On the Pebbling Threshold of Paths and the Pebbling Threshold Spectrum

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Abstract

A configuration of pebbles on the vertices of a graph is solvable if one can place a pebble on any given root vertex via a sequence of pebbling steps. A function is a pebbling threshold for a sequence of graphs if a randomly chosen configuration of asymptotically more pebbles is almost surely solvable, while one of asymptotically fewer pebbles is almost surely not. In this paper we tighten the gap between the upper and lower bounds for the pebbling threshold for the sequence of paths in the multiset model. We also find the pebbling threshold for the sequence of paths in the binomial model. Finally, we show that the spectrum of pebbling thresholds for graph sequences in the multiset model spans the entire range from $n^{1/2}$ to n, answering a question of Czygrinow, Eaton, Hurlbert and Kayll. What the spectrum looks like above n remains unknown.

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1 Introduction

Let $G = (V, E)$ be a connected graph on n vertices and let D be a *configuration* of t unlabeled pebbles on V (formally D is multiset of t elements from V, with $D(v)$ the number of pebbles on vertex v). A *pebbling step* consists of removing two pebbles from a vertex v and placing one pebble on a neighbor of v. A configuration is called r*-solvable* if it is possible to move at least one pebble to vertex r by a sequence of pebbling steps. A configuration is called *solvable* if it is r-solvable for every vertex $r \in V$. The *pebbling number* of G is the smallest integer $\pi(G)$ such that every configuration of $t = \pi(G)$ pebbles on G is solvable. Pebbling problems have a rich history and we refer to [6] for a thorough discussion. Standard asymptotic notation will be used in the paper. For two functions $f = f(n)$ and $g = g(n)$, we write $f \ll g$ (or $f \in o(g)$) if f/g approaches zero as n approaches infinity, $f \in O(g)$ ($f \in \Omega(g)$) if there exist positive constants c, k such that $f < cg$ ($f > cg$) whenever $n > k$. In addition, $f \in \Theta(g)$ when $f \in O(g)$ and $g \in O(f)$. We will also use $f \sim g$ if f/g approaches 1 as n approaches infinity. Finally to simplify the exposition we shall always assume, whenever needed, that our functions take integer values.

We will be mainly interested in the following random model considered in [2]. A configuration D of t pebbles assigned to G is selected randomly and uniformly from all $\binom{n+t-1}{t}$ configurations. The problem to investigate, then, is to find what values of t, as functions of the number of vertices $n = n(G)$, make D almost surely solvable. More precisely, a function $t = t(n)$ is called a *threshold* of a graph sequence $\mathcal{G} = (G_1, \ldots, G_n, \ldots)$, where G_n has n vertices, if the following conditions hold as n tends to infinity:

1. for $t_1 \ll t$ the probability that a configuration of t_1 pebbles is solvable tends to zero, and
2. for $t_2 \gg t$ the probability that a configuration of t_2 pebbles is solvable tends to one.

We denote by $\tau_{\mathbf{M}}(\mathcal{G})$ the set of all threshold functions of $\mathcal{G}$ in the multiset model. It is not immediately clear, however, that $\tau_{\mathbf{M}}(\mathcal{G})$ is nonempty for all $\mathcal{G}$. Nonetheless it is proven to be the case in [1]. In this paper, we will study thresholds in the case when $\mathcal{G}$ is the family of paths. First let us note that the pebbling number of a path on n vertices is equal to 2^{n-1}. However, most of the configurations on t pebbles with t much smaller than 2^{n-1} will still be solvable and so not surprisingly the threshold of the family of paths is much smaller than 2^{n-1}. Let $\mathcal{P} = (P_n)_{n=1}^{\infty}$ be the sequence of paths. In [1] it is showed that

$$\tau_{\mathbf{M}}(\mathcal{P}) = O(n2^{2\sqrt{\lg n}}) \tag{1}$$

and

$$\tau_{\mathbf{M}}(\mathcal{P}) = \Omega(n2^{c\sqrt{\lg n}}) \tag{2}$$

for any constant $c < 1/\sqrt{2}$. The upper bound (1) was improved by Godbole, et al. [5], to

$$\tau_{\mathbf{M}}(\mathcal{P}) = O(n2^{C\sqrt{\lg n}}) \tag{3}$$

for any constant $C > 1$. Our main result of the paper improves the lower bound from [1], showing a lower bound which almost matches the upper bound from [5].

