Two Pebbling Theorems

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Abstract

Given a distribution D of pebbles on the vertices V of a graph G, a pebbling step consists of removing two pebbles from a vertex u and placing one pebble on an adjacent vertex v. For a vertex r, D is r-solvable if it is possible to place a pebble on r after a sequence of pebbling steps. Then D is solvable if it is r-solvable for all r. The pebbling number $f(G)$ is the least t so that every distribution of t pebbles on V is solvable. A well known conjecture due to Graham is that $f(G_1 \square G_2) \leq f(G_1)f(G_2)$, where $\square$ denotes the cartesian product. In this paper we prove a result involving a more general product, generalizing a technique of Chung. We also prove a result regarding the pebbling numbers of regular bipartite graphs.

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1 Introduction

Suppose t pebbles are distributed onto the vertices of a graph G. A pebbling step $[u, v]$ consists of removing two pebbles from one vertex u and then placing one pebble at an adjacent vertex v. We say a pebble can be *moved* to a vertex r, the *root* vertex, if we can repeatedly apply pebbling steps so that in the resulting distribution r has at least one pebble.

For a graph G, we define the *pebbling number*, $f(G)$, to be the smallest integer t such that for any distribution of t pebbles to the vertices of G, one pebble can be moved to any specified root vertex r. If D is a distribution of pebbles on the vertices of G and it is possible to move a pebble to the root vertex r, then we say that D is *r-solvable*. Otherwise, D is *r-unsolvable*. Then D is *solvable* if it is r-solvable for all r, and *unsolvable* otherwise. We denote by $D(v)$ the number of pebbles on vertex v in D and let the *size*, $|D|$, of D be the total number of pebbles in D, that is $|D| = \sum_v D(v)$. Thus $f(G)$ is one more than the maximum t such that there exists an unsolvable pebbling distribution D of size t.

Define the *support* of a distribution D to be the set of vertices v for which $D(v) > 0$ and let $q = q(D)$ be the number of such vertices. We say that a graph G has the *2-pebbling property* if, whenever D satisfies $|D| \geq 2f(G) - q + 1$, one can move two pebbles to any vertex r by a sequence of pebbling steps.

The origins of graph pebbling stem from an attempt by Lagarias and Saks to solve a problem of Erdős and Lemke by a different method than found in [8]. One can read [1, 3] for more on the subject. A nice generalization is found in [4].

Throughout this paper G will denote a simple connected graph, where $n(G) = |V(G)|$, and $f(G)$ will denote the pebbling number of G. For any two graphs G_1 and G_2, we define the *cartesian product* $G_1 \Box G_2$ to be the graph with vertex set $V(G_1 \Box G_2) = \{(v_1, v_2) | v_1 \in V(G_1), v_2 \in V(G_2)\}$ and edge set $E(G_1 \Box G_2) = \{((v_1, v_2), (w_1, w_2)) | (v_1 = w_1 \text{ and } (v_2, w_2) \in E(G_2)) \text{ or } (v_2 = w_2 \text{ and } (v_1, w_1) \in E(G_1))\}$. Thus the m-dimensional cube Q^m can be written as the cartesian product of an edge with itself m times.

We are concerned in this paper with a more general kind of product. Given two graphs G_1 and G_2, denote by $B(G_1, G_2)$ the set of all bipartite graphs F such that $E(F) \subseteq V(G_1) \times V(G_2)$ and such that F has no isolated vertices. We let $\mathcal{M}(G_1, G_2)$ be the set of graphs $\{H | H = (G_1 + G_2) \cup F \text{ for some } F \in B(G_1, G_2)\}$, where $+$ denotes the vertex disjoint graph union. Clearly, $G_1 \Box G_2 \in \mathcal{M}(G_1, G_2)$. In section 2.1 we prove the following generalization of a theorem of Chung [1].

