Enforcement in Private vs. Public Externalities

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Abstract

Law and enforcement policy is among the key elements of a civil society that ensures the achievement of a higher social welfare. In this paper, I study socially optimal law and enforcement policy making under two different environments. In the first environment, private externalities, an activity a person engages harms equally likely everyone in the society. In the second environment, public externalities, it harms the whole society. I show that social welfare function of these two problems are the same under certain conditions. Polinsky and Shavell [4] show that the optimal level of punishment in equilibrium is such that expected level of punishment is less than the harm it causes. I generalize their result to public and private externality environments where all agents are either risk neutral or risk averse with respect to uncertainties in harms they face. On the other hand, by allowing private and public externality acts in the same environment, I show that even though contribution of agents to the public harm is greater than harm they may cause by choosing private externalities, the punishment level of a private externality may be greater than the punishment level of public externality if agents are sufficiently risk averse, which is different from a result in Shavell [5]. This result shows that the distinction between private and public externalities is important.

1 Introduction

In civil societies, there are some rules for the protection of both individuals' and society's interests against individuals' or groups' actions harming those interests. A particular way of implementing these rules is through deterrence by punishment. These rules, as well as the ways and levels of punishment for rule violation are usually formed in various ways like copying and adapting them from another society or coming up with new ways and forms through common sense in a particular society.

In economics, beginning with Becker [1], several papers have been written suggesting how to choose the level of punishment and the level of detection so that it would be optimal from the law maker's point of view. Becker [1] says that the aim can be minimizing social cost, which consists of harms caused by agents, costs of detection, and the costs of punishment to the criminals less gains of criminals. Individuals and their choices are implicitly explained, and they get utility only from consumption, and face uncertainty only because of the possibility of getting caught. He shows that the probability of detecting the crime should be set to minimum possible and the level of punishment should be as high as possible, for example total wealth of an individual. His result is based on the risk neutrality of agents. In Polinsky and Shavell [3], they study risk averse agents as well as risk neutral agents. In their model, an agent derives utility from his consumption. If he does not engage in the externality creating activity then his consumption consists of his wealth remaining from taxes and insurance premium which covers all losses due to externalites caused by individuals' actions in the society. If he engages in the activity then his consumption, in addition to taxes and insurance premium paid, will increase with his gain from the activity and will decrease with fine paid unless he does not get caught. Externalities created is felt equally likely by everyone in the society but if agents are risk neutral then premium paid can also be seen as the risk they bear due to possible harms caused by others in the society. Since Polinsky and Shavell [3] use continuum (measure one) of people, it is possible to see their model as the one in which the whole society faces externalities without any uncertainty, and the level of harm they face is given by the harm a single activity causes, and the ratio of people engaging in the activity. The objective of the law maker in Polinsky and Shavell [3] is to maximize the total expected utility of individuals in the society[1]. They show that, if individuals are risk neutral it is optimal to set probability of detection to a minimum level below which it is not possible to detect any crime, and optimal to set the punishment level as high as possible which is constrained by wealth of individuals. They have

[1]Cooter and Ulen [2] (p. 443) suggest that the aim of the law maker should be minimization of the social cost which consists of costs of protection and the net harm caused while crime is committed.

two results when agents are risk averse. They say that these results explain why the previous result is not realistic. The first one is that, probability of detecting should be set to 1 when individuals are risk averse, and cost of catching is sufficiently small. The second one is that, it may not be optimal to set low probability to detecting, and setting the punishment levels very high even if cost of catching is very large when individuals are risk averse.

In Polinsky and Shavell [4], the objective of the law maker is to maximize total expected utility of individuals in the society. The expected utility of an agent consists of gain from engaging in the activity, expected loss due to the possibility of getting caught, expected loss due to the possibility of being a victim of a crime, and the per capita cost of enforcement. If the agent does not engage in the activity, then the expected utility will not include the gain from engaging in the activity, and the expected punishment. They show that in equlibrium, expected punishment for activity will always be less than the harm caused by it.

In this paper I define public and private externality environments formally, and find what utility functions of agents in these environments become under certain assumptions (these assumptions greatly simplifies utility functions, and make it possible to analyze models, and derive results). I also show that these two environments are equivalent from social planner's point of view.

2 The Single-Act Model

A significant reason for societies to have laws and enforcement is that people in the society fear from possible negative externalities that may be caused by others in the society. Laws exist, of course together with the enforcement, to achieve a socially optimal state of the world by decreasing the possibility of individuals being hurt, i.e. by deterring agents from taking certain actions. An agent's utility function is $u_i(c_i, a_i, h_i)$ where $c_i \in \mathbb{R}_+^S$ is the consumption plan of the individual. A state $s \in S$ is determined by the set of all offender-sufferer pairs and the set of all offenders captured. $a_i \in \mathcal{A} = \{a_0, a\}$ [2] is the action taken by agent i. Agent i can either choose action a that gives pleasure to him/her and causes negative externality to others (to either one person or to all society, depending on the type of the action), or choose action a_0 that does not change anybody's utility. The externality i faces is denoted by a random variable h_i. Each realization of h_i determines the group of people harming (taking action a against) agent i. For any agent i, the set of people taking action a he/she faces, i.e. the set of people of who are likely to hurt agent i, is denoted by N_a^i and is defined as $N_a^i = N_a \setminus \{i\}$ where N_a is the set of people who are taking action a in the society. If the action is public externality then h_i is the same for all agents as everyone suffers from the same externality. There is a single consumption good, and it is assumed that utility is an increasing function of consumption. Consumption of an agent consist of his initial endowment, and transfers made to him less per capita cost of enforcement if he did not take any illegal action. Otherwise, if the agent takes an illegal action and he gets caught then his consumption is the amount of endowment, and transfers made to him remaining from the punishment, and per capita cost of enforcement. If the agent takes an illegal action and he does not get caught then he consumes his endowment and transfers made to him.

2.1 Private Externalities

Private externality is a negative externality that affects any and only one person. Property crime is a good example for that kind of externality. DWI is another good example. In both of these examples, at any point in time, although offender threatens every individual in the society, it will possibly

[2] $\mathcal{A}$ stands for the set of all possible actions. The set of illegal actions will be chosen by the law maker, and it is a subset of $\mathcal{A}$. Note that I asssume that an individual can choose only one action at a time, as it has always been done in the literature, so multiple actions are not allowed. An action is illegal if and only if its punishment is positive.

hurt only one individual. If actions agents take cause *private externalities* , then agent j will offend agent i by taking action a with given probability [3] π_{ji}. Note that $\sum_{i \in N} \pi_{ji} = 1$ for any j. So the probability that agent i will be offended by a group of $R \subseteq N_a^i$ people is $\beta(R, N_a^i) = \prod_{j \in R} \pi_{ji} \prod_{l \in N_a^i \setminus R}(1 - \pi_{li})$. If N_a^i is the set of offenders agent i faces, and r of them offend agent i, then $\alpha(p, r, k) = p^k(1-p)^{r-k}\binom{r}{k}$ is the probability that any k of these r people will be captured, and punished. Revenue from punishments of these k agents then will be transferred to agent i. So $c_i^{(k,0)}$ stands for the consumption level of agent i when he/she gets caught for taking the action a. If the agent i takes the action a_0 or takes the action a but he/she does not get caught then his/her consumption is $c_i^{(k,1)}$. Note that k stands for the transfers from those k people who are caught after hurting agent i. His budget constraint is $c_i^{(k,0)} + f \leq w_i - c(p) + kf$ if he takes the action a, and gets caught, and it is $c_i^{(k,1)} \leq w_i - c(p) + kf$ if he takes action a_0, or he takes action a and does not get caught. In both cases agent i gets compensated by k criminals' punishments, kf as these are the only criminals caught among those who hurt agent i. Each agent choosing action a will cause harm e. So, given N_a^i, the set of offenders, other than agent i himself, $|R|e$ is the total externality i faces when people in $R \subseteq N_a^i$ offend him. The subset of offenders R agent i faces, the set of agents among R caught, and whether agent i caught if he took action a determine the state agent i faces.