Theorem 1 *Let $\mathcal{P} = (P_n)_{n=1}^{\infty}$ be the sequence of paths. For any $\delta > 0$, let $w = (1-\delta)\sqrt{\lg n}$. Then $\tau_{\mathbf{M}}(\mathcal{P}) = \Omega\left(n2^w\right)$.*

Clearly the random pebbling model from [2] is only one of many that can be considered. In particular, if pebbles are distinguishable and each of them selects independently at random a vertex to be placed on then we obtain a completely different model, which we call the binomial model. We can define the threshold $\tau_{\mathbf{B}}(\mathcal{G})$ in this model in the same way that $\tau_{\mathbf{M}}(\mathcal{G})$ is defined for the multinomial model. Then it is easy to see that $\tau_{\mathbf{B}}(\mathcal{P}) = O(n \ln n)$ (since the probability that every vertex contains a pebble tends to 1) but in fact the threshold is slightly smaller.

Theorem 2 *Let $\mathcal{P} = (P_n)_{n=1}^{\infty}$ be the sequence of paths. Then*

$$\tau_{\mathbf{B}}(\mathcal{P}) = \left(\frac{1}{2} + o(1)\right) n \frac{\ln n}{\lg \ln n}.$$

It turns out that to prove Theorem 1 it is convenient to consider one more model, the geometric one. In this model each vertex on a path generates the number of pebbles that it contains according to the geometric distribution with $p = t/(t+n)$, where t is some function of n — that is, the probability that exactly C pebbles sit on a fixed vertex equals $p^C(1-p)$. Conveniently, the geometric model can be used to approximate the multinomial one from [2]. It is this observation that allows us to generalize the technique from [1] and prove a better lower bound.

The rest of the paper is organized as follows. We prove Theorem 1 in Section 2, and in Section 3 we show Theorem 2. Finally, Section 4 is devoted to investigating which functions $t = t(n)$ can be pebbling thresholds in the multiset model for some sequence of graphs. In particular, we verify the following conjecture posed in [2].

Conjecture 3 *For every $\Omega(n^{1/2}) \ni t_1 \ll t_2 \in O(n)$ there exists a graph sequence $\mathcal{G} = (G_1, \ldots, G_n, \ldots)$ such that $\tau_{\mathbf{M}}(\mathcal{G}) \subset \Omega(t_1) \cap O(t_2)$.*

Let m be an positive integer, $P = \{v_1, v_2, \ldots v_m\}$ and, $S = \{v_{m+1}, \ldots, v_n\}$. Consider the graph $B_{m,n} = (V, E)$, where the set of vertices $V = P \cup S$ and the set of edges E is defined as follows: for every $i = 1, \ldots, m-1$, $\{v_i, v_{i+1}\} \in E$ and for $i = m+1, \ldots, n$, $\{v_i, v_m\} \in E$. In other words $B_{m,n}$ is a path on m vertices with the center of a star on $n - m + 1$ vertices identified with one of its endpoints. (These graphs are called *brooms* in [4], with *handle* P and *bristles* S.) Finally, for m a function of n, define the graph sequence $\mathcal{B}_m = (B_{m,1}, \ldots, B_{m,n}, \ldots)$.