Theorem 1.1 *Let G_1 and G_2 have the 2-pebbling property and suppose $H \in \mathcal{M}(G_1, G_2)$. Then $f(H) \leq f(G_1) + f(G_2)$. Furthermore, if $f(H) = f(G_1) + f(G_2)$ then H has the 2-pebbling property.*

We say that a graph G is of *Class* i if $f(G) = n(G) + i$. It is natural to look for properties which guarantee that a graph G is of Class 0. Some examples of Class 0 graphs include cliques, cubes, the 5-cycle, the Petersen graph, and most diameter two graphs (see Result 1.9). For a family $\mathcal{G}$ of graphs, we say that $\mathcal{G}$ is of Class 0 if every graph $G \in \mathcal{G}$ is of Class 0.

For $2 \leq k \leq m$ define $\mathcal{R}(m, k)$ to be the set of connected k-regular bipartite graphs on $n = 2m$ vertices. Let $r(m)$ be the minimum r such that $\mathcal{R}(m, k)$ is of Class 0 for all $k \geq r$. Because $\mathcal{R}(m, 2)$ consists of the $2m-cycle$, $r(2) = 2$ and $r(m) > 2$ for all $m > 2$. It is trivial that $r(m) \leq m$ for all $m \geq 2$, and it is shown in [2] that $r(m) \leq m - 1$ for all $m \geq 4$. Hence $r(3) = r(4) = 3$. The vertex-transitive graph in $\mathcal{R}(5, 3)$ is of Class 0 but the other graph in $\mathcal{R}(5, 3)$ is not, and hence $r(5) = 4$. We prove in section 2.2 the following result, improving the upper bound to roughly $2m/3$.

Theorem 1.2 *For $m \geq 6$, let $m = 3a + \epsilon$ with $\epsilon \in \{0, 1, 2\}$. Then $r(m) \leq 2a + \epsilon + 1$.*

We begin with some introductory results.

1.1 Past Results

If one pebble is placed at each vertex other than the root vertex, r, then no pebble can be moved to r. Also, if w is at distance l from r, and $2^l - 1$ pebbles are placed at w, then no pebble can be moved to r. On the other hand, if more than $(2^d - 1)(n - 1)$ pebbles are placed on the vertices of a graph of diameter d then either every vertex has at least one pebble on it or some vertex w has at least 2^d pebbles on it. In either case one can immediately pebble from w to any vertex r. We record these observations as

Fact 1.3 *Let $d = diam(G)$ and $n = n(G)$. Then $max\{n, 2^d\} \leq f(G) \leq (2^d - 1)(n - 1) + 1$.*

Of course this means that $f(K_n) = n$, where K_n is the complete graph on n vertices.

Let P_n denote the path on $n + 1$ vertices. A simple weight function method shows that $f(P_n) = 2^n$. For a given distribution D and leaf root r define the weight $w(D) = \sum_v w(v)$, where $w(v) = D(v)/2^{dist(v,r)}$. Because the weight of a distribution is preserved under pebbling steps in the direction of r, D is an r-unsolvable distribution if and only if $w(D) < 1$. Because pebbling reduces the size of a distribution, if D has maximum size with respect to r-unsolvable distributions then all its pebbles lie on the leaf opposite from r, implying $|D| = 2^n - 1$. Finally, for any other choice of root r, one applies the above argument to both sides of r and notices that $(2^a - 1) + (2^b - 1) < 2^{a+b} - 1$.

The pebbling number of a tree T on n vertices is more complicated. One should consult [9] for the relevant definitions.

Result 1.4 *[9] Let $(q_1, q_2, \ldots, q_m)$ be the nonincreasing sequence of path lengths of a maximum path partition $Q = (Q_1, \ldots, Q_m)$ of a tree T. Then $f(T) = \left(\sum_{i=1}^{m} 2^{q_i}\right) - m + 1$.*

Let C_n be the cycle on n vertices. The pebbling numbers of cycles is derived in [10].

