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Graham's pebbling conjecture on product of complete bipartite graphs

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Abstract The pebbling number of a graph G, $f(G)$, is the least n such that, no matter how n pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. Graham conjectured that for any connected graphs G and H, $f(G\times H)\leqslant f(G)f(H)$. We show that Graham's conjecture holds true of a complete bipartite graph by a graph with the two-pebbling property. As a corollary, Graham's conjecture holds when G and H are complete bipartite graphs.

Keywords: pebbling, Graham's conjecture, Cartesian product, complete bipartite graph.

A pebbling of a connected graph is a placement of pebbles on the vertices of the graph. A pebbling move consists of removing two pebbles from a vertex, throwing one pebble away, and moving the other pebble to an adjacent vertex. The pebbling number of a vertex v in a graph G is the smallest number $f(G,v)$ with the property that from every placement of $f(G,v)$ pebbles on G, it is possible to move a pebble to v by a sequence of pebbling moves. The pebbling number of a graph G, denoted by $f(G)$, is the maximum $f(G,v)$ over all the vertices of G.

There are some known results regarding $f(G)$ (see refs. [1—4]). If one pebble is placed on each vertex other than the vertex v, then no pebble can be moved to v. Also, if w is at distance d from v, and 2^d-1 pebbles are placed on w, then no pebble can be moved to v. So it is clear[1] that $f(G)\geqslant\max\{|V(G)|, 2^D\}$, where $|V(G)|$ is the number of vertices of G and D is the diameter of the graph G. Furthermore, we know from ref. [1] that $f(K_n)=n$, where K_n is the complete graph on n vertices, and $f(P_n)=2^{n-1}$, where P_n is the path on n vertices.

We say a graph is demonic if $f(G)=|V(G)|$. Given a pebbling of G, let q be the number of vertices with at least one pebble, and let r be the number of vertices with an odd number of pebbles. We say that G satisfies the two-pebbling property (respectively, weak two-pebbling property) if it is possible to move two pebbles to any specified target vertex when the total starting number of pebbles is $2f(G)-q+1$ (respectively, $2f(G)-r+1$). Note that any graph which satisfies the two-pebbling property also satisfies the weak two-pebbling property. Given a pebbling of G, a transmitting subgraph of G is a path $x_0, x_1, \cdots, x_k$ such that there are at least two pebbles on x_0 and at least one pebble on each of the other vertices in the path, except possibly x_k. In this case, we can transmit a pebble from x_0 to x_k.

This paper explores the pebbling number of the Cartesian product of complete bipartite graphs. The idea for Cartesian products comes from a conjecture of Graham[1]. The conjecture states that for any graphs G and H, $f(G\times H)\leqslant f(G)f(H)$. There are a few results that verify Graham's conjecture. Among them, the conjecture holds to a tree by a tree[2], a cycle by a cycle[1)], and a complete graph by a graph with the two-pebbling property[1]. In the present paper we show that Graham's conjecture holds to a complete bipartite graph by a graph with the two-pebbling property. Furthermore, we prove that the product of two complete bipartite graphs is demonic, i.e. $f(K_{m,n}\times K_{s,t})=(m+n)(s+t)$.

Throughout this paper G will denote a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex v of G, $p(v)$ will refer to the number of pebbles on v.

1 Pebbling $K_{m,n}$

Relating the ideas above to $K_{m,n}$, we have the following lemmas.

Lemma 1. Let $K_{m,n}$ be the complete bipartite graph. Then $f(K_{m,n})=m+n$ if $m>1$ and $n>1$.

Proof. Label the vertices of $K_{m,n}$ by $v_1,\cdots,v_m;w_1,\cdots,w_n$ such that every v_i is adjacent to every w_j for $i=1,\cdots,m$ and $j=1,\cdots,n$. Without loss of generality, we assume the target vertex is v_1 and $p(v_1)=0$. We break down the possible configuration of $(m+n)$ pebbles on $K_{m,n}$ as the following three cases.

(1) If $p(w_j)\geqslant 2$ for some j, then one pebble can be moved to v_1 from w_j.

(2) If $p(w_{j_0})=1$ for some j_0 and $p(w_j)\leqslant 1$ for all j, then $p(v_i)\geqslant 2$ for some i and $\{v_i, w_{j_0}, v_1\}$ form a transmitting subgraph.

(3) If $p(w_j)=0$ for all j, then $m+n$ pebbles are placed on the vertices $v_2,\cdots,v_m$. There must be some v_i with $p(v_i)\geqslant 2$. Using two pebbles on v_i, we can put one pebble on w_1. Since $m+n-2\geqslant m$ pebbles remain on vertices $v_2,\cdots,v_m$, there exists some vertex v_{i_0} with $p(v_{i_0})\geqslant 2$. Therefore $\{v_{i_0}, w_1, v_1\}$ forms a transmitting subgraph.