The expected utility function of agent i will then be

$$\begin{aligned} Eu_i(c_i, a_i, h_i) = p &\sum_{R \subseteq N_a^i} \beta(R, N_a^i) \sum_{k=0}^{|R|} \alpha(p, |R|, k) u_i(c_i^{(k,0)}, a_i, e|R|) \\ &+ (1-p) \sum_{R \subseteq N_a^i} \beta(R, N_a^i) \sum_{k=0}^{|R|} \alpha(p, |R|, k) u_i(c_i^{(k,1)}, a_i, e|R|) \end{aligned} \tag{1}$$

Note that if agent i takes action a_0 then the expected utility function above will become

$$Eu_i(c_i, a_i, h_i) = \sum_{R \subseteq N_a^i} \beta(R, N_a^i) \sum_{k=0}^{|R|} \alpha(p, |R|, k) u_i(c_i^{(k,1)}, a_i, e|R|) \tag{2}$$

as $c_i^{(k,0)} = c_i^{(k,1)}$ *for all* k.

[3]The distribution of crime is to be determined endogeneously according to ratios of individuals who choose actions. Note that the uncertainty presented here is endogeneous. It may be an interesting problem and may be worth studying decision making under this type of uncertainty.

2.2 Public Externalities

Public externality is a negative externality that affects everybody in the society. Air pollution is a good example for public externality. If actions agents take cause *public externalities* then each one of these agents can cause a harm level of e_P, which will affect everybody. So the harm level agent i faces is $h_i = e_P|N_a|$, e.g. total level of harm.

Let $\alpha(p, N_a^i, r) = p^r(1-p)^{|N_a^i|-r}\binom{|N_a^i|}{r}$ denote the probability that any r of the $|N_a^i|$ people commiting crime will be caught, and punished. Revenue from punishment of an offender is transferred to every other agent in the society equally. Agent's budget constraint is $c_i^{(r,0)} + f \leq w_i - c(p) + \frac{rf}{N-1}$ if he takes the action a, and gets caught , and it is $c_i^{(r,1)} \leq w_i - c(p) + \frac{rf}{N-1}$ if he takes action a_0, or he takes action a, and does not get caught. In both cases agent i gets compensated by r criminals' punishments, $\frac{rf}{N-1}$ as these are the only criminals caught who hurt agent i together with the rest of the society. Agent i takes N_a, the number of people commiting crime, i.e. taking action a, other than agent i himself, as given. w_i is the wealth of the agent, p is the objective probability[4] of getting caught if an illegal action is taken, $c(p)$ is the per capita cost of detection [5] [6] [7] with probability p, and f is the punishment [8] for action a. The punishment $f \in \mathbb{R}_+$ is positive if the action a is considered as illegal.

[4] In a more general model, the probability of detection may depend on the action taken, and it can be denoted by a function $p : \mathcal{A} \to [0, 1]$.

[5] $c(p)$ is assumed to be an increasing function of p.

[6] As the number of people is fixed througout the model, the cost of maintaining probability of detection p is also fixed. It may be possible to think that the cost of maintaining probability p depends on the number of people both in the society and those commiting acts.

[7] Fines criminals pay are transferred to victims. This is assumed as it makes analysis simpler. If fines are considered as a part of the cost of enforcement then the taxes will be nondeterministic when there are finite number of people in the society. If agents have additively separable utility functions and they are risk neutral in consumption then this won't affect the optimal policy as fines paid by offender will be exactly offset by the gain of the individuals receiving the fine revenue.

[8] The punishment can very well depend on the income of the agent. Then the constraint in law maker's problem will become $0 \leq f(w_i) \leq w_i$ where w_i is agent i's income level. Whether the punishment should depend the level of income can be analyzed from the welfare and incentive compatibility point of views.

The expected utility function of an agent is [9] [10]

$$
\begin{aligned}
Eu_i(c_i, a_i, h_i) &= E[u_i(c_i, a_i, e_P|N_a|)] \\
&= p \sum_{r=0}^{|N_a^i|} \alpha(p, N_a^i, r) u_i(c_i^{(r,0)}, a_i, e_P|N_a|) \\
&+ (1-p) \sum_{r=0}^{|N_a^i|} \alpha(p, N_a^i, r) u_i(c_i^{(r,1)}, a_i, e_P|N_a|)
\end{aligned} \tag{3}
$$

2.3 Agent's Problem

Given the punishment level f, the probability of detection p, and the externality level h_i agent i faces, action a_i^* and the consumption plan c_i^* solve agent i's problem which is

$$\max_{a_i \in \mathcal{A}, c_i \in \mathbb{R}_+^S} Eu_i(c_i, a_i, h_i) \tag{4}$$

subject to the budget constraints for each state he faces.

2.4 Equilibrium

Given the punishment level and probability of detection, equilibrium is a sequence $\{c_i^*, a_i^*\}_{i \in N}$ of allocations and actions such that for each agent i, (c_i^*, a_i^*) is a solution to his/her problem defined above. The equilibrium utility level of an agent i is defined as $v_i(p, f) = Eu_i(c_i^*, a_i^*, h_i)$ where c_i^*,and a_i^* are solutions to agent i's problem given in 4.

2.5 Law Maker's Problem

I assume that the aim of the law maker is to choose the punishment level f for action a, and a probability of detection p so that total expected utility of individuals in the society in equilibrium will be maximized[11]. The level

[9] If the utility function is linear and additively separable in its arguments then the consumption can be discarded and the utility function can be redefined as a function of only actions taken, the punishment level for that action, and externality of other people.

[10] A more general model would be to consider not total harm caused by agents but actions of other agents separately.

[11] It may be easier (or not?) to use policy maker's problem in which the aim is to minimize the cost of enforcement subject to a certain social welfare

of punishment[12] f for action a is constrained by the net wealth of agents, and the probability of detection p is in $[0,1]$.

$$\max_{f,p} \sum_{i=1}^{N} Eu_i(c_i^*, a_i^*, h_i)$$

$$\textit{subject to} \quad 0 \leq f \leq \overline{w_i} \ \textit{for all } i$$

$$\textit{and} \quad 0 \leq p \leq 1$$

where $\overline{w_i}$ is the net wealth of individual i[13] , and a_i^* is the best response of individual i to punishment level f, and probability of detection p, and the externality he faces, h_i. It will be assumed that f is given through the paper.

[12] Other types of punishments are possible too. The constraint on the wages can be relaxed by implementing different levels of punishment on avegare life expectancy, e.g. different levels of restrictions on freedom. The aim here would be not only punish offenders but also to protect society if the offender is seen as a threat to the society. The tradeoff will be solved through social welfare maximization. Nonmonetary based punishments may be used for fairness purposes. For example, only giving ticket to high speeding may create an incentive for wealthy people to high speed. It is worth studying this problem.

[13] Here it will be equal to $w_i - c(p)$.

3 Separable Preferences

In this problem, given the probability of detection and punishment level, each agent faces a choice among different alternatives that causes externatilities to someone in the society or to the whole society. The aim here is to determine the set of actions which are to be treated as illegal and a socially optimal detection and punishment levels for this set of actions. An agent may choose not to choose any action. For simplicity, and for now I assume that agents can choose only one action[14] (other than not doing anything), and their utility functions are additively separable and they are risk neutral in consumption. In the next section, I will show the derivation of objective functions used in Polinsky and Shavell [4] model. The main differences are that I consider finite number of people model and that i make distinguish between Private and Public externalities.