Result 1.5 *[10] For $k \geq 1$, $f(C_{2k}) = 2^k$ and $f(C_{2k+1}) = 2\lfloor \frac{2^{k+1}}{3} \rfloor + 1$.*

The following conjecture has generated a great deal of interest.

Conjecture 1.6 *(Graham) For all G_1 and G_2, $f(G_1 \Box G_2) \leq f(G_1)f(G_2)$.*

Several results support Graham's conjecture. Among them, the conjecture holds for a tree by a tree [9], a cycle by a cycle (with possibly some small exceptions: it holds for $C_5 \Box C_5$ [7], and otherwise for $C_m \Box C_n$, provided m and n are not both from the set $\{5, 7, 9, 11, 13\}$ [10], and a clique by a graph with the 2-pebbling property [1]. Also we find the pebbling numbers of cubes in [1].

Result 1.7 *[1] For all $m \geq 0$, $f(Q^m) = 2^m$.*

In [10] we find the following theorem.

Result 1.8 *[10] If $diam(G) = 2$ then $f(G) \leq n(G) + 1$.*

It was characterized in [3] presicely which diameter two graphs are of Class 0 and which are not. Let $\kappa = \kappa(G)$ be the connectivity of G. It is not difficult to see that $f(G) > n(G)$ whenever $k = 1$, so we will assume that $\kappa \geq 2$.

Define a graph H to be in the family $\mathcal{F}$ as follows (see Figure 1). Every vertex of the nonempty (not necessarily connected) subgraph H_r' is adjacent to both a and b, every vertex of the (not necessarily connected, possibly empty) subgraph H_c' is adjacent to c and to at least one of a or b, and every vertex of the nonempty (connected) subgraph H_p' (resp. H_q') is adjacent to both a and c (resp. b and c). Also, at least two edges exist between a, b, and c.

Result 1.9 *[3] Let $diam(G) = 2$ and $\kappa(G) \geq 2$. Then G is of Class 0 if and only if $G \notin \mathcal{F}$.*

We also find in [3] the following corollary and interesting conjecture.

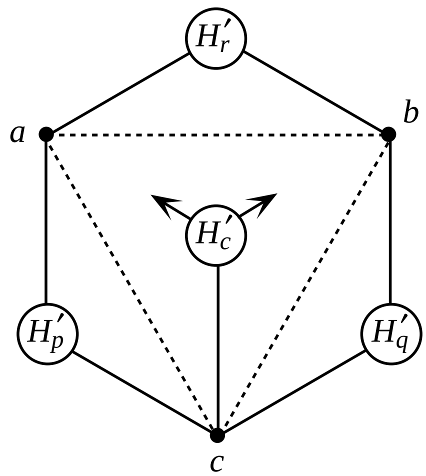

Figure 1: A schematic diagram of graphs $H \in \mathcal{F}$.

Corollary 1.10 *[3] Let diam$(G) = 2$. If $\kappa(G) \geq 3$ then G is of Class 0.*

Conjecture 1.11 *[3] There is a function k such that if diam$(G) = d$ and $\kappa(G) \geq k(d)$ then G is of Class 0.*

This conjecture is open for all $d > 2$. Other results can be found in [1, 2, 3, 5, 6, 7].

2 Proofs

2.1 General Product Graphs

Our proof of Theorem 1.1 follows the strategy employed in [1]. The results of this section can also be found in [2].

Proof of Theorem 1.1. Assume that we have a distribution D of size $d = f(G_1) + f(G_2)$. Let $f = f(H)$, $f_i = f(G_i)$, $q = q(D)$, and $q_i = q(D_{G_i})$. Without loss of generality let the root vertex $r \in V_1$ and then choose $r' \in V_2 \cap N(r)$, where $N(v)$ is the *neighborhood* (set of adjacent vertices) of a vertex v.

