Pebbling numbers of some graphs

FENG Rongquan (冯荣权)[1] & Ju Young Kim[2]

1. School of Mathematical Sciences, Peking University, Beijing 100871, China
2. Department of Mathematics, Catholic University of Daegu, Kyongsan 713-702, Korea
Correspondence should be addressed to Feng Rongquan (email: fengrq@sxx0.math.pku.edu.cn)

Received April 9, 2001

Abstract Chung defined a pebbling move on a graph G as the removal of two pebbles from one vertex and the addition of one pebble to an adjacent vertex. The pebbling number of a connected graph G, $f(G)$, is the least n such that any distribution of n pebbles on G allows one pebble to be moved to any specified but arbitrary vertex by a sequence of pebbling moves. Graham conjectured that for any connected graphs G and H, $f(G \times H) \leqslant f(G)f(H)$. In the present paper the pebbling numbers of the product of two fan graphs and the product of two wheel graphs are computed. As a corollary, Graham's conjecture holds when G and H are fan graphs or wheel graphs.

Keywords: pebbling, Graham's conjecture, Cartesian product, fan graph, wheel graph.

Pebbling in graphs was first considered by Chung[1]. Consider a connected graph with a fixed number of pebbles distributed on its vertices. A pebbling move consists of the removal of two pebbles vertex, and the placement of one of those two pebbles on an adjacent vertex. The pebbling number of a vertex v in a graph G is the smallest number $f(G,v)$ with the property that from every placement of $f(G,v)$ pebbles on G, it is possible to move a pebble to v by a sequence of pebbling moves. The pebbling number of a graph G, denoted by $f(G)$, is the maximum $f(G,v)$ over all the vertices of G.

A graph G is called demonic if $f(G)$ is equal to the number of its vertices. Very little is known of $f(G)$ (see refs. [1—6]). If one pebble is placed on each vertex other than the vertex v, then no pebble can be moved to v. Also, if w is at a distance d from v, and $2^d - 1$ pebbles are placed on w, then no pebble can be moved to v. So it is clear[1] that $f(G) \geqslant \max\{|V(G)|, 2^D\}$, where $|V(G)|$ is the number of vertices of G, and D is the diameter of the graph G. Furthermore, we know that K_n and $K_{s,t}$ are demonic when $s > 1$ and $t > 1$ (see refs. [1, 2]), where K_n is the complete graph on n vertices, and $K_{s,t}$ is the complete bipartite graph whose two partition sets have s and t vertices respectively. But $f(P_n) = 2^{n-1}$ (see ref. [1]), i.e. the graph P_n is not demonic when $n > 2$, where P_n is the path on n vertices.

Given a pebbling of G, let p be the number of pebbles, q be the number of vertices with at least one pebble, and let r be the number of vertices with an odd number of pebbles. We say that G satisfies the two-pebbling property (respectively, weak two-pebbling property) if it is possible to move two pebbles to any specified target vertex whenever p and q satisfy the inequality $p + q > 2f(G)$ (respectively, whenever p and r satisfy $p + r > 2f(G)$). Note that any graph which satisfies the two-pebbling property also satisfies the weak two-pebbling property. Given a pebbling of G, a transmitting subgraph of G is a path $x_1, x_2, \cdots, x_k$ such that there are at least two pebbles on x_1 and at least one pebble on each of the other vertices in the path, except possibly x_k. In this case, we can transmit a pebble from x_1 to x_k.

This paper explores the pebbling number of the Cartesian product of fan graphs. The idea for Cartesian products comes from a conjecture of Graham[1]. The conjecture states that for any connected graphs G and H, $f(G \times H) \leqslant f(G)f(H)$. It is worth mentioning that there are a few results which verify Graham's conjecture. Among them, the conjecture holds of a tree by a tree[3], a cycle by a cycle1), a clique by a graph with the two-pebbling property[1] and a complete bipartite graph by a graph with the two-pebbling property[2]. In the present paper we show that Graham's conjecture holds of the product of two fan graphs or two wheel graphs. To be exactly, we prove that the product of two fan graphs or two wheel graphs are demonic, i.e. $f(F_n \times F_m) = f(W_n \times W_m) = nm$.

Throughout this paper G denotes a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex v of a graph G, $p(v)$ refers to the number of pebbles on v and $d(v)$ denotes the degree of v.