Theorem 4 *Let $\epsilon = \epsilon(n) > 1/2$ be any function such that $n^\epsilon \ll n$. Then for $m = (2\epsilon - 1) \lg n$ we have $\tau_{\mathbf{M}}(\mathcal{B}_m) = \Theta(n^\epsilon)$.*

Note that Theorem 4 implies Conjecture 3. Indeed, for given $t \in \Omega(t_1) \cap O(t_2)$ it is enough to consider $\mathcal{B}_m$ with $m = \lg \frac{t^2}{n}$.

2 Paths in the multinomial model

In this section, we will prove Theorem 1. As mentioned in the introduction, it is convenient to introduce a different probabilistic pebbling model. For $t = t(n)$, in the *geometric* model the number of pebbles on a vertex v has the geometric distribution with probability $p = t/(t+n)$ and the random variables are independent. Therefore,

$$\mathbf{Pr}_{\mathbf{G}}[D(v) = C] \; = \; \left(\frac{t}{t+n}\right)^C \left(\frac{n}{t+n}\right) \; ,$$

where $C = 0, 1, \ldots$. On the other hand in the multinomial model, we have

$$\mathbf{Pr}_{\mathbf{M}}[D(v) = C] \; = \; \frac{\binom{t+n-C-2}{t-C}}{\binom{t+n-1}{t}}$$

and the random variables are dependent. Let F_i denote the event that $D(v_i) = C_i$.

Lemma 5 *Let $w = w(n)$ and let $t = n2^w$. If $k \ll \sqrt{n}$ and $k + \sum_{i=1}^k C_i \ll n2^{2w}$ then*

$$\mathbf{Pr_M}[\wedge_{i=1}^k F_i] \;=\; (1+o(1))\mathbf{Pr_G}[\wedge_{i=1}^k F_i] \;.$$

Proof. We will prove the lower bound. The upper bound can be proved in a similar way. First define $S_k = \sum_{i=1}^k C_i$ and note that

$$\mathbf{Pr_M}[\wedge_{i=1}^k F_i] \;=\; \frac{\binom{t+n-1-k-S_k}{t-S_k}}{\binom{t+n-1}{t}} \;. \tag{4}$$

Further, by repeatedly using the inequality $\frac{a}{b} \geq \frac{a-1}{b-1}$ for $0 < a < b$, we can bound the right hand side of (4) as follows

$$\mathbf{Pr_M}[\wedge_{i=1}^k F_i] \;\geq\; \left(\frac{t-(k+S_k)}{t+n-(k+S_k)}\right)^{S_k} \left(\frac{n-k}{t+n-k}\right)^{k} \;.$$

Therefore,

$$\begin{aligned}\mathbf{Pr_M}[\wedge_{i=1}^k F_i] \;&\geq\; e^{-(k+S_k)^2 n/t^2 - k^2/n} \left(\frac{t}{t+n}\right)^{S_k} \left(\frac{n}{t+n}\right)^{k} \\ &=\; (1-o(1)) \left(\frac{t}{t+n}\right)^{S_k} \left(\frac{n}{t+n}\right)^{k} \;.\end{aligned}$$

□

Let L_i denote the event that $D(v_i) \leq C_i$.

Corollary 6 *Let $w = w(n)$ and let $t = n2^w$. If $k \ll \sqrt{n}$ and $k + \sum_{i=1}^k C_i \ll n2^{2w}$ then*

$$\mathbf{Pr_M}[\wedge_{i=1}^k L_i] \;=\; (1+o(1))\mathbf{Pr_G}[\wedge_{i=1}^k L_i] \;.$$

□

In the argument from [1] the authors consider blocks on a path and use the second moment method to prove that at least one of the blocks is empty. We need a natural generalization of an empty block. Let $m = 2(a+w)$

and consider a contiguous block of m vertices $B = \{u_1, \ldots, u_m\}$. Let k be a positive integer such that $k|w$. We define an (a,k)-*partition* of B by $\Pi = \{A, J_0, \ldots, J_{k-1}\}$, where $v_j \in A$ if and only if $|(a+w+1/2)-j| < a$, and $v_j \in J_i$ if and only if $a + \frac{iw}{k} < |(a+w+1/2)-j| \leq a + \frac{(i+1)w}{k}$ (see Figure 1). The block B is called an (a,k)-*bowl* for the configuration D if

- $D(v) = 0$ for $v \in A$, and
- $D(v) \leq C_i$ for $v \in J_i$, where $C_i = 2^{iw/k}$ for $0 \leq i < k$.