Then for some integer x we have $d_1 = f_1 - x$ and $d_2 = f_2 + x$. Since G_2 has the 2-pebbling property we may assume

$$x \leq f_2 - q_2. \tag{1}$$

Otherwise we could move two pebbles to r' and then one to r. From equation 1 it follows that

$$q_2 \leq f_2 - x. \tag{2}$$

Now we will move as many pebbles as possible from G_2 to G_1. We can move at least

$$\frac{f_2 + x - q_2}{2} \geq \frac{f_2 + x - (f_2 - x)}{2} = x \tag{3}$$

pebbles to G_1, yielding f_1 pebbles on G_1, and then we can move a pebble to r. Therefore $f \leq f_1 + f_2$.

Now to show the second part of the theorem, assume that $f = f_1 + f_2$ and that $d = 2f - q + 1$. If $d_1 > 2f_1 - q_1$ then since G_1 has the 2-pebbling property we are done. If instead $f_1 < d_1 \leq 2f_1 - q_1$ then we can move one pebble to r by the definition of f_1, and since $d_2 \geq 2(f_1 + f_2) - q - (2f_1 - q_1) + 1 = 2f_2 - q_2 + 1$ we can move two pebbles to r' and then one to r. Finally, if $d_1 < f_1$ then $d_1 = f_1 - x$ for some x. Notice that, for $d_2 \geq q_2 + 2t$, t pebbles can be moved to G_1, while $d_2 - 2t$ pebbles remain on G_1. Then $d_2 = d - d_1 = (2f - q + 1) - (f_1 - x) = f_1 + 2f_2 - q_1 - q_2 + x + 1$ and so

$$d_2 \geq 2f_2 - q_2 + 2x + 1 \geq q_2 + 2x. \tag{4}$$

The last inequality follows since $q_1 \leq f_1 - x$ and $q_2 \leq f_2$. Thus we can move x pebbles to G_1, yielding f_1 pebbles on G_1. Therefore we can move one pebble to r. Also, $d_2 - 2x \geq 2f_2 - q_2 + 1$ pebbles remain on G_2. Hence we can move two pebbles to r' and then one to r. This puts two pebbles on r. □

It is important to observe the following corollary.

Corollary 2.1 *Let G_1 and G_2 have the 2-pebbling property and suppose $H \in \mathcal{M}(G_1, G_2)$. If G_i s of Class 0 for each i then H is of Class 0 and has the 2-pebbling property.*

This corollary can be used inductively to prove Theorem 1.7. It can be used also to prove not only that the Petersen graph P has pebbling number 10 (which can be proved by ad hoc methods) but that P has the 2-pebbling property as well. This is because C_5 has the 2-pebbling property [10], $P \in \mathcal{M}(C_5, C_5)$, and $f(C_5) = 5$.

We can generalize our product as follows. Denote by $\mathcal{M}(G_1, \ldots, G_t)$ the set of all graphs H such that $H[V_i \cup V_j] \in \mathcal{M}(G_i, G_j)$ for all $i \neq j$, where $V_i = V(G_i)$. For example $G \square K_t \in \mathcal{M}(G_1, \ldots, G_t)$, with each $G_i \cong G$. Theorem 1.1 is also the base case in an induction argument which proves the following result.

Theorem 2.2 *Let G_i have the 2-pebbling property for $1 \le i \le t$ and let $H \in \mathcal{M}(G_1, \ldots, G_t)$. Then $f(H) \le \sum_{i=1}^{t} f(G_i)$. Moreover, if $f(H) = \sum_{i=1}^{t} f(G_i)$ then H has the 2-pebbling property.*

Proof. The case $t = 2$ is Theorem 1.1. We suppose that the statements are true for all $2 \le t < m$ and consider the case $t = m$. Let $H' = H[V_1 \cup \cdots \cup V_{m-1}]$ and notice that $H \in \mathcal{M}(H', G_m)$. Thus $f(H) \le f(H') + f(G_m) \le \sum_{i=1}^{m} f(G_i)$ by cases $t = 2$ and $t = m - 1$. Furthermore, if $f(H) = \sum_{i=1}^{m} f(G_i)$ then $f(H') = \sum_{i=1}^{m-1} f(G_i)$ and so H' has the 2-pebbling property by case $t = m - 1$. Finally, this means that H has the 2-pebbling property by case $t = 2$. □

Analogously, the following corollary is proved easily.