1 Fan graphs

A fan graph, denoted by F_n, is a path P_{n-1} plus an extra vertex connected to all vertices of the path P_{n-1}. Throughout this paper, a fan graph with vertices $v_0, v_1, \cdots, v_{n-1}$ in order means the fan graph F_n whose vertices of the path P_{n-1} are $v_1, \cdots, v_{n-1}$ in order and whose extra vertex is v_0. The fan graph F_n satisfies the two-pebbling property because the diameter of F_n is 2 (see ref. [6]).

Theorem 1. The fan graph F_n is demonic, i.e. $f(F_n) = n$.

Proof. Suppose that there are n pebbles distributing on the vertices of F_n, where F_n is the fan graph with vertices $v_0, v_1, \cdots, v_{n-1}$ in order. First, let the target vertex be v_0. If $p(v_0) = 0$, then there exists some $i \in \{1, \cdots, n-1\}$ with $p(v_i) \geqslant 2$. So we can move one pebble to v_0 from v_i. Next, suppose the target vertex is v_k and $p(v_k) = 0$ where $k \in \{1, \ldots, n-1\}$. We consider the following three cases.

(i) If $p(v_0) \geqslant 2$ or $p(v_i) \geqslant 4$ for some $i \geqslant 1$, then $\{v_0, v_k\}$ or $\{v_i, v_0, v_k\}$ forms a transmitting subgraph. So one pebble can be moved to v_k.

(ii) If $p(v_0) = 1$ and $p(v_i) \leqslant 3$ for all $i \geqslant 1$, then there exists some v_i with $p(v_i) \geqslant 2$. So $\{v_i, v_0, v_k\}$ forms a transmitting subgraph.

(iii) If $p(v_0) = 0$ and $p(v_i) \leqslant 3$ for all $i \geqslant 1$, then there must be at least one vertex v_j with $p(v_j) \geqslant 2$. If in addition there are at least two vertices v_j and v_ℓ with $2 \leqslant p(v_j), p(v_\ell) \leqslant 3$, then we can move one pebble from v_ℓ to v_0. So $\{v_j, v_0, v_k\}$ is a transmitting subgraph. Otherwise, there is only one vertex v_j with $2 \leq p(v_j) \leqslant 3$. Therefore, for every v_i, $1 \leqslant i \leqslant n-1$ and $i \neq k$, we have $p(v_i) \geqslant 1$. So $\{v_j, v_{j-1}, \cdots, v_k\}$ or $\{v_j, v_{j+1}, \cdots, v_k\}$ is a transmitting subgraph according to $j > k$ or $j < k$.

1) Herscovici, D. S., Graham's Pebbling Conjecture on Product of Cycles, preprint.

2 Cartesian product

Let G and H be two graphs, the (Cartesian) product of G and H, denoted by $G \times H$, is the graph whose vertex set is the Cartesian product

$$V(G \times H) = V(G) \times V(H) = (x, y)|x \in V(G), y \in V(H),$$

and two vertices (x, y) and (x', y') are adjacent if and only if $x = x'$ and $\{y, y'\} \in E(H)$, or $\{x, x'\} \in E(G)$ and $y = y'$. We can depict $G \times H$ pictorially by drawing a copy of H at every vertex of G and connecting each vertex in one copy of H to the corresponding vertex in an adjacent copy of H. We write $\{x\} \times H$ (respectively, $G \times \{y\}$) for the subgraph of vertices whose projection onto $V(G)$ is the vertex x (respectively, whose projection onto $V(H)$ is y). If the vertices of G are labelled by x_i, then for any distribution of pebbles on $G \times H$, we write p_i for the number of pebbles on $\{x_i\} \times H$, q_i for the number of occupied vertices of $\{x_i\} \times H$ and r_i for the number of vertices of $\{x_i\} \times H$ with an odd number of pebbles. Note that p_i and r_i have the same parity and $r_i \leqslant p_i$. Let $\{x_i, x_j\}$ be an edge in G. If $p(x_i, y_k)$ is even, then $p(x_i, y_k)/2$ pebbles can be moved to (x_j, y_k) from (x_i, y_k) and if $p(x_i, y_k)$ is odd, then $(p(x_i, y_k) - 1)/2$ pebbles can be moved to (x_j, y_k), where y_k is a vertex of the graph H. Anyhow we can move $(p_i - r_i)/2$ pebbles to $\{x_j\} \times H$ from $\{x_i\} \times H$.