3.1 Private Externalities

Theorem 1. *Assume that an agent has a utility function given in 1, and his utility function is additively separable and he is risk neutral with respect to uncertainties in consumption.*

Then his expected utility function becomes:

$$Eu_i(c_i, a_i, h_i) = v_i(a_i) - pf(a_i) - c(p) + pf \sum_{j \in N_a^i} \pi_{ji} - E[h_i] \tag{5}$$

where $v_i(a_i)$ is the utility i gets by taking action a_i, and $v_i(a_0) = 0$.

Lemma 1. *For any finite set of probabilities $\{\pi_j\}_{j \in J}$,*

$$\sum_{R \subseteq J} \Big(\prod_{j \in R} \pi_j \prod_{l \in J \setminus R} (1 - \pi_l) \Big) = 1.$$

[14] When there are more than one action, and it can be assumed that the ordering of harms caused by these actions is the same among agents. In this case, agents are generally deterred from more harmful actions to less harmful actions. These actions may also stand for different levels of a certain action, e.g. a factory that causes air pollution. This is called *marginal deterrence* (See Stigler [6]). The aim of the law maker would be to draw a line between illegal and legal actions, and determining levels of punishments and the level of detection (if it is optimal to use the same enforcement for them) so that social welfare will be maximized.

Proof. For any k, $k \in \{1, .., |J|\}$,

$$\begin{aligned}\sum_{R\subseteq J}\Big(\prod_{j\in R}\pi_j \prod_{l\in J\setminus R}(1-\pi_l)\Big) =&\pi_{j_k} \sum_{R\subseteq J\setminus\{j_k\}}\Big(\prod_{j\in R}\pi_j \prod_{l\in (J\setminus\{j_k\})\setminus R}(1-\pi_l)\Big)\\ &+(1-\pi_{j_k})\sum_{R\subseteq J\setminus\{j_k\}}\Big(\prod_{j\in R}\pi_j \prod_{l\in (J\setminus\{j_k\})\setminus R}(1-\pi_l)\Big)\\ =&\sum_{R\subseteq J\setminus\{j_k\}}\Big(\prod_{j\in R}\pi_j \prod_{l\in (J\setminus\{j_k\})\setminus R}(1-\pi_l)\Big).\end{aligned}$$

So,

$$\begin{aligned}\sum_{R\subseteq J}\Big(\prod_{j\in R}\pi_j \prod_{l\in J\setminus R}(1-\pi_l)\Big) =&\sum_{R\subseteq J\setminus\{j_1,j_2,..,j_{|J|-1}\}}\Big(\prod_{j\in R}\pi_j \prod_{l\in (J\setminus\{j_1,j_2,..,j_{|J|-1}\})\setminus R}(1-\pi_l)\Big)\\ =&\pi_{|J|}+1-\pi_{|J|}=1.\end{aligned}$$

□

Lemma 2. *For any finite set of probabilities* $\{\pi_j\}_{j\in J}$,

$$\sum_{R\subseteq J}\Big(\prod_{j\in R}\pi_j \prod_{l\in J\setminus R}(1-\pi_l)\Big)|R| = \sum_{j\in J}\pi_j.$$

Proof. For $|J| = 1$,

$$\sum_{R\subseteq J}\Big(\prod_{j\in R}\pi_j \prod_{l\in J\setminus R}(1-\pi_l)\Big)|R| = \pi_1.$$

Assume that for $|J| = k-1$,

$$\sum_{R\subseteq J}\Big(\prod_{j\in R}\pi_j \prod_{l\in J\setminus R}(1-\pi_l)\Big)|R| = \sum_{j=1}^{|J|}\pi_j.$$

For $|J| = k$,

$$\begin{aligned}\sum_{R\subseteq J}\Big(\prod_{j\in R}\pi_j \prod_{l\in J\setminus R}(1-\pi_l)\Big)|R| =&\pi_{j_{|J|}}\sum_{\{j_{|J|}\}\subseteq R\subseteq J}\Big(\prod_{j\in R\setminus\{j_{|J|}\}}\pi_j \prod_{l\in J\setminus R}(1-\pi_l)\Big)|R|\\ &+(1-\pi_{j_{|J|}})\sum_{R\subseteq J\setminus\{j_{|J|}\}}\Big(\prod_{j\in R}\pi_j \prod_{l\in (J\setminus\{j_{|J|}\})\setminus R}(1-\pi_l)\Big)|R|\\ =&\pi_{j_{|J|}}\sum_{R\subseteq J\setminus\{j_{|J|}\}}\Big(\prod_{j\in R}\pi_j \prod_{l\in (J\setminus\{j_{|J|}\})\setminus R}(1-\pi_l)\Big)\\ &+\sum_{R\subseteq J\setminus\{j_{|J|}\}}\Big(\prod_{j\in R}\pi_j \prod_{l\in (J\setminus\{j_{|J|}\})\setminus R}(1-\pi_l)\Big)|R|\\ =&\pi_{j_{|J|}}+\sum_{j=1}^{|J|-1}\pi_j\\ =&\sum_{j=1}^{|J|}\pi_j.\end{aligned}$$

□

Proof of Theorem 1. First note that $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$ for any integer n, and $\sum_{r=0}^{n} \binom{n}{r} (1-p)^{n-r} p^r r = pn \sum_{r=0}^{n-1} \binom{n-1}{r} (1-p)^{n-1-r} p^r$. By substitution of budget constraint, the expected utility function given in 1, by additive separability, and risk neutrality, becomes

$$\begin{aligned}
& p \sum_{R \subseteq N_a^i} \beta(R, N_a^i) \sum_{k=0}^{|R|} \alpha(p, |R|, k)(w_i - c(p) + kf - f(a_i) + v(a_i)) \\
& + (1-p) \sum_{R \subseteq N_a^i} \beta(R, N_a^i) \sum_{k=0}^{|R|} \alpha(p, |R|, k)((w_i - c(p) + kf + v(a_i)) - E[h_i] \\
= & \sum_{R \subseteq N_a^i} \beta(R, N_a^i) \sum_{k=0}^{|R|} \alpha(p, |R|, k)(w_i - c(p) + kf + v(a_i)) - pf(a_i) \sum_{R \subseteq N_a^i} \beta(R, N_a^i) \sum_{k=0}^{|R|} \alpha(p, |R|, k) - E[h_i] \\
= & (w_i - c(p) + v(a_i) - pf(a_i)) + \sum_{R \subseteq N_a^i} \beta(R, N_a^i) \sum_{k=0}^{|R|} \alpha(p, |R|, k)(kf) - E[h_i] \\
= & (w_i - c(p) + v(a_i) - pf(a_i)) + f \sum_{R \subseteq N_a^i} \beta(R, N_a^i) \sum_{k=0}^{|R|} \alpha(p, |R|, k) k - E[h_i] \\
= & (w_i - c(p) + v(a_i) - pf(a_i)) + f \sum_{R \subseteq N_a^i} \beta(R, N_a^i) p |R| \sum_{k=0}^{|R|-1} \alpha(p, |R|-1, k) - E[h_i] \\
= & (w_i - c(p) + v(a_i) - pf(a_i)) + pf \sum_{R \subseteq N_a^i} \beta(R, N_a^i) |R| - E[h_i] \\
= & (w_i - c(p) + v(a_i) - pf(a_i)) + pf \sum_{j \in N_a^i} \pi_{ji} - E[h_i] \\
= & (w_i - c(p) + v(a_i) - pf(a_i)) + pf \sum_{j \in N_a^i} \pi_{ji} - E[h_i] \\
\equiv & v(a_i) - pf(a_i) - c(p) + pf \sum_{j \in N_a^i} \pi_{ji} - E[h_i]
\end{aligned}$$

□

3.1.1 Risk Neutral Agents

Corollary 1. *Assume that an agent has a utility function given in 1, and his utility function is additively separable and he is risk neutral with respect to uncertainties in consumption and in harm he faces, the level of private externality an agent can cause over another agent is* e*, and is the same for all agents. Then his expected utility function becomes:*

$$Eu_i(c_i, a_i, h_i) = v_i(a_i) - pf(a_i) - c(p) - e \sum_{j \in N_a^i} \pi_{ji} + pf \sum_{j \in N_a^i} \pi_{ji} \quad (6)$$

where $v_i(a_i)$ is the utility i gets by taking action a_i, and $v_i(a_0) = 0$.