Lemma 2. The complete bipartite graph satisfies the two-pebbling property.

Proof. Let p be the number of pebbles on the complete bipartite graph $K_{m,n}$, q be the number of vertices with at least one pebble and $p+q=2(m+n)+1$. Assume the target vertex is v_1. If $p(v_1)=1$, then the number of pebbles on all the vertices except v_1 is $2(m+n)+1-q-1\geqslant m+n$. Since $f(K_{m,n})=m+n$, we can put one more pebble on v_1 using $2(m+n)+1-q-1$ pebbles. If $p(v_1)=0$, then we consider the following three cases.

(1) Suppose that $p(w_j)\geqslant 2$ for some w_j. We can move one pebble on v_1 from w_j. Using the remaining $2(m+n)+1-q-2$ pebbles, we can put one more pebble on v_1.

(2) Suppose that $p(w_{j_0})=1$ for some vertex w_{j_0} and $p(w_j)\leqslant 1$ for all j.

(2.1) If $q\leqslant m+n-2$, then there is some v_i with $p(v_i)\geqslant 2$. $\{v_i, w_{j_0}, v_1\}$ forms a transmitting subgraph. Hence we can move one pebble to v_1 using three pebbles on v_i and w_{j_0}. Using the remaining $2(m+n)+1-q-3\geqslant m+n$ pebbles, we can have one more pebble on v_1.

(2.2) If $q=m+n-1$, then $m+2$ pebbles are placed on $v_2,\cdots,v_m$. We have two transmitting subgraphs $\{v_i, w_1, v_1\}$ and $\{v_{i'}, w_2, v_1\}$ for some i and i'. Thus we are done.

(3) Suppose that $p(w_j)=0$ for all j. Since there are $2(m+n)+1-q\geqslant m+2n+2$

1) Herscovici, D. S., Graham's Pebbling Conjecture on Product of Cycles, preprint.

pebbles on $v_2, \cdots, v_m$, we can spend 4 pebbles to move one to v_1. Using the remaining $2(m+n)+1-q-4 \geqslant m+n$ pebbles, we can put one more pebble on v_1.

2 Cartesian product

Let G and H be two graphs. The (Cartesian) product of G and H, denoted by $G \times H$, is the graph whose vertex set is the Cartesian product

$$V(G \times H) = V(G) \times V(H) = \{(x, y) \mid x \in V(G), y \in V(H)\},$$

and two vertices (x, y) and (x', y') are adjacent if and only if $x = x'$ and $\{y, y'\} \in E(H)$, or $\{x, x'\} \in E(G)$ and $y = y'$. We can depict $G \times H$ pictorially by drawing a copy of H at every vertex of G and connecting each vertex in one copy of H to the corresponding vertex in an adjacent copy of H. We write $\{x\} \times H$ (respectively, $G \times \{y\}$) for the subgraph of vertices whose projection onto $V(G)$ is the vertex x (respectively, whose projection onto $V(H)$ is y). If the vertices of G are labelled by x_i, then for any distribution of pebbles on $G \times H$, we write p_i for the number of pebbles on $\{x_i\} \times H$, q_i for the number of occupied vertices of $\{x_i\} \times H$ and r_i for the number of vertices of $\{x_i\} \times H$ with an odd number of pebbles.

The following conjecture, by Ronald Graham, suggests a constraint on the pebbling number of the product of two graphs.

Conjecture (Graham). The pebbling number of $G \times H$ satisfies $f(G \times H) \leqslant f(G) \cdot f(H)$.

Lemma 3[3]. Let $\{x_i, x_j\}$ be an edge in G. Suppose that in $G \times H$, we have p_i pebbles occupying q_i vertices of $\{x_i\} \times H$. If $q_i - 1 \leqslant k \leqslant p_i$, and if k and p_i have the same parity, then k pebbles can be retained on $\{x_i\} \times H$, while moving $(p_i - k)/2$ pebbles onto $\{x_j\} \times H$. If k and p_i have opposite parity, we must leave $k+1$ pebbles on $\{x_i\} \times H$, so we can only move $(p_i - k - 1)/2$ pebbles onto $\{x_j\} \times H$, in particular, we can always move at least $(p_i - q_i)/2$ pebbles onto $\{x_j\} \times H$.

Lemma 4[1]. Let K_2 be the complete graph on two vertices. Suppose that G satisfies the two-pebbling property. Then $f(K_2 \times G) \leqslant 2f(G)$.

Lemma 5. Suppose that G satisfies the two-pebbling property. Consider the graph $K_{1,n} \times G$, where $n > 1$. Let v_0 be the vertex of $K_{1,n}$ with degree n. To pebble a target vertex on $\{v_0\} \times G$, it suffices to start with $(n+1)f(G)$ pebbles on $K_{1,n} \times G$.