The following conjecture, by Ronald Graham, suggests a constraint on the pebbling number of the product of two graphs.

Conjecture (Graham). The pebbling number of $G \times H$ satisfies $f(G \times H) \leqslant f(G)f(H)$.

Lemma 1. Let G be a connected graph. If $p = |V(G)| + 1$ and $q = |V(G)|$, then we can put two pebbles on any vertex of G.

Lemma 2. Let P_3 be the path with vertices x_1, x_2 and x_3 in order and let G be a connected graph with the weak two-pebbling property. Consider the graph $P_3 \times G$. If $p_1 + r_1 > 2f(G)$ and $p_3 + r_3 > 2f(G)$, then we can move two pebbles on any vertex (x_2, v) in $\{x_2\} \times G$.

Lemmas 1 and 2 are straightforward. The following lemmas are necessary in proving our main theorem.

Lemma 3.[1] Let K_2 be the complete graph on two vertices. Suppose that G satisfies the two-pebbling property. Then $f(K_2 \times G) \leqslant 2f(G)$.

Lemma 4. Let P_3 be the path with vertices x_1, x_2 and x_3 in order and let F_m be the fan graph with vertices $v_0, v_1, \cdots, v_{m-1}$ in order. Consider the graph $P_3 \times F_m$. If $p_1 + r_1 > 2f(F_m)$ and $p(x_2, v_0) \geqslant 2$, then we can pebble any vertex (x_3, v_i) in $\{x_3\} \times F_m$.

Proof. If $v_i = v_0$, then one pebble can be moved to (x_3, v_0) from (x_2, v_0). Next, suppose that $v_i \neq v_0$. Since $p_1 + r_1 > 2f(F_m)$, we can put two pebbles on (x_1, v_i) and these two pebbles can be used to pebble (x_2, v_i). Therefore $\{(x_2, v_0), (x_2, v_i), (x_3, v_i)\}$ forms a transmitting subgraph.

Lemma 5. Let K_2 be the complete graph on two vertices x_1 and x_2, and let F_m be the fan graph with vertices $v_0, v_1, \cdots, v_{m-1}$ in order. If $p_1 + r_1 = 2f(F_m)$, $p_1 - r_1 = 2$ and $p(x_1, v_0) = 0$ in $K_2 \times F_m$, then we can pebble any vertex (x_2, v_i) for $i = 1, \cdots, m-1$.

Proof. By assumption, $p_1 = m + 1$ and $r_1 = m - 1$, so all vertices (x_1, v_i) except (x_1, v_0) in $\{x_1\} \times F_m$ are occupied and there exists some (x_1, v_j) such that $p(x_1, v_j) \geqslant 2$. So we can put two pebbles on any (x_1, v_i) except (x_1, v_0) in $\{x_1\} \times F_m$. Therefore one pebble can be moved to (x_2, v_i) from (x_1, v_i) for $i = 1, \cdots, m-1$.

By Lemma 5 and the definition of two-pebbling property, we have

Lemma 6. Let P_3 be the path with vertices x_1, x_2 and x_3 in order and let F_m be the fan graph with vertices $v_0, v_1, \cdots, v_{m-1}$ in order. If $p_1 + r_1 > 2f(F_m)$, $p(x_2, v_0) \geqslant 1$, $p_3 + r_3 = 2f(F_m)$ and $p_3 - r_3 = 2$, then two pebbles can be put on any vertex (x_2, v_i) in $\{x_2\} \times F_m$.

Lemma 7. Let $K_{1,3}$ be the star graph with vertices x_0, x_1, x_2 and x_3 such that $d(x_0) = 3$ and let F_m be the fan graph with vertices $v_0, v_1, \cdots, v_{m-1}$ in order. Consider $K_{1,3} \times F_m$. If $p_1 + r_1 > 2f(F_m)$, $p_i + r_i = 2f(F_m)$ and $p_i - r_i = 2$ for $i = 2, 3$, then we can put two pebbles on any vertex (x_0, v_j) in $\{x_0\} \times F_m$.

Proving Lemma 7 involves examining various cases of $p(x_2, v_0)$ and $p(x_3, v_0)$. We omit the details here.