We define $E(a,k)$ to be the event that the block B is an (a,k)-bowl.

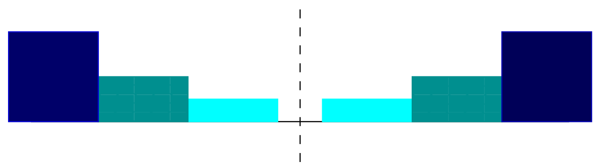

Figure 1: (a,k)-bowl.

Lemma 7 *Let $0 < \delta < 1$ and $w = (1-\delta)\sqrt{\lg n}$. In addition let k be a positive integer such that $k \geq (1-\delta)/\delta$ and let $a \ll w$. Then if $t = n2^w$ pebbles are distributed in the multiset model then $\mathbf{Pr_M}[E(a,k)] = \Omega(n^{-1+\delta/2})$.*

Proof. By Corollary 6, $\mathbf{Pr_M}[E(a,k)] = (1+o(1))\mathbf{Pr_G}[E(a,k)]$. In the geometric model,

$$\mathbf{Pr_G}[D(v) \leq C] \;=\; 1 - \left(\frac{t}{t+n}\right)^{C+1} \;. \qquad (5)$$

As $(1-a)^l \leq 1 - la + l^2a^2$ for nonnegative integer l and $0 < a < 1$,

$$\begin{aligned} \mathbf{Pr_G}[D(v) \leq C] &= 1 - \left(1 - \frac{n}{t+n}\right)^{C+1} \\ &\geq \frac{(C+1)n}{t+n}\left(1 - \frac{(C+1)n}{t+n}\right) \;, \end{aligned}$$

and for $C = 2^{(1-\alpha)w}$ with $0 < \alpha < 1$, one gets

$$\mathbf{Pr_G}[D(v) \leq C] \ \geq \ \frac{1}{2^{\alpha w+1}} \ , \tag{6}$$

In addition, for $C = 0$,

$$\mathbf{Pr_G}[D(v) \leq 0] \ = \ \frac{n}{n+t} \ . \tag{7}$$

Combining (6) and (7) with the independence of the $D(v)$'s in the geometric model shows that $\mathbf{Pr_G}[E(a,k)]$ is at least

$$\left(\frac{n}{t+n}\right)^{2a} \prod_{i=0}^{k-1} 2^{-2((1-i/k)w+1)w/k} \ . \tag{8}$$

We can further simplify (8) to get that the above probability is at least

$$\begin{aligned} &\left(\frac{n}{t+n}\right)^{2a} 2^{-\sum_{i=0}^{k-1} 2((1-i/k)w+1)w/k} \qquad (9)\\ &\quad = \left(\frac{n}{t+n}\right)^{2a} 2^{-2(wk-(k-1)w/2+k)w/k} \\ &\quad = \left(\frac{n}{t+n}\right)^{2a} 2^{-((1+1/k)w^2+2w)} \ . \end{aligned}$$

Since $\left(\frac{n}{t+n}\right)^{2a} = \Theta\left(2^{-2aw}\right)$, we have

$$\mathbf{Pr_G}[E(a,k)] \ = \ \Omega\left(2^{-w^2(1+1/k)-2w(1+a)}\right) .$$

But $a \ll w$ and $w = (1-\delta)\sqrt{\lg n}$, so by the assumption on k

$$\mathbf{Pr_G}[E(a,k)] \ = \ \Omega\left(n^{-1+\delta} 2^{-2w(1+a)}\right) \geq n^{-1+\delta/2} \ ,$$