Proof. Label the vertices of $K_{1,n}$ by $v_0, v_1, \cdots, v_n$, where v_0 is the vertex with degree n. The target vertex in $K_{1,n} \times G$ is then (v_0, y). Suppose that in $K_{1,n} \times G$, we have p_i pebbles occupying q_i vertices of $\{v_i\} \times G$ for each $i = 0, 1, \cdots, n$. If

$$p_0 + \sum_{i=1}^{n} \frac{p_i - q_i}{2} \geqslant f(G),$$

then we can use Lemma 3 to put $f(G)$ pebbles on $\{v_0\} \times G$. Since this subgraph is isomorphic to G, we can then put a pebble on (v_0, y). Also, since G has the two-pebbling property, if $(p_i + q_i)/2 > f(G)$ for some $i \in \{1, \cdots, n\}$, then we can put two pebbles on (v_i, y), and then use a pebbling move to pebble (v_0, y). Hence the distributions from which we cannot pebble the target vertex satisfy the inequalities

$$p_0 + \sum_{i=1}^{n} \frac{p_i - q_i}{2} < f(G),$$

$$\frac{p_i + q_i}{2} \leqslant f(G), \quad i = 1, \cdots, n.$$

Adding these inequalities together gives

$$p_0 + p_1 + \cdots + p_n < (n+1)f(G).$$

Thus any distribution of pebbles from which we cannot reach some vertex on $\{v_0\} \times G$ must begin with fewer than $(n+1)f(G)$ pebbles.

As a corollary, we can get the following result.

Corollary 1[3]. Suppose that G satisfies the two-pebbling property and consider the graph $P_3 \times G$. To pebble a target vertex on the middle copy of G, it suffices to start with $3f(G)$ pebbles on $P_3 \times G$.

Lemma 6. Suppose that G satisfies the weak two-pebbling property and p pebbles are placed on $K_{1,n} \times G$ in such a way that there are r vertices with an odd number of pebbles. Let v_0 be the vertex of $K_{1,n}$ with degree n. If $p + r > 2(n+1)f(G)$, then two pebbles can be moved to (v_0, y) by a sequence of pebbling moves.

Proof. Label the vertices of $K_{1,n}$ by $v_0, v_1, \cdots, v_n$. For $K_{1,n} \times G$, let p_i be the number of pebbles on the graph $\{v_i\} \times G$, and let r_i be the number of vertices with an odd number of pebbles for $i = 0, 1, \cdots, n$. Note that $p_i + r_i$ must be even. Suppose that there are $p_0 > 2f(G) - r_0$ pebbles assigned to $\{v_0\} \times G$. Then the result holds. So we may assume that $p_0 + r_0 \leqslant 2f(G)$ and consider the following three cases.

(1) If at least two of $p_1 + r_1, \cdots, p_n + r_n$ are larger than $2f(G)$, say, $p_1 + r_1 > 2f(G)$ and $p_2 + r_2 > 2f(G)$, then two pebbles can be moved to each of (v_1, y) and (v_2, y). Since (v_0, y) is adjacent to both (v_1, y) and (v_2, y), two pebbles can be moved to (v_0, y) from those 4 pebbles on (v_1, y) and on (v_2, y).

(2) If only one of $p_1 + r_1, \cdots, p_n + r_n$, say, $p_1 + r_1$, is larger than $2f(G)$, then we can move

$$\frac{(p_1 + r_1) - (2f(G) + 2)}{2}$$

pebbles to $\{v_0\} \times G$ while keeping $(2f(G) - r_1 + 2)$ pebbles on $\{v_1\} \times G$. From $p_1 + r_1 \geqslant 2f(G) + 2$, $p_2 + r_2 \leqslant 2f(G)$, $\cdots$, $p_n + r_n \leqslant 2f(G)$ and $(p_0 + r_0) + (p_1 + r_1) \geqslant 4f(G) + 2$, we have

$$p_0 + \frac{p_1 + r_1 - (2f(G) + 2)}{2} \geqslant p_0 + \frac{2f(G) - p_0 - r_0}{2} = f(G) + \frac{p_0 - r_0}{2} \geqslant f(G).$$

So we can move one pebble to (v_0, y) in $\{v_0\} \times G$. From the remaining $(2f(G) - r_1 + 2)$ pebbles on $\{v_1\} \times G$, we can put two pebbles on (v_1, y). So we can move one more pebble to (v_0, y) from (v_1, y).

(3) If none of $p_1 + r_1, \cdots, p_n + r_n$ is larger than $2f(G)$, then

$$\begin{aligned} p_0 + r_0 &> 2(n+1)f(G) - (p_1 + r_1) - \cdots - (p_n + r_n) \\ &\geqslant 2(n+1)f(G) - 2nf(G) \\ &= 2f(G). \end{aligned}$$

Hence we are done.