Lemma 8[2]. Suppose that G satisfies the two-pebbling property. Consider the graph $K_{1,n} \times G$. Let v_0 be the vertex of $K_{1,n}$ with degree n. To pebble a target vertex on $\{v_0\} \times G$, it suffices to start with $(n+1)f(G)$ pebbles on $K_{1,n} \times G$.

3 Pebbling $F_n \times F_m$

Lemma 9. Let F_m be the fan graph with vertices $y_0, y_1, \cdots, y_{m-1}$ in order. Given a pebbling of F_m, let p be the number of pebbles and r be the number of vertices with an odd number of pebbles. Then at least $(p-r)/2$ pebbles can be put on the vertex y_0 and at least $(p+r)/2$ pebbles are still on F_m.

Proof. Suppose that $p(y_0) = t$. For any $i \in \{1, \cdots, m-1\}$, if $p(y_i)$ is even, then $p(y_i)/2$ pebbles can be moved to y_0 from y_i and if $p(y_i)$ is odd, then $(p(y_i) - 1)/2$ pebbles can be moved to y_0. Therefore $t + [(p-t) - (r-\delta)]/2 \geqslant (p-r)/2$ pebbles can be put on y_0 and $p - [(p-t) - (r-\delta)]/2 \geqslant (p+r)/2$ pebbles are still on F_m, where $\delta = 0$ or 1 according as t is even or t is odd.

Lemma 10. Let F_n and F_m be two fan graphs with vertices $v_0, v_1 \cdots, v_{n-1}$ in order and $y_0, y_1, \cdots, y_{m-1}$ in order respectively. Suppose that $f(F_{n'} \times F_m) = n'm$ for all $n' < n$ and there are nm pebbles assigned to vertices of $F_n \times F_m$. If $p_{n-2} = r_{n-2} = m$ in $F_n \times F_m$, then we can put one pebble on any vertex (v_{n-1}, y_i) in $\{v_{n-1}\} \times F_m$.

Proof. If $p_{n-1} \geqslant m$, then we can pebble (v_{n-1}, y_i) since $f(\{v_{n-1}\} \times F_m) = m$. Now suppose that $p_{n-1} < m$. Let F_{n-1} be the fan graph with vertices $v_0, v_1, \cdots, v_{n-2}$ in order. Then $F_{n-1} \times F_m$ is a subgraph of $F_n \times F_m$. Since $p_{n-1} < m$, there are at least $(n-1)m + 1$ pebbles on $F_{n-1} \times F_m$. Note that in $\{v_{n-2}\} \times F_m$, every vertex is occupied by exactly one pebble. By assumption, $f(F_{n-1} \times F_m) = (n-1)m$. Keeping one pebble on (v_{n-2}, y_i) and remaining $(n-1)m$ pebbles on $F_{n-1} \times F_m$, we can put one more pebble on (v_{n-2}, y_i). Using these two pebbles on (v_{n-2}, y_i), we can pebble (v_{n-1}, y_i).

Lemma 11. Let F_n and F_m be fan graphs with vertices $v_0, v_1, \cdots, v_{n-1}$ in order and with vertices $y_0, y_1, \cdots, y_{m-1}$ in order respectively. Suppose that there are nm pebbles assigned to vertices of $F_n \times F_m$. If $p_1 + r_1 > 2m$ and

$$p_0 + p_{n-1} + \sum_{i=1}^{n-2} \frac{p_i - r_i}{2} < 2m,$$

then

$$\sum_{i=2}^{n-2} (p_i + r_i) \geqslant 2(n-3)m - [p_1 - (2m + 2 - r_1)].$$

Proof. Set

$$k_1 = \frac{p_1 - (2m+2-r_1)}{2} \quad \text{and} \quad k_2 = \sum_{i=2}^{n-2} \frac{p_i - r_i}{2}.$$

Then the assumption can be written as

$$p_0 + p_{n-1} + k_1 + k_2 < m - 1 + r_1.$$

From

$$\begin{aligned} nm = \sum_{i=0}^{n-1} p_i &= p_0 + [(2m+2-r_1) + 2k_1] + \sum_{i=2}^{n-2} p_i + p_{n-1} \\ &= p_0 + [(2m+2-r_1) + 2k_1] + \sum_{i=2}^{n-2} r_i + 2k_2 + p_{n-1} \\ &= p_0 + p_{n-1} + (k_1 + k_2) + (2m + 2 - r_1) + \left(k_1 + k_2 + \sum_{i=2}^{n-2} r_i \right), \end{aligned}$$

we have

$$\begin{aligned} k_1 + k_2 + \sum_{i=2}^{n-2} r_i &= nm - (p_0 + p_{n-1} + k_1 + k_2) - (2m + 2 - r_1) \\ &> nm - (m - 1 + r_1) - (2m + 2 - r_1) \\ &= (n-3)m - 1. \end{aligned}$$