Note that an agent, say i, will take the action which causes an externality if and only if $v_i(a) > pf(a)$.

3.1.2 Risk Averse Agents

In previous section, I assumed that agents are risk neutral both in consumption and harm they face. Now, I will study the case where agents are risk averse in harms they face.

Corollary 2. *Assume that an agent has a utility function given in 1, and his utility function is additively separable and he is risk neutral with respecto to uncertainties in consumption, risk averse with respect to uncertainties in harm he faces, and the level of private externality an agent can cause over another agent is e, and is the same for all agents.*

Then his expected utility function becomes:

$$Eu_i(c_i, a_i, h_i) = v_i(a_i) - pf(a_i) - c(p) + pf \sum_{j \in N_a^i} \pi_{ji} - \sum_{R \subseteq N_a^i} \beta(R, N_a^i) h_i(|R|e) \tag{7}$$

where $v_i(a_i)$ is the utility i gets by taking action a_i, and $v_i(a_0) = 0$.

3.2 Public Externalities

The model here is the same as the previous one except that an action taken by an individual will affect everyone in the society.

Theorem 1. *Assume that an agent has a utility function given in 3, and his utility function is additively separable and he is risk neutral in consumption, the level of public externality an agent can cause is e_P, and is the same for all agents. Then his expected utility function becomes:*

$$Eu_i(c_i, a_i, h_i) = v_i(a_i) - pf(a_i) - c(p) + \frac{|N_a^i|}{|N-1|} pf - h_i(|N_a| e_P) \tag{8}$$

where $v_i(a_i)$ is the utility i gets by taking action a_i, $v_i(a_0) = 0$, and in equilibrium

$$N_a = \begin{cases} N_a^i \cup \{i\} & \text{if } v_i > pf + (h_i((|N_a^i| + 1)e_P) - h_i(|N_a^i| e_P)), \\ N_a^i & \text{otherwise.} \end{cases}$$

Proof. First note that $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$ for any integer n, and $\sum_{r=0}^{n} \binom{n}{r}(1-p)^{n-r}p^r r = pn\sum_{r=0}^{n-1}\binom{n-1}{r}(1-p)^{n-1-r}p^r$. By substitution of budget constraint, the expected utility function given in 3, by additive separability, and risk neutrality, becomes

$$
\begin{aligned}
& p\sum_{r=0}^{|N_a^i|} \alpha(p,|N_a^i|,r)(w_i - c(p) + \frac{rf}{N-1} - f(a_i) + v(a_i)) \\
& + (1-p)\sum_{r=0}^{|N_a^i|} \alpha(p,|N_a^i|,r)(w_i - c(p) + \frac{rf}{N-1} + v(a_i)) - E[h_i] \\
=& \sum_{r=0}^{|N_a^i|} \alpha(p,|N_a^i|,r)(w_i - c(p) + \frac{rf}{N-1} + v(a_i)) - pf(a_i)\sum_{r=0}^{|N_a^i|}\alpha(p,|N_a^i|,r) - E[h_i] \\
=& (w_i - c(p) + v(a_i) - pf(a_i)) + \sum_{r=0}^{|N_a^i|}\alpha(p,|N_a^i|,r)(\frac{rf}{N-1}) - E[h_i] \\
=& (w_i - c(p) + v(a_i) - pf(a_i)) + \frac{1}{|N-1|} f\sum_{r=0}^{|N_a^i|}\alpha(p,|N_a^i|,r)r - E[h_i] \\
=& (w_i - c(p) + v(a_i) - pf(a_i)) + \frac{|N_a^i|}{|N-1|}pf\sum_{r=0}^{|N_a^i|-1}\alpha(p,|N_a^i|-1,r) - E[h_i] \\
=& (w_i - c(p) + v(a_i) - pf(a_i)) + \frac{|N_a^i|}{|N-1|}pf - E[h_i] \\
\equiv& v(a_i) - pf(a_i) - c(p) + pf\frac{|N_a^i|}{|N-1|} - E[h_i]
\end{aligned}
$$

□

3.2.1 Risk Neutral Agents

The model here is the same as the previous one except that an action taken by an individual will affect everyone in the society.

Theorem 3. *Assume that an agent has a utility function given in 3, and his utility function is additively separable and he is risk neutral with respect to uncertainties in consumption and in harm he faces, the level of public externality an agent can cause is e_P, and is the same for all agents. Then his expected utility function becomes:*

$$Eu_i(c_i, a_i, h_i) = v_i(a_i) - pf(a_i) - c(p) - |N_a|e_P + \frac{|N_a^i|}{|N-1|}pf \tag{9}$$

where $v_i(a_i)$ is the utility i gets by taking action a_i, $v_i(a_0) = 0$, and in equilibrium

$$N_a = \begin{cases} N_a^i \cup \{i\} & \text{if } v_i > pf + e_P, \\ N_a^i & \text{otherwise.} \end{cases}$$

Note that an agent, say i, will take the action which causes an externality if and only if $v_i(a) > pf(a) + e_P$.

3.3 Multiplicity of Equilibria

When agents are risk averse with respect to uncertainties in harms they face and risk neutral with respect to uncertainties in consumption, multiple equilibria may arise. Consider the following example.

Example: There are two agents with distinct valuations and same convex harm functions. Their valuations are sufficiently close, i.e. $v_2 - v_1 < h(2) - 2h(1)$. If $v_2 - (h(2e_P) - h(e_P)) \leq pf < v_1 - h(e_P)$ then in one equilibrium agent 1 is deterred while agent 2 is an offender whereas in the other equilibrium agent 2 is deterred while agent 1 is an offender. This happens because expected punishment is large enough to deter only one person and not large enough to deter everybody.Convexity of harm functions guarantees that as expected level of punishment increases, number of offenders do not increase. Although an efficient enforcement policy is unlikely to result in an inefficient equilibria, existence of multiple equilibria under certain policies is likely to arise as long as valuations of agents are sufficiently close. It is possible though to have multiple equilibria under an efficient policy if there are agents with identical valuations.

4 Monotonicity of Harm Probabilities

This section explores how probability of being offended is affected by changes in the number of offenders in the society and by changes in the number of offenders an agent faces.

If the number of offenders in a society increases then addition of a new offender will affect all other agents. The number of states an agent faces will increase because of the addition of this new offender. On the other hand, probability of each state (being offended by a certain number of agents) will decrease as for a given number of offenders an agent faces, number of offenders who are not offending that agent is increasing. It is shown below that, in general, effect of more number states an agent faces will outweight the effect of lower probability of facing each state, and this will cause an agent to face a higher probability of being offended by a certain number of agents. This obvious result has an exception. If there are high number of offenders in the society, more precisely $N - 2$, then an increase in the number offenders will not affect any agent as those two effects will cancel each other and an agent will face the same probability of being offended by a certain number of offenders.