Theorem 2.3 *Let G_i have the 2-pebbling property for $1 \le i \le t$ and let $H \in \mathcal{M}(G_1, \ldots, G_t)$. If G_i is of Class 0 for each i then H is of Class 0 and has the 2-pebbling property.*

We also obtain the following.

Result 2.4 *[1] If G has the 2-pebbling property then $f(G \square K_t) \le t f(G)$.*

2.2 Regular Bipartite Graphs

Proof of Theorem 1.2. We begin with some notation. Let $G \in \mathcal{R}(m, k)$ with $m = 3a + \epsilon \ge 6$ ($a \ge 2$, $\epsilon \in \{0, 1, 2\}$) and $k = 2a + \epsilon + 1$. Suppose $V(G)$ has the bipartition $X \cup Y$ with $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_m\}$. For any subset $I \subset \{1, \ldots, m\}$ denote by $Y(I)$ its set of common neighbors, that is the set of vertices of Y which are adjacent to x_i for every $i \in I$. Define $X(I)$ analogously. Because $3k > 2m$, we know that if $|I| = 2$ then $|Y(I)| \ge a + \epsilon + 2$, and if $|I| = 3$ then $|Y(I)| \ge 1$ (and likewise for $|X(I)|$ in each case).

Suppose that D is an unsolvable distribution of size $2m$. We argue for a contradiction. Without loss of generality, let D be r-unsolvable for the root $r = x_1$. For any set of vertices S, we define D_S to be the restriction of D to S, and denote its size by $|D_S|$. Of course $D(r) = 0$, and because $diam(G) = 3$ we know that $\max D < 8$. Also $\max D_X < 4$ and $\max D_{Y(1)} < 2$.

Claim 1: $|D_X| \le m + 1$.

Otherwise there are vertices x_{α_1} and x_{α_2} with each $D(x_{\alpha_i}) > 1$. In this case we find $y \in Y(\alpha_1, \alpha_2)$ and solve D by pebbling $[x_{\alpha_1}, y][x_{\alpha_2}, y][y, r]$.

Thus $|D_Y| \geq m-1$.

Claim 2: $\max D_X \leq 1$.

Otherwise, let $D(x_\alpha) > 1$. If for some $y \in Y(1,\alpha)$ we find $D(y) > 0$ then $[x_\alpha, y][y, r]$ solves D. Thus $|D_{Y(1,\alpha)}| = 0$. Because $|Y(1,\alpha)| \geq a+\epsilon+2$ we have $|Y(1)-Y(1,\alpha)| \leq a-1$, and since $\max D_{Y(1)} \leq 1$ we obtain $|D_{Y(1)}| \leq a-1$. This implies that $|D_{Y(1)-Y(1,\alpha)}| \geq (m-1)-(a-1) = 2a+\epsilon$. Hence there is some $y \in Y(\alpha)-Y(1,\alpha)$ with $D(y) \geq 2$. It must be that $D(x_\alpha) = 2$, or else, for $y' \in Y(1,\alpha)-Y(\alpha)$ we solve D by pebbling $[y, x_\alpha][x_\alpha, y']^2[y', r]$. Therefore we have $|D_X| \leq m$, $|D_Y| \geq m$, and $|D_{Y(1)-Y(1,\alpha)}| \geq 2a+\epsilon+1$.

We can pebble similarly if there are vertices $y_{\beta_1}, y_{\beta_2} \in Y(\alpha)-Y(1,\alpha)$ with each $D(y_{\beta_j}) > 1$, so we may assume that y is unique. This means that $|D_{Y(\alpha)-Y(1,\alpha)}| \leq |Y(\alpha)-Y(1,\alpha)|+2 \leq a+1$, which contradicts that $|D_{Y(\alpha)-Y(1,\alpha)}| \geq 2a+\epsilon+1$, thereby proving the claim.