Therefore $k_1 + k_2 + \sum_{i=2}^{n-2} r_i \geqslant (n-3)m$ and

$$\sum_{i=2}^{n-2} (p_i + r_i) = 2k_2 + 2\sum_{i=2}^{n-2} r_i \geqslant 2(n-3)m - 2k_1.$$

So the result follows.

Theorem 2. The graph $F_n \times F_m$ is demonic, i.e. $f(F_n \times F_m) = nm$.

Proof. The proof is by induction on n. It is trivially true for $n = 1$. Suppose that it is true for $n' < n$. In fact, if $n \leqslant 5$, then $f(F_n \times F_m) = nm$ (see the footnote on page 471). Let $n \geqslant 6$, F_n the fan graph with vertices $v_0, v_1, \cdots, v_{n-1}$ in order and F_m the fan graph with vertices $y_0, y_1, \cdots, y_{m-1}$ in order. Suppose that there are nm pebbles assigned to vertices of $F_n \times F_m$.

First, suppose that the target vertex is (v_i, y_j), where $i = 0, 2, \cdots, n-2$ and $j = 0, 1, \cdots, m-1$. For the vertex (v_0, y_j), by Lemma 8, $f(F_n \times F_m, (v_0, y_j)) = nm$. Now let $2 \leqslant i \leqslant n-2$. If $p_i + p_0 \geqslant 2m$, then we can move one pebble to (v_i, y_j) by Lemma 3. We may assume that $p_i + p_0 < 2m$. Let $p_i + p_0 = 2m - t$ with $t > 0$, $p_1 + \cdots + p_{i-1} = (i-1)m + t_1$ and $p_{i+1} + \cdots + p_{n-1} = (n-1-i)m + t_2$, where $t = t_1 + t_2$. We consider the following two cases.

(i) When $t_1 \geqslant t$ or $t_2 \geqslant t$, since

$$p_0 + p_1 + \cdots + p_{i-1} + p_i = (2m - t) + (i-1)m + t_1 \geqslant (i+1)m$$

or

$$p_0 + p_i + p_{i+1} + \cdots + p_{n-1} = (2m - t) + (n-1-i)m + t_2 \geqslant (n-i+1)m,$$

one pebble can be moved to (v_i, y_j) by induction on n.

(ii) When $t_1 < t$ and $t_2 < t$, we consider the following two subcases.

Subcase 1. If

$$\frac{1}{2}\sum_{k=i+1}^{n-1}(p_k - r_k) \geqslant t_2 \quad \text{or} \quad \frac{1}{2}\sum_{k=1}^{i-1}(p_k - r_k) \geqslant t_1,$$

then

$$\sum_{\ell=0}^{i} p_\ell + \frac{1}{2}\sum_{k=i+1}^{n-1}(p_k - r_k) \geqslant (i-1)m + t_1 + 2m - t + t_2 = (i+1)m$$

or

$$p_0 + \sum_{\ell=i}^{n-1} p_\ell + \frac{1}{2}\sum_{k=1}^{i-1}(p_k - r_k) \geqslant (2m - t) + (n-1-i)m + t_2 + t_1 = (n-i+1)m.$$

So, by induction on n, we can pebble (v_i, y_j).

Subcase 2. If

$$\frac{1}{2}\sum_{k=i+1}^{n-1}(p_k - r_k) < t_2 \quad \text{and} \quad \frac{1}{2}\sum_{k=1}^{i-1}(p_k - r_k) < t_1,$$

then

$$\sum_{k=i+1}^{n-1}(p_k + r_k) = \sum_{k=i+1}^{n-1} p_k + \sum_{k=i+1}^{n-1} r_k > 2\sum_{k=i+1}^{n-1} p_k - 2t_2 = 2(n-1-i)m$$

and

$$\sum_{k=1}^{i-1}(p_k + r_k) = \sum_{k=1}^{i-1} p_k + \sum_{k=1}^{i-1} r_k > 2\sum_{k=1}^{i-1} p_k - 2t_1 = 2(i-1)m.$$

