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PEBBLING NUMBERS OF GRAPH PRODUCTS

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ABSTRACT. Let G be a connected graph. A pebbling move on a graph G is taking two pebbles off one vertex and placing one of them on an adjacent vertex. The pebbling number of a connected graph G, $f(G)$, is the least n such that any distribution of n pebbles on the vertices of G allows one pebble to be moved to any specified, but arbitrary vertex by a sequence of pebbling moves. In this paper, the pebbling numbers of the lexicographic products of some graphs are computed.

1. Introduction

Pebbling in graphs was first considered by Chung[1]. Consider a connected graph with a fixed number of pebbles distributed on its vertices. We define a *pebbling move* as the process of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. We say that we can pebble to a vertex v, the target vertex, if we can apply pebbling moves repeatedly so that it is possible to reach a configuration with at least one pebble at v. We define the *pebbling number of a vertex* v in a graph G, denoted $f(G, v)$, to be the smallest integer m which guarantees that any starting pebble configuration with m pebbles allows pebbling to v. We define the *pebbling number of* G, denoted f(G) as the maximum of $f(G, v)$, over all vertices v.

A graph G is called *demonic* if $f(G)$ is equal to the number of its vertices. So far, very little is known regarding $f(G)$(See [1] -[6]). If one pebble is placed on each vertex other than the vertex v, then no

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pebble can be moved to v. Also, if w is at distance d from v, and $2^d - 1$ pebbles are placed on w, then no pebble can be moved to v. So it is clear [1] that $f(G) \geq max\{|V(G)|, 2^D\}$, where $|V(G)|$ is the number of vertices of G and D is the diameter of the graph G. Furthermore, we know that K_n and $K_{s,t}$ are demonic when $s > 1$ and $t > 1$(See [1] and [2]), where K_n is the complete graph on n vertices, and $K_{s,t}$ is the complete bipartite graph such that two partition sets have s and t vertices respectively. But $f(P_n) = 2^{n-1}$ (See [1]), *i.e.*, the graph P_n is not demonic when $n > 2$, where P_n is the path on n vertices. Given a pebbling of G , a *transmitting subgraph of* G is a path $x_1, x_2, \ldots x_k$ such that there are at least two pebbles on x_1, and at least one pebble on each of the other vertices in the path, except possibly x_k. In this case, we can transmit a pebble from x_1 to x_k.

In this paper, we study the pebbling number of the lexicographic product of some graphs. Throughout this paper, G will denote a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex v of a graph G, $p(v)$ will refer to the number of pebbles on v.

2. Lexicographic Product

We now define the lexicographic product of two graphs, and discuss some results on the pebbling number of such graphs.

DEFINITION : If $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are two graphs, the *lexicographic product* of G and H is the graph $G * H$, whose vertex set is the Cartesian product.

$$V_{G*H} = V_G \times V_H = \{(x, y) : x \in V_G, y \in V_H\}$$

and whose edge are given by

$$E_{G*H} = \{((x, y), (x', y')) : either\ (x, x') \in E_G \text{ and } y \neq y',\\ \text{or } x = x' \text{ and } (y, y') \in E_H\}$$

If the vertices of G are labelled by x_i, then for any distribution of pebbles on $G * H$, we write p_i for the total number of pebbles on $\{x_i\} \times H$, q_i for the total number of vertices of $\{x_i\} \times H$ with pebbles.

THEOREM 1. *Let P_3 be the path with vertices x_1, x_2 and x_3 in order and let H be any graph with vertices $y_1, \ldots y_n (n \geq 4)$. Then $f(P_3 * H) \leq 3f(H)$*

Proof. Suppose there are $3f(H)$ pebbles assigned to the vertices of $P_3 * H$.

First, suppose that the target vertex is (x_1, y_i), for some i, where $i \in \{1, \ldots, n\}$. If $p(x_1, y_i) \geq 1$ or $p_1 \geq f(H)$, then we are done. Therefore, we may assume that $p(x_1, y_i) = 0$ and $p_1 < f(H)$. Then $p_2 + p_3 \geq 2f(H) + 1$. We consider the following two cases.

Case 1. $p_2 \geq f(H) + 1$.
(1.1). If $p(x_2, y_j) \geq 1$ for some $j \neq i$, then two pebbles can be moved to (x_2, y_j) because we keep one pebble on (x_2, y_j) and move one more pebble to (x_2, y_j) by using the remaining $f(H)$ pebbles on $\{x_2\} \times H$. Since (x_1, y_i) and (x_2, y_j) are adjacent in $P_3 * H$, we can take two pebbles from (x_j, y_j) and move one pebble to (x_2, y_j).