Theorem 4. *Assume that in a private externalities environment number of offenders is* N_a*,* $0 \leq N_a \leq N-2$*,* $N \geq 2$.[15] *Let* $\pi(r, N_a) = (\frac{1}{N-1})^r(1-\frac{1}{N-1})^{N_a-r}\binom{N_a}{r}$ *denote the probability that an agent will be harmed by* r *of* N_a *offenders. Then*

1. $\pi(0, N_a) > \pi(0, N_a+1)$.

2. *For* $N_a < N-2$*,* $\pi(r, N_a) < \pi(r, N_a+1)$ *for* $r \geq 1$.

3. *For* $N_a = N-2$*,* $\pi(r, N_a) < \pi(r, N_a+1)$ *for* $r > 1$*, and* $\pi(1, N_a) = \pi(1, N_a+1)$

Proof. If $N = 2$ then $N_a = 0$ by assumption. So $\pi(0, N_a) = 1 > \pi(0, N_a+1) = 0$. Now assume that $N > 2$.

Case 1. $r = 0$

For $0 \leq r \leq N_a$, $\frac{\pi(r,N_a+1)}{\pi(r,N_a)} = \frac{(\frac{1}{N-1})^r(1-\frac{1}{N-1})^{N_a+1-r}\binom{N_a+1}{r}}{(\frac{1}{N-1})^r(1-\frac{1}{N-1})^{N_a-r}\binom{N_a}{r}} = (1-\frac{1}{N-1})(\frac{N_a+1}{N_a+1-r})$. So $\frac{\pi(0,N_a+1)}{\pi(0,N_a)} = (1-\frac{1}{N-1}) < 1$.

Case 2.1 $N_a < N-2$

$N_a < N-2 \Leftrightarrow 0 < \frac{N_a+1}{N-1} < 1$. This implies that for $r \geq 1 \Rightarrow r > \frac{N_a+1}{N-1} \Leftrightarrow (N-1)r > N_a + 1 \Leftrightarrow (N-2)(N_a+1) > (N-1)(N_a+1) - (N-1)r \Leftrightarrow (\frac{N-2}{N-1})(\frac{N_a+1}{N+a+1-r}) > 1 \Leftrightarrow \pi(r, N_a) < \pi(r, N_a+1)$.

Case 2.2 $N_a = N-2$

$\frac{\pi(1,N_a+1)}{\pi(1,N_a)} = 1$, and for $r > 1$, $\frac{\pi(r,N_a+1)}{\pi(r,N_a)} = (1-\frac{1}{N-1})(\frac{N_a+1}{N_a+1-r}) = (\frac{N-2}{N-1})(\frac{N-1}{N-1-r}) > 1$. □

The following theorem shows that, in general, the probability of being offended by a certain number of agents is higher than the probability of being offended by a bigger number of agents. There cases when this is not true. Probabilities of being not offended and being offended by only one agent are equal when total number of offenders is very high, $N-2$ to be more precise. Moreover probability of being not offended is less than the probability of being offended by only one agent when total number of offenders is $N-1$.

Theorem 5. *Assume that in a private externalities environment number of offenders is* N_a*,* $0 \leq N_a \leq N-1$*, and* $N \geq 2$.[16] *Let* $\pi(r, N_a) = (\frac{1}{N-1})^r(1-\frac{1}{N-1})^{N_a-r}\binom{N_a}{r}$ *denote the probability that an agent will be harmed by* r*,* $r < N_a$*, of* N_a *offenders. Then*

[15] $N_a \leq N-2$ as the maximum number of people that can hurt an agent is $N-1$.

[16] $N_a \leq N-1$ as the maximum number of people that can hurt an agent is $N-1$.

1. *For* $N_a < N-2$, $\pi(r+1, N_a) < \pi(r, N_a)$ *for* $r \geq 0$.

2. *For* $N_a = N-2$, $\pi(r+1, N_a) < \pi(r, N_a)$ *for* $r \geq 1$, *and* $\pi(0, N_a) = \pi(1, N_a)$.

3. *For* $N_a = N-1$, $\pi(r+1, N_a) < \pi(r, N_a)$ *for* $r \geq 1$, *and* $\pi(0, N_a) < \pi(1, N_a)$.

Proof. If $N = 2$ then $N_a = 1$ is the only case to consider, and $\pi(0,1) = 0 < \pi(1,1) = 1$. Now assume that $N > 2$.
$\frac{\pi(r+1,N_a)}{\pi(r,N_a)} = \frac{(\frac{1}{N-1})^{(}r+1)(1-\frac{1}{N-1})^{N_a-r-1}\binom{N_a}{r+1}}{(\frac{1}{N-1})^{r}(1-\frac{1}{N-1})^{N_a-r}\binom{N_a}{r}} = \frac{(\frac{1}{N-1})}{(1-\frac{1}{N-1})}(\frac{N_a-r}{r+1})$. So $\pi(r+1, N_a) \leq \pi(r, N_a) \Leftrightarrow (N_a - r) \leq (N-2)(r+1) \Leftrightarrow \frac{N_a-(N-2)}{N-1} \leq r$. This implies that if $N_a < N-2$ then $\pi(r+1, N_a) < \pi(r, N_a)$ for $r \geq 0$. If $N_a = N-2$ then $\pi(r+1, N_a) < \pi(r, N_a)$ for $r \geq 1$, and $\pi(0, N_a) = \pi(1, N_a)$. Finally, if $N_a = N-1$ then $\pi(r+1, N_a) < \pi(r, N_a)$ for $r \geq 1$, and $\pi(0, N_a) < \pi(1, N_a)$.

□

5 Social Welfare

In this section, I will look at the social welfare of the society, and socially optimal level of enforcement. Without loss of generality assume that *if* $i < j$ *then* $v_i \leq v_j$. For notational simplicity, I will use f for $f(a), a \neq a_0$, and v_i for $v_i(a_i), a_i \neq a_0$, and assume that $f \geq v_N$, where v_N is the highest level of valuation of the action in the society.

It is also assumed that an individual will not take any action if he is indifferent between taking and not taking it. Otherwise, social planner's problem may not have a solution. $\sum_{i \in N} c_i(p)$ is the total cost of enforcement, and the level of enforcement p is in $[0,1]$. Individuals' wealth are identical, i.e. $w_i = w$ for all $i \in \{1, .., N\}$. The punishment level f is given, and it does not bind budget constraint of individuals. Optimal value of probability of detection is denoted by p^*. Social welfare of the society is a function of the level of enforcement, $W(p) = \sum_{i \in N} \alpha_i Eu_i^*$, $\sum_{i \in N} \alpha_i = 1, \alpha_i > 0$ for all $i \in N$. Eu_i^* is the equilibrium utility level of agent i. The following two theorems are generalizations of the main theorem in Polinsky and Shavell [4] where they show that the optimal level of punishment in equilibrium is such that expected level of punishment is less than the harm it causes.

Theorem 6. *Assume that each agent has an equal weight in a private externalities environment social planner's problem, and that each agent will hurt another agent with equal probability. Then the optimal level of enforcement* p^* *is such that* $N_{p^*f} p^* f < \sum_{i \in N} (\sum_{R \subseteq N_a^i \cup N_{p^*f} \setminus \{i\}} \beta(R, N_a^i \cup N_{p^*f} \setminus$

$\{i\})h_i(|R|e) - \sum_{R\subseteq N_a^i}\beta(R, N_a^i)h_i(|R|e))$ where N_{pf} is the set of all people whose valuations are equal to pf.