Hence there exist some α, β with $i+1 \leqslant \alpha \leqslant n-1$, $1 \leqslant \beta \leqslant i-1$, $p_\alpha + r_\alpha > 2m$ and $p_\beta + r_\beta > 2m$. Since F_m satisfies the weak two-pebbling property, we can put two pebbles on (v_0, y_j) by Lemma 2. Therefore, one pebble can be moved to (v_i, y_j)

Next, let the target vertex be (v_{n-1}, y_j). If $p_0 + p_2 + \cdots + p_{n-1} \geqslant (n-1)m$, then we can pebble (v_{n-1}, y_j) by induction hypothesis. So we may assume that $p_0 + p_2 + \cdots + p_{n-1} < (n-1)m$. Set $p_0 + p_2 + \cdots + p_{n-1} = (n-1)m - s$ with $s > 0$. Then $p_1 = m + s$. Consider the following two cases.

(i) Let $p_1 + r_1 \leqslant 2m$. Then $s \leqslant m - r_1$. From $2s \leqslant (m+s) - r_1 = p_1 - r_1$, we get $s \leqslant (p_1 - r_1)/2$. First we move $(p_1 - r_1)/2$ pebbles to $\{v_0\} \times F_m$ from $\{v_1\} \times F_m$. Since $p_0 + p_2 + \cdots + p_{n-1} + (p_1 - r_1)/2 \geqslant (n-1)m$, we can pebble (v_{n-1}, y_j) by induction on n.

(ii) Let $p_1 + r_1 > 2m$. If $p_i + r_i > 2m$ for some i with $2 \leqslant i \leqslant n-2$, then we can put two pebbles on (v_0, y_j) by Lemma 2. So one pebble can be moved to (v_{n-1}, y_j) from these two pebbles on (v_0, y_j). Now suppose that $p_i + r_i \leqslant 2m$ for all $2 \leqslant i \leqslant n-2$. If

$$p_0 + p_{n-1} + \frac{1}{2}[(p_1 - r_1) + \cdots + (p_{n-2} - r_{n-2})] \geqslant 2m,$$

then we can pebble (v_{n-1}, y_j) by Lemma 3. In the following, suppose that

$$p_0 + p_{n-1} + \frac{1}{2}[(p_1 - r_1) + \cdots + (p_{n-2} - r_{n-2})] < 2m.$$

We consider the following five possibilities.

(a) Let $p_1 - (2m + 2 - r_1) \geqslant 8$. It is clear that $r_1 \leqslant m$, so $p_1 - r_1 \geqslant 8$. By Lemma 9, we can put 4 pebbles on (v_1, y_0) and then 2 pebbles can be put on (v_0, y_0). Remember that at least $(2m + 2 - r_1)$ pebbles are still on $\{v_1\} \times F_m$. We can pebble arbitrary vertex (v_{n-1}, y_j) in $\{v_{n-1}\} \times F_m$ by Lemma 4.

For the next 4 possibilities, if one of the following 4 conditions is satisfied, then we are done.

Condition 1. $p_{n-2} = r_{n-2} = m$, the result follows from Lemma 10.

Condition 2. There are at least 2 i's in $\{2, \cdots, n-2\}$ with $p_i + r_2 = 2m$ and $p_i - r_i = 2$. The result follows from Lemma 7.

Condition 3. There are at least 2 i's in $\{2, \cdots, n-2\}$ with $p_i - r_i \geqslant 4$. The result follows from Lemmas 9 and 4.

Condition 4. There is only one $i \in \{2, \cdots, n-2\}$ with $p_i - r_i \geqslant 4$ and there is at least one $\ell \in \{2, \cdots, n-2\}$ with $p_\ell + r_\ell = 2m$ and $p_\ell - r_\ell = 2$. The result follows from Lemmas 9 and 6.

Therefore for the next 4 possibilities we consider only the following conditions.

Condition 5. There is only one $i \in \{2, \cdots, n-2\}$ with $p_i - r_i \geqslant 4$ and $p_\ell + r_\ell = 2m$ implies $p_\ell = r_\ell = m$ for all $\ell \in \{2, \cdots, n-2\}$.

Condition 6. $p_i - r_i \leqslant 2$ for all $i \in \{2, \cdots, n-2\}$ and there is some $\ell \in \{2, \cdots, n-2\}$ with $p_\ell + r_\ell = 2m$ and $p_\ell - r_\ell = 2$.