since $2w(1+a) < (\delta \lg n)/2$ for large enough n. □

Let $a = \lg k + \lg \ln n + 2$ where $k \geq (1-\delta)/\delta$. With $m = 2(a+w)$ we partition the path on n vertices into $\lfloor n/m \rfloor$-blocks $B_1, \ldots, B_{n/m}$, each of length m, and the final block B_∞ of the remaining $n \bmod m$ vertices. We show that with probability tending to one there is a block which is an (a,k)-bowl. To that end let $X_i = 1$ if B_i is an (a,k)-bowl and $X_i = 0$ otherwise. We will need the following correlation inequality.

Lemma 8 *For $i \neq j$,*

$$\mathbf{E}[X_i X_j] \ \leq \ \mathbf{E}[X_i]\mathbf{E}[X_j] \ .$$

As our proof requires tedious but trivial computations we will present it in the appendix.

Lemma 9 *If $X = \sum_{i=1}^{\lfloor n/m \rfloor} X_i$ then $\mathbf{Pr_M}[X = 0] \to 0$.*

Proof. First observe that by Lemma 7.

$$\mathbf{E}[X] \ = \ \Omega\left(\frac{n}{2(a+w)} n^{-1+\delta/2}\right) \ = \ \Omega\left(\frac{n^{\delta/2}}{2(a+w)}\right) \ \to \ \infty \ .$$

Since $\mathbf{E}[X_i X_j] \ \leq \ \mathbf{E}[X_i]\mathbf{E}[X_j]$, by Lemma 8, the second moment method applies. Consequently $\mathbf{Pr_M}[X = 0] \to 0$. □

The next lemma shows that with large probability every vertex will have at most $t(\ln n)/n$ pebbles in the multiset model.

Lemma 10 *Let $\alpha > 0$ and let $C = (1+\alpha)(\frac{t}{n}) \ln n$, with $t = n2^w$ as above. Then*

$$\mathbf{Pr_M}[\exists_v D(v) \geq C] \ \to \ 0 \ .$$

Proof. For fixed v,

$$\mathbf{Pr_M}[D(v) \geq C] \ = \ \frac{\binom{t+n-2-C}{t-C}}{\binom{t+n-1}{t}} \ \leq \ \left(\frac{t}{t+n}\right)^C \ .$$

Therefore, the probability that there is a vertex v with $D(v) \geq C$ is at most

$$n\left(\frac{t}{t+n}\right)^C \ \approx \ e^{\ln n - Cn/(t+n)} \ \to \ 0 \ .$$

□

Finally, we can show that it is not possible to pebble to the middle vertex of a (a, k)-bowl. Indeed, suppose B_j is a block which is a (a, k)-bowl. If $V(B_j) = u_1, \ldots u_m$ then let $u = u_w$ and $v = u_{w+2a+1}$. We show that we cannot accumulate too many pebbles on u and on v.

Lemma 11 *With probability tending to one (in the multiset model), we can accumulate on each of u and v at most $2k + 4\ln n$ pebbles.*

Proof. Consider the vertex u (the proof for v is identical). By the definition of an (a,k)-bowl and by Lemma 10, we can accumulate at most

$$1 \;+\; \sum_{i=0}^{k-1} \sum_{j \geq iw/k} \frac{2^{iw/k}}{2^j} \;+\; 2\ln n \sum_{j \geq w} \frac{t}{n2^j}$$

pebbles on u. This quantity is at most $2k + 4\ln n$, since $t = n2^w$. □

2.1 Proof of Theorem 1

By Lemma 7, with large probability, there is a block B_j on $m = 2(a+w)$ vertices that is an (a,k)-bowl. If $V(B_j) = u_1, \ldots u_m$ and $u = u_w$, $v = u_{w+2a+1}$ then by Lemma 11, we can accumulate at most $2k + 4\ln n$ pebbles on each u and v. However, $a = \lg k + \lg\ln n + 2$, and so $a > \lg(2k + 4\ln n)$ when $\delta < 1/2$ (which we may assume is the case). Thus it is not possible to pebble to one of the middle vertices (u_{w+a}) of B_j. □

3 Proof of Theorem 2

The proof follows the lines of path threshold proof in [1]. We will need the following two Chernoff tail bounds for the tails of random variable with binomial distribution $\mathbf{Bi}[m,p]$.