(1.2). If $p(x_2, y_j) = 0$ for all $j \neq i$, then $p(x_2, y_i) = p_2$. So $[\frac{p_2}{2}]$ pebbles can be moved from (x_2, y_i) to (x_2, y_k), where $(y_i, y_k) \in E_H$. Moreover $[\frac{p_2}{2}] \geq [\frac{(f(H)+1)}{2}] \geq 2$. Thus one pebble can be moved to (x_1, y_i) from (x_2, y_k).

Case 2. $p_2 \leq f(H)$.
In this case, $p_3 \geq f(H) + 1$.
Consider the following two possibilities.

(2.1). If $p_2 = 1$, then $p_3 \geq 2f(H)$.

(2.1.1). If $q_3 = 1$, then $[\frac{p_3}{2}]$ pebbles can be moved to $\{x_2\} \times H$ from $\{x_3\} \times H$. Since $[\frac{p_3}{2}] \geq f(H)$, $\{x_2\} \times H$ comes to at least $f(H) + 1$ pebbles. Thus one pebble can be moved to (x_1, y_i) as in the case 1.

(2.1.2). If $q_3 \geq 2$, then there exists some vertex (x_3, y_k) with more than one pebbles. Let (x_3, y_j) be another vertex with pebbles. Keep two pebbles on (x_3, y_k). Then we can put two pebbles on (x_3, y_j) by using $(p_3 - 2)$ pebbles on $\{x_3\} \times H$ because $p_3 - 2 \geq 2f(H) - 2 \geq f(H) + 1$. Also we can move one pebble from (x_3, y_k) to (x_2, y_s), where $s \neq i, j$. Then $\{(x_3, y_j), (x_2, y_s), (x_1, y_i)\}$ forms a transmitting subgraph of $G * H$. So we are done.

(2.2). If $2 \leq p_2 \leq f(H)$, then $p_3 \geq 2f(H) + 1 - f(H) = f(H) + 1$. By using p_2 pebbles on $\{x_2\} \times H$, we can put one pebble on some vertex (x_2, y_j) such that $j \neq i$. Since $p_3 \geq f(H) + 1$, we can put two pebbles on some vertex (x_3, y_s), where $s \neq j$. So $\{(x_3, y_s), (x_2, y_j), (x_1, y_i)\}$ forms a transmitting subgraph of $G * H$. Thus we are done.
Next, the target vertex is (x_2, y_i), for some i. If $p_2 \geq f(H)$, then we can pebble (x_2, y_i) because $\{x_2\} \times H$ is isomorphic to H. If $p_2 < f(H)$, then $p_1 + p_3 \geq 2f(H) + 1$. So one of them is larger than $f(H)$. W.L.O.G, we may assume that $p_1 \geq f(H) + 1$. Then we can move one pebble from $\{x_1\} \times H$ to (x_2, y_i) as in case1.
Finally, if the target vertex is (x_3, y_i), then we can prove it in the same way as when the target vertex is (x_1, y_i). □

LEMMA 1. *Let H be any graph with $|V(H)| \geq 4$. Then $f(K_{1,n} * H) \leq (n + 1)f(H)$*

Proof. Suppose that $(n + 1)f(H)$ pebbles are assigned to the vertices of $K_{1,n} * H$. Label the vertices of $K_{1,n}$ by $x_0, x_1 \dots x_n$ such that the degree of x_0 is n.

First, the target vertex is (x_0, y) with $y \in V(H)$. If $p(x_0, y) \geq 1$ or $p_0 \geq f(H)$, then we are done. Thus we may assume that $p(x_0, y) = 0$ and $p_0 < f(H)$. So $\sum_{i=1}^{n} p_i \geq nf(H) + 1$ and $p_i \geq f(H) + 1$, for some $i \in \{1, \dots n\}$. Thus as case 1 in the proof of the theorem 1, we can pebble (x_0, y)

Second, the target vertex is (x_i, y), for some $i \in \{1, \dots n\}$. If $p(x_i, y) \geq 1$ or $p_i \geq f(H)$, then we are done. Thus we may assume that $p(x_i, y) = 0$ and $p_i < f(H)$. Then $p_0 + p_1 + \cdots + p_{i-1} + p_{i+1} + \cdots + p_n \geq nf(H) + 1$. If $p_0 \geq f(H) + 1$, then we can pebble (x_i, y) as case 1 in the proof of the theorem 1.

If $p_0 \leq f(H)$, then we consider the following two possibilities.

(1) If there exists unique $j \in \{1, \dots i-1, i+1, \dots n\}$ with $p_j \geq f(H)+1$ then $p_i + p_0 + p_j \geq 3f(H)$. By theorem 1, we can pebble (x_i, y).