Condition 7. $p_i - r_i \leqslant 2$ for all $i \in \{2, \cdots, n-2\}$ and $p_\ell + r_\ell = 2m$ implies $p_\ell = r_\ell = m$ for all $\ell \in \{2, \cdots, n-2\}$.

(b) Let $p_1 - (2m + 2 - r_1) = 6$. By Lemma 11, $p_k + r_k \geqslant 2m - 6$ for all $k \in \{2, \cdots, n-2\}$. If Condition 7 is satisfied, then $p_2 + r_2 = 2m$, $2m - 2$ or $2m - 4$ because $p_2 + r_2 = 2m - 6$ implies that $p_{n-2} = r_{n-2} = m$, i.e. Condition 1 holds. When $p_2 + r_2 = 2m$, $p_2 = r_2 = m$. We keep $(2m+2-r_1)$ or more pebbles on $\{v_1\} \times F_m$ and transfer one pebble to $\{v_2\} \times F_m$. Then we can put two pebbles on (v_1, y_j) and on (v_2, y_j). Therefore one pebble can be moved to (v_0, y_j) from (v_1, y_j) and then $\{(v_2, y_j), (v_0, y_j), (v_{n-1}, y_j)\}$ forms a transmitting subgraph. When $p_2 + r_2 = 2m - 2$, we transfer 3 pebbles to $\{v_2\} \times F_m$ from $\{v_1\} \times F_m$ while keeping $(2m+2-r_1)$ pebbles on $\{v_1\} \times F_m$. Now in $\{v_2\} \times F_m$, we have $p_2 + q_2 \geqslant (2m - 2) + 3$. Thus we can put two pebbles on any vertex in $\{v_1\} \times F_m$ and on any vertex in $\{v_2\} \times F_m$. So we can pebble any vertex in $\{v_{n-1}\} \times F_m$ as above. When $p_2 + r_2 = 2m - 4$, $p_{n-2} + r_{n-2} = 2m - 2$, $p_3 + r_3 = 2m$ and $p_3 = r_3 = m$. We transfer 3 pebbles to $\{v_2\} \times F_m$ from $\{v_1\} \times F_m$ while keeping $(2m+2-r_1)$ pebbles on $\{v_1\} \times F_m$. Therefore there are at least $(m + 1)$ pebbles on $\{v_2\} \times F_m$. We can transfer one pebble from $\{v_2\} \times F_m$ to $\{v_3\} \times F_m$. Thus two pebbles can be put on any vertex in $\{v_1\} \times F_m$ and on any vertex in $\{v_3\} \times F_m$. In the sequel, we can pebble any vertex in $\{v_{n-1}\} \times F_m$. In the following, we may assume that Condition 5 or Condition 6 is satisfied. Since $p_1 = (2m + 2 - r_1) + 6$, i.e. $p_1 - r_1 > 6$, we can put two pebbles on (v_1, y_0) by Lemma 9 and these two pebbles can be used to put one pebble on (v_0, y_0). Remember that there are still $(2m+2-r_1)$ or more pebbles on $\{v_1\} \times F_m$. If Condition 5 is satisfied, then we can put two pebbles on (v_i, y_0) by Lemma 9. Using these two pebbles we can put one more pebble on (v_0, y_0). Therefore we can pebble any vertex in $\{v_{n-1}\} \times F_m$ by Lemma 4. If Condition 6 is satisfied, then we can put two pebbles on any vertex in $\{v_0\} \times F_m$ by Lemma 6. So one pebble can be moved to any vertex in $\{v_{n-1}\} \times F_m$ by using these two pebbles.

(c) Let $p_1 - (2m+2-r_1) = 4$. Then $p_k + r_k \geqslant 2m-4$ for all $k \in \{2, \cdots, n-2\}$ by Lemma 11. The same process in (b) is used.