Hence $|D_X| \leq m-1$ and $|D_Y| \geq m+1$. Also, some y_{β_1} satisfies $D(y_{\beta_1}) > 1$.

Claim 3: $|D_{X(\beta_1)}| = 0$.

Otherwise, let $D(x_{\alpha_1}) > 0$ for some $x_{\alpha_1} \in X(\beta_1)$. It must be that $|D_{Y(1,\alpha_1)}| = 0$, or else $[y_{\beta_1}, x_{\alpha_1}][x_{\alpha_1}, y_{\beta_2}][y_{\beta_2}, r]$ solves D for some $y_{\beta_2} \in Y(1,\alpha_1)$. Since $\max D_{Y(1)} \leq 1$ and $|Y(1)-Y(1,\alpha_1)| \leq a-1$, we derive $|D_{Y(\alpha_1)-Y(1,\alpha_1)}| \geq (m+1)-(a-1) = 2a+\epsilon+2$. Because $|Y(\alpha_1)-Y(1,\alpha_1)| \leq a-1$ there must be some y_{β_2} with $D(y_{\beta_2}) > 1$. Now we must have $|D_{Y(1)}| = 0$, or else $[y_{\beta_1}, x_{\alpha_1}][y_{\beta_2}, x_{\alpha_1}][x_{\alpha_1}, y]^2[y, r]$ solves D for some $y \in Y(1,\alpha_1)$. Therefore $|D_{Y(\alpha_1)-Y(1,\alpha_1)}| \geq m+1$.

If $D(y_{\beta_j}) > 3$ for either $j \in \{1,2\}$ (say $j=2$) then choose $x_{\alpha_2} \in X(\beta_2)$ and $y \in Y(1,\alpha_1,\alpha_2)$ and pebble $[y_{\beta_1}, x_{\alpha_1}][y_{\beta_2}, x_{\alpha_2}]^2[x_{\alpha_1}, y][x_{\alpha_2}, y][y, r]$ to solve D. Otherwise there is some y_{β_3} with $D(y_{\beta_3}) > 1$. Now solve D by pebbling $[y_{\beta_1}, x_{\alpha_1}][y_{\beta_2}, x_{\alpha_1}][y_{\beta_3}, x_{\alpha_1}][x_{\alpha_1}, y]^2[y, r]$, where $y \in Y(1,\alpha_1)$. This proves Claim 3.

Now we know that $|D_Y| \geq 2m-(a-2) = 5a+2\epsilon+2$, and $|D_{Y-Y(1)}| \geq |D_Y|-(2a+\epsilon+1) = 3a+\epsilon+1$. Because $|Y-Y(1)| = a-1$, we can find some y_{β_1} such that $D(y_{\beta_1}) > 3$. If some $y_{\beta_2} \in Y(1)$ has $D(y_{\beta_2}) > 0$ then for any $x \in X(\beta_1,\beta_2)$ $[y_{\beta_1}, x]^2[x, y_{\beta_2}][y_{\beta_2}, r]$ solves D. Thus $|D_Y(1)| = 0$, which implies that $|D_{Y-Y(1)-y_{\beta_1}}| = |D_Y|-D(y_{\beta_1}) \geq 5a+2\epsilon-5 = 5(a-2)+(2\epsilon+5)$ (since $D(y_{\beta_1}) \leq 7$). Because $|Y-Y(1)-y_{\beta_1}| = a-2$ we can find some y_{β_2} with $D(y_{\beta_2}) > 5$. Now we can solve D by choosing $y_{\beta_3} \in Y(1)$ and $x \in X(\beta_1,\beta_2,\beta_3)$ and pebbling $[y_{\beta_1}, x][y_{\beta_2}, x]^3[x, y_{\beta_3}]^2[y_{\beta_3}, r]$. This is the final contradiction which proves Theorem 1.2. □

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