(2) If there exist s and t such that $s, t \in \{1, \dots i-1, i+1, \dots n\}$ with $p_s \geq f(H) + 1$ and $p_t \geq f(H) + 1$, then we can pebble some vertex (x_0, y'), $y \neq y'$ by using p_t pebbles on $\{x_t\} \times H$. By using p_s pebbles on $\{x_s\} \times H$, we can move one more pebble on (x_0, y') from $\{x_s\} \times H$. Hence we can pebble (x_i, y) from (x_0, y'). □

In the case of $|V(H)| < 4$, we have the following results which we can prove easily. Let g_n be the number of unlabelled connected graphs with n vertices. Then $g_1 = 1$, $g_2 = 1$ and $g_3 = 2$ by corollary 5.4 in [2]. So H is one of the following graphs P_1, P_2, P_3 and C_3 when $|V(H)| \leq 3$.

FACT. *Let C_3 be cycle with three vertices. Then*

(1) $f(P_3 * C_3) \leq 3f(C_3)$

(2) $f(P_3 * P_i) \leq 3f(P_i)$, *for* $i = 1, 2, 3$

(3) $f(K_{1,n} * C_3) \leq (n+1)f(C_3)$

(4) $f(K_{1,n} * P_i) \leq (n+1)f(P_i)$, for $i = 1, 2, 3$

By Lemma 1 and the above Fact, we have the following Theorem.

THEOREM 2. *Let H be any graph*
Then $f(K_{1,n} * H) \leq (n+1)f(H)$

COROLLARY 1. *Label the vertices of $K_{1,n}$ as $x_0, x_1, \ldots x_n$ such that the degree of x_0 is n. Consider $K_{1,n} * H$. If $p_0 + p_1 + \ldots. + p_{i-1} + p_{i+1} + \cdots + p_n \geq nf(H) + 1$ for each $i \in \{1, \ldots n\}$, then we can pebble any vertex (x_i, y) of $K_{1,n} * H$.*

COROLLARY 2. *If H is demonic, then $P_3 * H$ is also demonic.*

3. Pebbling $G * H$ with diameter$(G) = 2$.

In this section, we show that the pebbling number of $G * H$ with diameter $(G) = 2$ is not larger than $f(G)f(H)$.

DEFINITION : A *tree* is a connected acyclic graph. Let G and H be graphs. If $V(H) = V(G), E(H) \subset E(G)$, and H is a tree, then H is called a *spanning tree* of G. A vertex with degree one in a tree is called a *leaf.*

THEOREM 3. *Let G be a graph with diameter$(G) = 2$. Then $f(G * H) \leq f(G)f(H)$.*

Proof. Suppose that there are $f(G)f(H)$ pebbles assigned to the vertices of $G * H$ and diameter $(G) = 2$. Let $n = |V(G)|$ and label $V(G)$ as the following. Let the target vertex of $G * H$ be (x_1, y), $x_2, \ldots x_s$ be the vertices of G which are adjacent to x_1, and $x_{s+1}, \ldots x_n$ be the vertices of G which are not adjacent to x_1. So the distance of x_1 and $x_i (2 \leq i \leq s)$ is one and the distance of x_1 and $x_j (s+1 \leq j \leq n)$ is 2. If $p(x_1, y) \geq 1$ or $p_1 \geq f(H)$, then we are done. Therefore we may assume that $p(x_1, y) = 0$ and $p_1 < f(H)$. We consider the following two possibilities (1) and (2).

(1) If there exists some $x_i (2 \leq i \leq s)$ with $p_i \geq f(H)+1$, then we can pebble (x_1, y) as case 1 in the proof of theorem 1.

(2) $p_i \leq f(H)$, for all $i \in \{2, \ldots s\}$. Consider some spanning tree T of G such that x_1 is the root of T and $\{x_{s+1}, \ldots x_n\}$ is the set of all leaves of T. For each $i, j \in \{2, \ldots s\}$, let the subtree T_i of T consist of x_i and some leaves of T such that $V(T_i) \bigcap V(T_j) = \emptyset$ if $i \neq j$ and $\bigcup_{i=2}^{s} V(T_i) = V(G) - \{x_1\}$. Thus $1 + \Sigma_{i=2}^{s} |V(T_i)| = n$. Let $\Sigma_{x_t \in V(T_i)} p_t = n_i$. Then $p_1 + \sum_{i=2}^{s} n_i = f(G)f(H)$. There exists $i_0 \in \{2, \ldots s\}$ such that $n_{i_0} \geq |V(T_{i_0})| f(H) + 1$. Indeed, if $n_i \leq |V(T_i)| f(H)$ for all $i \in \{2, \ldots s\}$, then $f(G)f(H) = p_1 + \sum_{i=2}^{s} n_i < f(H) + \sum_{i=2}^{s} |V(T_i)| f(H) = (1 + \Sigma_{i=2}^{s} |V(T_i)|) f(H) = n f(H)$. This is a contradiction. Hence we can pebble (x_1, y) by corollary 1. $\square$

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