(d) Let $p_1 - (2m+2-r_1) = 2$. By Lemma 11, there is just one $k \in \{2, \cdots, n-2\}$ with $p_k + r_k = 2m-2$ and $p_s + r_s = 2m$ for any $s \in \{2, \cdots, n-2\}$ and $s \neq k$. If Condition 5 is satisfied, then either $p_k - r_k \geqslant 4$ or $p_k - r_k \leqslant 2$. Let $p_k - r_k \geqslant 4$, then $k = n-2$ because $k \neq n-2$ means that Condition 1 is satisfied. So $p_2 + r_2 = 2m$ and $p_2 = r_2 = m$. We keep $(2m+2-r_1)$ pebbles on $\{v_1\} \times F_m$ and move one pebble to $\{v_2\} \times F_m$. Thus we can put two pebbles on both (v_1, y_j) and (v_2, y_j). In the sequal, we can pebble our target vertex (v_{n-1}, y_j). Let $p_k - r_k \leqslant 2$, then there exists some $i \in \{2, \cdots, n-2\}$ such that $p_i + r_i = 2m$ and $p_i - r_i \geqslant 4$. If $|i-k| \geqslant 2$, then there is some $t \in \{2, \cdots, n-2\}$ such that $|i-t| = 1$, $p_t + r_t = 2m$ and $p_t = r_t = m$. We can move one pebble to $\{v_t\} \times F_m$ from $\{v_i\} \times F_m$. Therefore we can pebble our target vertex as before. If $|i-k| = 1$, then $\{i, k\} = \{n-3, n-2\}$ since Condition 1 does not hold. Thus $p_2 + r_2 = 2m$ and $p_2 = r_2 = m$. The same process as before could be done. If Condition 6 or Condition 7 is satisfied, then the same process as above could be done.

(e) Let $p_1 - (2m+2-r_1) = 0$. By Lemma 11, $p_k + r_k = 2m$ for all $k \in \{2, \cdots, n-2\}$. If Condition 5 (respectively Condition 6) is satisfied, then it must be $p_{n-2} - r_{n-2} \geqslant 4$ (respectively $p_{n-2} - r_{n-2} = 2$) and $p_k = r_k = m$ for all $k \in \{2, \cdots, n-3\}$. We can move one pebble to $\{v_{n-3}\} \times F_m$ from $\{v_{n-2}\} \times F_m$. By Lemma 1, we can put two pebbles on any vertex (v_{n-3}, y_j) in $\{v_{n-3}\} \times F_m$ and these two pebbles can be used to put one pebble on (v_0, y_j) in $\{v_0\} \times F_m$. On the other hand, two pebbles can be put on (v_1, y_j) since $p_1 + r_1 > 2m$. Therefore $\{(v_1, y_j), (v_0, y_j), (v_{n-1}, y_j)$ forms a transmitting subgraph. Note that Condition 7 cannot happen because $p_{n-2} = r_{n-2} = m$ is impossible.

Finally, if the target vertex is (v_1, y_j) for some y_j, then we can prove it in the same way as in the case in which the target vertex is (v_{n-1}, y_j).

A wheel graph, denoted by W_n, is a cycle C_{n-1} plus an extra vertex connected to all vertices of the cycle C_{n-1}. From ref. [1] we know that if G' is a spanning subgraph of G, then $f(G') \geqslant f(G)$. Since F_n is a spanning subgraph of W_n, and $F_n \times F_m$ is a spanning subgraph of $W_n \times W_m$, we can obtain the following theorem.

Theorem 3. Both the wheel graph and the product of two wheel graphs are demonic, i.e. $f(W_n) = n$ and $f(W_n \times W_m) = nm$.

From Theorems 2 and 3, we know that Graham's conjecture holds for the product of fan graphs or the product of wheel graphs.

Acknowledgements This work was supported by the National Natural Science Foundation of China

(Grant No. 10001005) and by RFDP of China.

References

1. Chung, F. R. K., Pebbling in hyercubes, SIAM J. Discrete Math., 1989, 2: 461—472.
2. Feng, R., Kim, J., Graham's pebbling conjecture of product of complete bipartite graphs, Science in China, Ser. A, 2001, 44(7): 817—822.
3. Moews, D., Pebbling graphs, J. Combin. Theory, Ser. B, 1992, 55: 244—252.
4. Herscovici, D. S., Higgins, A. W., The pebbling number of $C_5 \times C_5$, Discrete Math., 1998, 189: 123—135.
5. Pachter, L., Snevily, H. S., Voxman, B., On pebbling graphs, Congr. Numer., 1995, 107: 65—80.
6. Snevily, H. S., Foster, J. A., The 2-pebbling property and a conjecture of Graham, Graphs and Combinatorics, 2000, 16: 231—244.