be stopped by means of electrical shocks to the heart, either externally or by implantable defibrillator devices.

Sustained reentry in a loop can be modeled by means of a *higher order, nonlinear difference equation* that is obtained from a special system of partial difference equations called a "coupled map lattice".

1. The coupled map lattice (CML):

The duration or length of each beat or cycle n is the time that it takes the pulse to complete one turn around the loop from a given node back to that node again.

For each node i this length of time is on the one hand:

$$APD_{i,n} + DI_{i,n}$$

and on the other hand it is:

$$\sum_{j=1}^{i-1} CT_{j,n+1} + \sum_{j=i}^{m} CT_{j,n}.$$

Here APD is the *action potential duration* and CT is the *conduction time* of the action potential pulse (or the wavefront) across a unit. Setting the two quantities above equal to each other, gives the coupled-map lattice.

2. The restitution functions:

The two quantities APD and CD are functions of DI. In the simplest case, we have:

$$\begin{aligned} APD_{i,n} &= A(DI_{i,n-1}) \\ CT_{i,n} &= \Delta L_i C(DI_{i,n-1}) \end{aligned}$$

where the single-variable functions A, C are, respectively, the *restitution of* APD and the *restitution of* CD. The function A is increasing and the function C is decreasing.

The numbers ΔL_i, $i = 1, \ldots, m$ are the physical lengths of the m units that make up the ring. The CML

$$\begin{aligned} &\sum_{j=1}^{i-1} \Delta L_j C(DI_{j,n}) + \sum_{j=i}^{m} \Delta L_j C(DI_{j,n-1}) \\ &= A(DI_{i,n-1}) + DI_{i,n}, \quad i = 1, \ldots, m \end{aligned}$$

is a system of m *partial difference equations*.

For *numerical simulations,* exponential type functions are commonly used to define A and C. For example,

$$\begin{aligned} A(t) &= a - be^{-\sigma t} \\ C(t) &= c + de^{-\omega t} \end{aligned}$$

where the positive real numbers $a, b, \sigma, c, d, \omega$ are parameters of the model that can be determined from curve fitting of experimental data or from the solutions of partial differential equations used to model the physiological aspects of each cell.

3. CML reduced to an ordinary difference equation:

Make a change of variables

$$x_{mn+i} = DI_{i,n}, \quad \delta_{mn+i} = \Delta L_i$$

for all $n = 0, 1, 2, \ldots$ and use the restitution functions in the CML to obtain the ordinary difference equation

$$x_{mn+i} = \sum_{j=mn-m+i}^{mn+i-1} \delta_j C(x_j) - A(x_{mn-m+i})$$

This can be re-written in a more conventional form by setting $k = mn + i$ to get

$$x_k = \sum_{j=k-m}^{k-1} \delta_j C(x_j) - A(x_{k-m}).$$

This is the *pulse propagation equation* in sustained reentry mode. Note that this equation is also a higher order, ordinary difference equation.