Proof. The social welfare function $W^R(p) = \sum \alpha_i^R E u_i^*$ for a private externalities problem for $v_j^R \leq pf < v_{j+1}^R$ (if such j exists) is

$$\begin{aligned}
W^R(p) &= \sum_{i\in N\setminus N_a} \alpha_i^R(-c_i(p) - E[h_i] + pf \sum_{k\in N_a^i} \pi_{ki}) \\
&\quad + \sum_{i\in N_a} \alpha_i^R(v_i^R - pf - c_i(p) - E[h_i] + pf \sum_{k\in N_a^i} \pi_{ki}) \\
&= \sum_{i\in N_a} \alpha_i^R v_i^R - \sum_{i\in N} \alpha_i^R c_i(p) - E[h_i] - pf \sum_{i\in N_a} \alpha_i^R + pf \sum_{i\in N} \alpha_i^R \sum_{k\in N_a^i} \pi_{ki}.
\end{aligned}$$

Since $\alpha_i = \frac{1}{N}$, and $\pi_{jk} = \frac{1}{N-1}$ for all $i, j, k \in N$,

$$\sum_{i\in N_a} \alpha_i^R + \sum_{i\in N} \alpha_i^R \sum_{k\in N_a^i} \pi_{ki} = 0.$$

Then social welfare becomes

$$W^R(p) = \sum_{i\in N_a} v_i^R - \sum_{i\in N} c_i(p) - \sum_{i\in N} E[h_i].$$

If $0 \leq pf < v_1^R$ then the social welfare function is $W^R(p) = \sum_{i\in N} \alpha_i^R v_i^R - \sum_{i\in N} \alpha_i^R c_i(p) - E[h_i]$, and it is $W^R(p) = -\sum_{i\in N} \alpha_i^R c_i(p)$ for $v_N^R \leq pf$. So the social welfare function for any $p \in [0, 1]$ is

$$W^R(p) = \sum_{i\in N_a} v_i^R - \sum_{i\in N} c_i(p) - \sum_{i\in N} E[h_i].$$

Now assume on the contrary that $p^* f \geq e$. If $p^* f \neq v_j$ for any $j \in N$ then a slight decrease in p will decrease the second part of the expression without changing the first one and this will lead to a increase in the social welfare, a contradiction to optimality of p^*. On the other hand, if $p^* f = v_j$ for some $j \in N$ then a slight decrease in p will increase the number of offenders, N_a, and increase the first sum, besides the second sum, in the equation above as $p^* f \geq e$, causing the welfare to increase, a contradiction. □

Theorem 7. *Assume that each agent has equal weight in a public externalities environment social planner's problem. Then the optimal level of enforcement $N_{p^*f} p^* f < \sum_{i\in N}(h_i(|N_a \cup N_{p^*f}|e_P) - h_i(|N_a|e_P))$.*

Proof. The social welfare function $W^P(p) = \sum \alpha_i^P E u_i^*$ for a public externalities problem for $v_j^P \leq pf + e_P < v_{j+1}^P$ (if such j exists) is

$$\begin{aligned}
W^P(p) &= \sum_{i\in N_a} \alpha_i^P(v_i^P - pf) - \sum_{i\in N} \alpha_i^P c_i(p) - \sum_{i\in N} \alpha_i^P h_i(|N_a|e_P) + \sum_{i\in N} \alpha_i^P \frac{|N_a|}{|N|} pf \\
&= \sum_{i\in N_a} \alpha_i^P v_i^P - \sum_{i\in N} \alpha_i^P c_i(p) + pf \frac{|N_a|}{|N|} - pf \sum_{i\in N_a} \alpha_i^P - \sum_{i\in N} \alpha_i^P h_i(|N_a|e_P)
\end{aligned}$$

As $\alpha_i = \frac{1}{N}$ for all $i \in N$ the social welfare function becomes

$$= \sum_{i \in N_a} v_i^P - \sum_{i \in N} c_i(p) - \sum_{i \in N} h_i(|N_a| e_P).$$

If $0 \leq pf + e_P < v_1^P$ then the social welfare function is $W^P(p) = \sum_{i=1}^N \alpha_i^P v_i^P - \sum_{i \in N} \alpha_i c_i(p) - \sum_{i \in N} h_i(|N_a| e_P)$, and it is $W^P(p) = -\sum_{i \in N} \alpha_i^P c_i(p)$ for $v_N^P \leq pf + e_P$. So the social welfare function is

$$W^P(p) = \sum_{i \in N_a} v_i^P - \sum_{i \in N} c_i(p) - \sum_{i \in N} h_i(|N_a| e_P).$$

Now assume on the contrary that $p^* f \geq e$. If $p^* f \neq v_j$ for any $j \in N$ then a slight decrease in p will decrease the second part of the expression without changing the first one and this will lead to a increase in the social welfare, a contradiction to optimality of p^*. On the other hand, if $p^* f = v_j$ for some $j \in N$ then a slight decrease in p will increase the number of offenders, N_a, and increase the first sum, besides the second sum, in the equation above as $p^* f \geq e$, causing the welfare to increase, a contradiction. $\square$

V stands for a partition of N, and defined as for any $V^k \in V$, for any $i, j \in N$, $i \in V^k$ and $j \in V^k$ if and only if $v_i = v_j$, and for any $k > l$, $i \in V^k$ and $j \in V^l$ if and only if $v_i > v_j$. The set of all sets of possible offenders for various deterrence levels is $V^* = \{\cup_{i \geq j} V^i | 0 \leq i \leq |V|\} \cup \{\emptyset\}$. So, for each $K \in V^*$, $K = \{i \in N | v_i > \underline{v}\ for\ some\ \underline{v}\}$, and the set of offenders is $N_a = \{i \in N | v_i > pf\} \in V^*$ where p is the enforcement level.

Theorem 8. *Social planner's problems in a private externalities problem with externality level e, and in a public externalities problem with the externality level e_P are the same whenever the following conditions are satisfied:*

1. *For all $i \in N$, $\alpha_i^R = \alpha_i^P$,*
2. *For all $K \in V^*$, $\sum_{i \in N} \alpha_i^R \sum_{k \in K \setminus \{i\}} \pi_{ki} = \frac{|K|}{|N|}$,*
3. *For all $i \in N$, $v_i^R = v_i^P - e_P$,*
4. *For all $K \in V^*$, $e = e_P(|N| - \frac{|N|}{|K|} \sum_{i \in K} \alpha_i^P)$.*

Proof. The social welfare function $W^R(p) = \sum \alpha_i^R E u_i^*$ for a private externalities problem for $v_j^R \leq pf < v_{j+1}^R$ (if such j exists), $N_a = \{j+1, j+2, ..., |N|\}$, is

$$\begin{aligned} W^R(p) &= \sum_{i=1}^{j} \alpha_i^R (-c_i(p) - e \sum_{k \in N_a^i} \pi_{ki} + pf \sum_{k \in N_a^i} \pi_{ki}) \\ &+ \sum_{i=j+1}^{N} \alpha_i^R (v_i^R - pf - c_i(p) - e \sum_{k \in N_a^i} \pi_{ki} + pf \sum_{k \in N_a^i} \pi_{ki}) \\ &= \sum_{i \in N_a} \alpha_i^R (v_i^R - pf) - \sum_{i \in N} \alpha_i^R c_i(p) - (e - pf) \sum_{i \in N} \alpha_i^R \sum_{k \in N_a^i} \pi_{ki}. \end{aligned}$$

If $0 \leq pf < v_1^R$ then the social welfare function is $W^R(p) = \sum_{i\in N} \alpha_i^R(v_i^R - e) - \sum_{i\in N} \alpha_i^R c_i(p)$, and it is $W^R(p) = -\sum_{i\in N} \alpha_i^R c_i(p)$ for $v_N^R \leq pf$.