Lemma 12 *If $X \in \mathbf{Bi}[m,p]$ then for $0 < \delta < 1$ we have*

$$\mathbf{Pr}[X < (1-\delta)\mathbf{E}[X]] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mathbf{E}[X]} .$$

Lemma 13 *If $X \in \mathbf{Bi}[m,p]$ then for $C \geq 7\mathbf{E}[X]$ we have*

$$\mathbf{Pr}[X \geq C] \;\leq\; e^{-C} .$$

3.1 Theorem 2 Lower Bound

To prove the lower bound, fix $\alpha > 2$ and let $t = \frac{n \ln n}{\alpha \lg \ln n}$. Let $2 < \beta < \alpha$ and partition path P_n into n/k blocks $B_1, \ldots B_{n/k}$ each of length $k = \beta \lg \ln n$. (Here we assume that k divides n by otherwise throwing away $n \bmod k$ vertices from the end.) Let Y_j be equal to one if the jth block has no pebbles, and zero otherwise. Then

$$\mathbf{Pr_B}[Y_j = 1] \;=\; \left(1 - \frac{k}{n}\right)^t \;\approx\; e^{-kt/n} \;, \tag{10}$$

and so the expected number of empty blocks is

$$\mathbf{E}\left[\sum_{j=1}^{n/k} Y_j\right] \;\approx\; \frac{n}{k} e^{-kt/n} \;\rightarrow\; \infty \;. \tag{11}$$

Also, for $j \neq i$ we have $\mathbf{E}[Y_j Y_i] < \mathbf{E}[Y_j]\mathbf{E}[Y_i]$, and so the second moment method applies: $\mathbf{Pr_B}[Y = 0] \leq \mathbf{Var}[Y]/\mathbf{E}[Y]^2 \leq 1/\mathbf{E}[Y]$, where $Y = \sum_{j=1}^{n/k} Y_j$. Consequently,

$$\mathbf{Pr_B}\left[\sum_{j=1}^{n/k} Y_j > 0\right] \;\rightarrow\; 1 \;. \tag{12}$$

For a vertex v, the number of pebbles on v, $D(v)$ has binomial distribution $\mathbf{Bi}[t, 1/n]$ with $\mathbf{E}[D(v)] = t/n = \frac{\ln n}{\alpha \lg \ln n}$. Thus, by Lemma 13

$$\mathbf{Pr_B}[D(v) \geq (\ln n)^{\beta/2}/2] \;\leq\; e^{-(\ln n)^{\beta/2}/2} \;. \tag{13}$$

Hence, with probability tending to one, all vertices v will have $D(v) < \frac{1}{2}(\ln n)^{\beta/2}$. Then (12) implies that there is a block B_j with no pebbles. The number of pebbles that can be accumulated on each of the endpoints of B_j is at most

$$\sum_{j \geq 0} \frac{D(v_j)}{2^j} \;<\; (\ln n)^{\beta/2} \;.$$

Because we have $2^{k/2} = (\ln n)^{\beta/2}$, it is not possible to pebble to the middle vertex of B_j. ◇

3.2 Theorem 2 Upper Bound

Consider the path $v_1, v_2, \ldots, v_n$. Fix $0 < \epsilon < 0.5$ and let $t = \left(\frac{1}{2} + \epsilon\right) \frac{n \ln n}{\lg \ln n}$. In addition let $\delta < 1$ be a positive number such that

$$\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}} \leq e^{-(1-\epsilon/2)} \ . \tag{14}$$