Lemma 7. Let K_2 be the complete graph on two vertices x_1 and x_2. Suppose that G satisfies the weak two-pebbling property. Let p pebbles be assigned to vertices of $K_2 \times G$ and r be the number of vertices with an odd number of pebbles. If $p + r > 4f(G)$, then two pebbles can be moved to any specified vertex of $K_2 \times G$ by a sequence of pebbling moves.

Proof. Without loss of generality, assume that the target is (x_1, y) for some y. If $p_1 + r_1$

$> 2f(G)$, then two pebbles can be moved to (x_1, y). We may assume $p_1 + r_1 \leqslant 2f(G)$. Then we can move

$$\frac{(p_2 + r_2) - (2f(G) + 2)}{2}$$

pebbles to $\{x_1\} \times G$ while keeping $(2f(G) - r_2 + 2)$ pebbles on $\{x_2\} \times G$. Since

$$\begin{aligned} p_1 + \frac{(p_2 + r_2) - (2f(G) + 2)}{2} &> p_1 + \frac{4f(G) - p_1 - r_1 - 2f(G) - 2}{2} \\ &= p_1 + \frac{2f(G) - p_1 - r_1 - 2}{2} \\ &= f(G) + \frac{p_1 - r_1}{2} - 1, \end{aligned}$$

the left side is larger than or equal to $f(G)$. Then we can move one pebble to (x_1, y) from $2f(G) - r_2 + 2$ pebbles on $\{x_2\} \times G$ and we can move one more pebble from $f(G)$ pebbles on $\{x_1\} \times G$.

3 Pebbling $K_{m,n} \times K_{s,t}$

In this section, we show that Graham's conjecture holds for the product of a complete bipartite graph and a graph with the two-pebbling property.

Theorem 1. Suppose that G satisfies the two-pebbling property. Then $f(K_{m,n} \times G) \leqslant f(K_{m,n}) f(G)$.

Proof. Label the vertices of $K_{m,n}$ by $v_1, \cdots, v_m; w_1, \cdots, w_n$ such that every v_i is adjacent to every w_j for $i = 1, \cdots, m$; $j = 1, \cdots, n$. Without loss of generality, assume that the target vertex is (v_1, y) for some y. Let $(m + n) f(G)$ pebbles be placed on $K_{m,n} \times G$. $K_{m,n} \times G$ can be partitioned into two parts, say, M_1 and M_2, as follows. M_1 is $A \times G$ and M_2 is $B \times G$ where A is the induced subgraph on the vertex subset $\{v_1, w_1, \cdots, w_{n-1}\}$ of $K_{m,n}$ and B is the induced subgraph on the vertex subset $\{w_n, v_2, \cdots, v_m\}$ of $K_{m,n}$. In fact, A is $K_{1,n-1}$ or K_2 and B is $K_{1,m-1}$ or K_2. Suppose that M_i contains p_i pebbles with r_i vertices having an odd number of pebbles for $i = 1, 2$. If $p_1 \geqslant nf(G)$, then one pebble can be moved to (v_1, y) by Lemmas 4 and 5. Now we assume that $p_1 < nf(G)$. Set $p_1 = nf(G) - t$ and $p_2 = mf(G) + t$ for some positive integer t.

(1) If $t \leqslant mf(G) - r_2$, then we apply pebbling steps to all vertices in M_2 and we can move at least $(p_2 - r_2)/2$ pebbles to vertices of M_1. Therefore, in M_1, we have altogether

$$p_1 + \frac{p_2 - r_2}{2} \geqslant nf(G) - t + \frac{mf(G) + t - mf(G) + t}{2} = nf(G)$$

pebbles. By Lemmas 4 and 5, we can move one pebble to (v_1, y).

(2) If $t > mf(G) - r_2$, then

$$p_2 + r_2 = mf(G) + t + r_2 > 2mf(G).$$

Therefore two pebbles can be moved to (w_n, y) by Lemmas 6 and 7. Because (v_1, y) and (w_n, y) are adjacent, one pebble can be moved to (v_1, y) from (w_n, y).

A complete bipartite graph satisfies the two-pebbling property. The following theorem is immediate.

Theorem 2. $f(K_{m,n} \times K_{s,t}) = (m + n)(s + t)$.

If G' is a spanning subgraph of G, then $f(G') \geqslant f(G)$ (see ref. [1]). Since $K_{m,n}$ is a spanning subgraph of K_{m+n}, we can obtain the following corollary which is Theorem 5 in ref. [1].

Corollary 2. $f(K_n \times G) \leqslant nf(G)$ if G satisfies the two-pebbling property.

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