The social welfare function $W^P(p) = \sum \alpha_i^P E u_i^*$ for a public externalities problem for $v_j^P \leq pf + e_P < v_{j+1}^P$ (if such j exists), $N_a = \{j+1, j+2, ..., |N|\}$, is

$$\begin{aligned} W^P(p) &= \sum_{i\in N_a} \alpha_i^P(v_i^P - pf) - \sum_{i\in N} \alpha_i^P c_i(p) - \sum_{i\in N} \alpha_i^P |N_a| e_P + \sum_{i\in N} \alpha_i^P \frac{|N_a|}{|N|} pf \\ &= \sum_{i\in N_a} \alpha_i^P(v_i^P - pf) - \sum_{i\in N} \alpha_i^P c_i(p) - (|N|e_P - pf)\frac{|N_a|}{|N|} \\ &= \sum_{i\in N_a} \alpha_i^P(v_i^P - e_P - pf) - \sum_{i\in N} \alpha_i^P c_i(p) + [e_P \sum_{i\in N_a} \alpha_i^P - |N_a| e_P] + pf\frac{|N_a|}{|N|} \end{aligned}$$

If $0 \leq pf + e_P < v_1^P$ then the social welfare function is $W^P(p) = \sum_{i=1}^N \alpha_i^P(v_i^P - |N|e_P) - \sum_{i\in N} \alpha_i c_i(p)$, and it is $W^P(p) = -\sum_{i\in N} \alpha_i^P c_i(p)$ for $v_N^P \leq pf + e_P$.

Hence, $W^P(p) = W^R(p)$ for any p if conditions in the statement of the theorem are satisfied. □

Corollary 9. *Social planner's problems in a private externalities problem with externality level e, and in a public externalities problem with the externality level e_P are the same whenever the following conditions are satisfied:*

1. $\alpha_i^R = \alpha_j^P$ *for all* $i, j \in N$,
2. *For all* $K \in V^*$, $\sum_{i\in N} \alpha_i^R \sum_{k\in K\setminus\{i\}} \pi_{ki} = \frac{|K|}{|N|}$,
3. *For all* $i \in N$, $v_i^R = v_i^P - e_P$,
4. $e = e_P(|N| - 1)$.

Lemma 3. *For all* $K \in V^*$ $\sum_{i\in N} \alpha_i^R \sum_{k\in K\setminus\{i\}} \pi_{ki} = \frac{|K|}{|N|}$ *if and only if for any* $V^m \in V$,$\sum_{i\in N} \alpha_i^R \sum_{k\in V^m\setminus\{i\}} \pi_{ki} = \frac{|V^m|}{|N|}$.

Proof. Suppose for any $K \in V^*$ $\sum_{i\in N} \alpha_i^R \sum_{k\in K\setminus\{i\}} \pi_{ki} = \frac{|K|}{|N|}$ holds. Then for $K = V^{|V^*|}$ $\sum_{i\in N} \alpha_i^R \sum_{k\in V^{|V^*|}\setminus\{i\}} \pi_{ki} = \frac{|V^{|V^*|}|}{|N|}$, and for $K = V^{|V^*|} \cup V^{|V^*|-1}$ $\sum_{i\in N} \alpha_i^R \sum_{k\in V^{|V^*|}\cup V^{|V^*|-1}\setminus\{i\}} \pi_{ki} = \frac{|V^{|V^*|}\cup V^{|V^*|-1}|}{|N|}$. As $\sum_{i\in N} \alpha_i^R \sum_{k\in V^{|V^*|}\cup V^{|V^*|-1}\setminus\{i\}} \pi_{ki} = \sum_{i\in N} \alpha_i^R \sum_{k\in V^{|V^*|-1}\setminus\{i\}} \pi_{ki} + \sum_{i\in N} \alpha_i^R \sum_{k\in V^{|V^*|}\setminus\{i\}} \pi_{ki} = \sum_{i\in N} \alpha_i^R \sum_{k\in V^{|V^*|-1}\setminus\{i\}} \pi_{ki} + \frac{|V^{|V^*|}|}{|N|} = \frac{|V^{|V^*|}\cup V^{|V^*|-1}|}{|N|}$ implies that $\sum_{i\in N} \alpha_i^R \sum_{k\in V^{|V^*|-1}\setminus\{i\}} \pi_{ki} = \frac{|V^{|V^*|-1}|}{|N|}$. Similarly, for $V^m \in V$, $\sum_{i\in N} \alpha_i^R \sum_{k\in V^m\setminus\{i\}} \pi_{ki} = \frac{|V^m|}{|N|}$. 'If' part of the proof is trivial. □

Lemma 4. *If for all k, for all $i, j, i \neq k, j \neq k$, $\pi_{ki} = \pi_{kj}$ then for any $V^m \in V$,$\sum_{i\in N} \alpha_i^R \sum_{k\in V^m\setminus\{i\}} \pi_{ki} = \frac{|V^m|}{|N|} \Rightarrow \sum_{i\in V^m} \alpha_i^R = \frac{|V^m|}{|N|}$, and for those agents who has different valuations from every other agents in the society, $\alpha_i^R = \frac{1}{|N|}$.*

Proof. For all k, for all $i, j, i \neq k, j \neq k$, $\pi_{ki} = \pi_{kj}$ implies that $\pi_k i = \frac{1}{N-1}$ for all $k, i \in N, k \neq i$. For any $V^m \in V, \sum_{i \in N} \alpha_i^R \sum_{k \in V^m \setminus \{i\}} \pi_{ki} = \sum_{\substack{i \in N \\ i \notin V^m}} \alpha_i^R \sum_{k \in V^m} \pi_{ki} + \sum_{\substack{i \in N \\ i \in V^m}} \alpha_i^R \sum_{k \in V^m \setminus \{i\}} \pi_{ki} = \sum_{\substack{i \in N \\ i \notin V^m}} \alpha_i^R \frac{|V^m|}{N-1} + \sum_{\substack{i \in N \\ i \in V^m}} \alpha_i^R \frac{|V^m|-1}{N-1} = \frac{|V^m|}{N-1} \sum_{i \in N} \alpha_i^R - \frac{1}{N-1} \sum_{\substack{i \in N \\ i \in V^m}} \alpha_i^R = \frac{|V^m|}{|N|} \Rightarrow 1 - \frac{\sum_{i \in V^m}}{V^m} = \frac{N-1}{N} \Rightarrow \sum_{i \in V^i} \alpha_i^R = \frac{|V^i|}{|N|}$. If i has different valuations from every other agents in the society then $\alpha_i^R = \frac{1}{|N|}$ as $V^m = \{v_i\}$ for some $V^m \in V$. □

Lemma 5. *If for all k, for all $i, j, i \neq k, j \neq k$, $\pi_{ik} = \pi_{jk} = \pi_k$ then for any $V^m \in V, \sum_{i \in N} \alpha_i^R \sum_{k \in V^m \setminus \{i\}} \pi_{ki} = \frac{|V^m|}{|N|} \Rightarrow \frac{\sum_{i \in V^m} \alpha_i^R \pi_i}{|V^m|} = \sum_{i \in N} \alpha_i \pi_i - \frac{1}{|N|}$, and if every agent has a distinct valuation in the society, $\alpha_i^R \pi_i = \frac{1}{|N|(|N|-1)}$.*