We will also assume that n is sufficiently large whenever needed. Let $k = \lg \ln n - \lg \lg \ln n + \lg(1-\delta) - 1$ and let A be the event that at least one of the vertices from $\{v_1, \ldots, v_k, v_n, \ldots, v_{n-k+1}\}$ has zero pebbles. Clearly, $P(A) \leq 2k(1-1/n)^t \sim 2ke^{-t/n} \to 0$. For v_i with $k < i < n-k$ let $B(i)$ denote the block $v_{i-k+1}, \ldots, v_i, \ldots, v_{i+k-1}$ of length $2k-1$ and let $T(i)$ be the number of pebbles in $B(i)$. Then $T(i) \in \mathbf{Bi}[t, 2k/n]$ and in particular

$$\mathbf{E}[T(i)] = t\left(\frac{2k}{n}\right) = (1+2\epsilon)\frac{k \ln n}{\lg \ln n}. \tag{15}$$

Consequently, we have

$$(1+\epsilon)\ln n \leq \mathbf{E}[T(i)] \leq (1+2\epsilon)\ln n \ . \tag{16}$$

We apply Lemma 12 with δ defined in (14) to conclude that

$$\begin{aligned} \mathbf{Pr_B}\left[T(i) < (1-\delta)\ln n\right] &\leq \mathbf{Pr_B}\left[T(i) < (1-\delta)\mathbf{E}[T(i)]\right] && (17)\\ &\leq e^{-(1-\epsilon/2)(1+\epsilon)\ln n} \ . && (18) \end{aligned}$$

The right-hand side of (18) is less than or equal to $e^{-(1+\epsilon/4)\ln n} = 1/n^{1+\epsilon/4}$. Hence, the probability that there is a vertex v_i such that $T(i) < (1-\delta)\ln n$ goes to zero. In addition, observe that $(2k-1)2^k \leq (1-\delta)\ln n$ and so with probability tending to one for every i, $T(i) \geq (2k-1)2^k$. Thus there is a vertex v in $B(i)$ such that $D(v) \geq 2^k$, and so we can place at least one pebble on v_i. $\diamond$

This completes the proof of Theorem 2. $\square$

4 Proof of Theorem 4

We return to the multiset model and divide the argument into two propositions. In the first one we show the upper bound, and in the second we show the lower bound. Let $B_{m,n} = (P \cup S, E)$ be a broom, as defined in the introduction. Assume that $t \ll n$ and let D be a configuration of t pebbles on B. Since $D(v_1) + D(v_2) + \ldots D(v_n) = t$, we have the expectation

$$\mathbf{E}[D(v_i)] \;=\; \frac{t}{n}\;. \tag{19}$$

First, for a fixed vertex v and $i \geq 1$, we compute the probability

$$\mathbf{Pr_M}[D(v) = i] \;=\; \frac{\binom{n+t-i-2}{t-i}}{\binom{n+t-1}{t}}\;.$$

We next compute

$$\binom{n+t-i-2}{t-i} \;=\; \left[\left(\frac{t-i+1}{n+t-i-1}\right)\cdots\left(\frac{t}{n+t-2}\right)\right]\binom{n+t-2}{t}$$

$$=\; \left[\left(\frac{t-i+1}{n+t-i-1}\right)\cdots\left(\frac{t}{n+t-2}\right)\right]\left(\frac{n-1}{n+t-1}\right)\binom{n+t-1}{t}\;.$$

This yields

$$\begin{aligned}\left(\frac{n-1}{n+t-1}\right)\left(\frac{t-i}{n+t-i-2}\right)^i\binom{n+t-1}{t} \;&\leq\; \binom{n+t-i-2}{t-i}\\ &\leq\; \left(\frac{t}{n}\right)^i\binom{n+t-1}{t}\;.\end{aligned}$$

Therefore,

$$\left(\frac{n-1}{n+t-1}\right)\left(\frac{t-i}{n+t-i-2}\right)^i \;\leq\; \mathbf{Pr}[D(v) = i] \;\leq\; \left(\frac{t}{n}\right)^i\;. \tag{20}$$

Proposition 14 *Let $\epsilon = \epsilon(n) \leq 1$ and let $\omega = \omega(n) \to \infty$ be such that $t = \omega(n)n^{\epsilon} \ll n$. Let $m = (2\epsilon - 1)\lg n$ and let C be a random configuration of t pebbles on $B_{m,n}$. Then*

$$\mathbf{Pr_M}[D \text{ is solvable}] \to 1$$

as $n \to \infty$.