Proof. For any $V^m \in V, \sum_{i \in N} \alpha_i^R \sum_{k \in V^m \setminus \{i\}} \pi_{ki} = \sum_{\substack{i \in N \\ i \notin V^m}} \alpha_i^R \sum_{k \in V^m} \pi_{ki} + \sum_{\substack{i \in N \\ i \in V^m}} \alpha_i^R \sum_{k \in V^m \setminus \{i\}} \pi_{ki} = \sum_{\substack{i \in N \\ i \notin V^m}} \alpha_i^R |V^m| \pi_i + \sum_{\substack{i \in N \\ i \in V^m}} \alpha_i^R (|V^m|-1) \pi_i = |V^m| \sum_{i \in N} \alpha_i^R \pi_i - \sum_{i \in V^m} \alpha_i^R \pi_i = \frac{|V^m|}{|N|} \Rightarrow \frac{\sum_{i \in V^m} \alpha_i^R \pi_i}{|V^m|} = \sum_{i \in N} \alpha_i \pi_i - \frac{1}{|N|}$. If for all $V^m \in V$, $|V^m| = 1$ then for all $i \in N$, $\alpha_i \pi_i = \sum_{i \in N} \alpha_i \pi_i - \frac{1}{|N|}$. Adding over all $i \in N$, $\sum_{i \in N} \alpha_i \pi_i = \frac{1}{|N|-1}$, and this shows that for all $i \in N$, $\alpha_i \pi_i = \frac{1}{|N|(|N|-1)}$. □

Note that $W(v_i) > W(pf)$ whenever $v_i < pf < v_{i+1}$. So, the planner has actually $N+1$ alternative expected punishment levels, including the legalization of the action, $pf = 0$, to choose from.

If $e \leq v_1$ then it is socially optimal to legalize that action, i.e. $p^* = f^* = 0$. If $w < v_j$ for some j then it is impossible to deter any agent i for $i \geq j$. So, $w < v_1$ implies that it is socially optimal to legalize that action, i.e. $p^* = f^* = 0$. Whenever all agents are identical then the optimal policy is complete deterrence, i.e. $p^* f^* = v$, if $c(\frac{v}{w}) + v \leq e$; and it is optimal to legalize the action, i.e. $p^* f^* = 0$, if $c(\frac{v}{w}) + v > e$. It is never optimal to deter everbody if $\overline{v} > Nh$ where $\overline{v}$ is the average of valuations of agents involving in the activity.

If there is a unit measure of agents in the society, $g(v)$ is the probability density of people who gain utility v from the action, and agents do not know their types, and *ex ante* they are identical then the expected utility of a representative agent in a unit measure society will be

$$Eu(c^*, a^*, h) = \int_{pf}^{\overline{v}} v g(v) \mathrm{d}v - e \int_{pf}^{\overline{v}} g(v) \mathrm{d}v - c(p). \tag{10}$$

This is exactly the same objective function used in Polinsky and Shavell [4].

Individuals do not know their types when the law and enforcement policy is made but they learn their types before they engage in any activity.

Theorem 10. *Let p_R^* be a solution to social planner's problems in a private externalities problem with externality level e, and p_P^* be a solution in a public externalities problem with the externality level e_P. Then $p_R^* \geq p_P^*$ if*

1. *for all $i \in N$, $\alpha_i^R = \alpha_i^P$,*
2. *for all $i \in N$, $\pi_{ki} = \frac{1}{N-1}$,*
3. *for all $i, k \in N$, $v_i^R = v_i^P - e_P$,*
4. *for all $K \in V^*$, $e = e_P(|N| - 1)$,*
5. *agents are sufficiently risk averse.*

Proof. The social welfare function $W^R(p) = \sum \alpha_i^R Eu_i^*$ for a private externalities problem for $v_j^R \leq pf < v_{j+1}^R$ (if such j exists), $N_a = \{j+1, j+2, ..., |N|\}$, is

$$\begin{aligned} W^R(p) = & \sum_{i=1}^{j} \alpha_i^R(-c_i(p) - \sum_{R \subseteq N_a^i} \beta(R, N_a^i) h_i(|R|e) + pf \sum_{k \in N_a^i} \pi_{ki}) \\ & + \sum_{i=j+1}^{N} \alpha_i^R(v_i^R - pf - c_i(p) - \sum_{R \subseteq N_a^i} \beta(R, N_a^i) h_i(|R|e) + pf \sum_{k \in N_a^i} \pi_{ki}) \\ & = \sum_{i \in N_a} \alpha_i^R(v_i^R - pf) - \sum_{i \in N} \alpha_i^R c_i(p) + pf \sum_{i \in N} \alpha_i^R \sum_{k \in N_a^i} \pi_{ki} - \sum_{i \in N} \alpha_i^R \sum_{R \subseteq N_a^i} \beta(R, N_a^i) h_i(|R|e). \end{aligned}$$

If $0 \leq pf < v_1^R$ then the social welfare function is $W^R(p) = \sum_{i \in N} \alpha_i^R(v_i^R - \sum_{R \subseteq N_a^i} \beta(R, N_a^i) h_i(|R|e)) - \sum_{i \in N} \alpha_i^R c_i(p)$, and it is $W^R(p) = -\sum_{i \in N} \alpha_i^R c_i(p)$ for $v_N^R \leq pf$.

The social welfare function $W^P(p) = \sum \alpha_i^P Eu_i^*$ for a public externalities problem for $v_j^P \leq pf + e_P < v_{j+1}^P$ (if such j exists), $N_a = \{j+1, j+2, ..., |N|\}$, is

$$\begin{aligned} W^P(p) &= \sum_{i \in N_a} \alpha_i^P(v_i^P - pf) - \sum_{i \in N} \alpha_i^P c_i(p) - \sum_{i \in N} \alpha_i^P h_i(|N_a|e_P) + \sum_{i \in N} \alpha_i^P \frac{|N_a|}{|N|} pf \\ &= \sum_{i \in N_a} \alpha_i^P(v_i^P - pf) - \sum_{i \in N} \alpha_i^P c_i(p) + pf \frac{|N_a|}{|N|} - \sum_{i \in N} \alpha_i^P h_i(|N_a|e_P) \\ &= \sum_{i \in N_a} \alpha_i^P(v_i^P - e_P - pf) - \sum_{i \in N} \alpha_i^P c_i(p) - \sum_{i \in N} \alpha_i^P h_i((|N_a| - 1)e_P) + pf \frac{|N_a|}{|N|} \end{aligned}$$

If $0 \leq pf + e_P < v_1^P$ then the social welfare function is $W^P(p) = \sum_{i=1}^{N} \alpha_i^P(v_i^P - h_i(|N|e_P)) - \sum_{i \in N} \alpha_i c_i(p)$, and it is $W^P(p) = -\sum_{i \in N} \alpha_i^P c_i(p)$ for $v_N^P \leq pf + e_P$.

Hence, $W^P(p) = W^R(p)$ for any p if conditions in the statement of the theorem are satisfied. □

References

[1] Gary S. Becker. Crime and punishment. *J. Pol. Econ.*, 76:169–217, 1968.

[2] Robert Cooter and Thomas Ulen. *Law and Economics*. 3rd edition, 2000.

[3] Mitchell Polinsky and Steven Shavell. The optimal tradeoff between the probability and magnitude of fines. *American Econ. Rev.*, 69:880–891, 1979.

[4] Mitchell Polinsky and Steven Shavell. The optimal use of fines and imprisonment. *J. Pub. Econ.*, 24:89–99, 1984.

[5] Steven Shavell. A note on marginal deterrence. *Int. Rev. Law Econ.*, 12:345–355, 1992.

[6] George J. Stigler. The optimum enforcement of laws. *J. Pol. Econ.*, 78:526–536, 1970.