Proof. Let $B_{m,n} = (P \cup S, E)$, where $m = (2\epsilon - 1)\lg n$, $P = \{v_1, \ldots, v_m\}$ and $S = \{v_{m+1}, \ldots, n\}$. Let $L_2 = \{v \mid D(v) = 2\}$ and consider $X = |S \cap L_2|$. Then $X = \sum_{i=m+1}^{n} X_i$, where $X_i = 1$ if and only if $D(v_i) = 2$. By (20),

$$\mathbf{E}[X] \ \leq \ |S| \left(\frac{t}{n} \right)^2$$

and

$$\mathbf{E}[X] \ \geq \ |S| \left(\frac{n-1}{n+t-1} \right) \left(\frac{t-2}{n+t-2} \right)^2 .$$

Since $t \ll n$, we have

$$\mathbf{E}[X] \ \sim \ (n - (2\epsilon - 1)\lg n)(\omega(n) n^{\epsilon - 1})^2 \ \sim \ \omega(n)^2 n^{2\epsilon - 1} . \tag{21}$$

Recall that v_m denotes the center of the set S. We shall show that $\mathbf{Pr}[X \geq n^{2\epsilon-1}] \to 1$. Then we can accumulate $n^{2\epsilon-1}$ pebbles on v_m, and since $m = (2\epsilon - 1)\lg n$ we can pebble from v_m to any other vertex of $B_{m,n}$. Indeed,

$$\sigma_X^2 \ = \ \mathbf{E}[X^2] - \mathbf{E}[X]^2 \ = \ \sum_{i=m+1}^{n} \mathbf{E}[X_i^2] + \sum_{i \neq j} \mathbf{E}[X_i X_j] - \mathbf{E}[X]^2 ,$$

and since $\mathbf{E}[X_i X_j] \leq \mathbf{E}[X_i]\mathbf{E}[X_j]$ we obtain

$$\sigma_X^2 \ \leq \ \sum_{i=m+1}^{n} \mathbf{E}[X_i] = \mathbf{E}[X] .$$

Using (21), we have $\mathbf{Pr}[X < n^{2\epsilon-1}] \leq \mathbf{Pr}[\ |X - \mathbf{E}[X]| > \mathbf{E}[X]/2\ ]$, which by Chebyshev's inequality is at most

$$\frac{4}{\mathbf{E}[X]} \ \to \ 0 .$$

□

Proposition 15 *Let $1/2 < \epsilon = \epsilon(n) \leq 1$ and $\omega = \omega(n) \to \infty$. Let $t = \lfloor \frac{n^\epsilon}{\omega} \rfloor$ and $m = (2\epsilon - 1)\lg n$ and let D be a random configuration of t pebbles on $B_{m,n}$. Then*

$$\mathbf{Pr}_{\mathbf{M}}[D \text{ is solvable}] \to 0$$

as n approaches infinity.

Proof. Let $B_{m,n} = (P \cup S, E)$, where $m = (2\epsilon - 1)\lg n$, $P = \{v_1, \ldots, v_m\}$ and $S = \{v_{m+1}, \ldots, v_n\}$. Set $L_i = \{v \mid D(v) = i\}$. Then $\mathbf{E}[|L_i \cap S|] \leq |S|(\frac{t}{n})^i$ and so

$$\mathbf{E}[|L_i \cap S|] \;\leq\; \frac{n - (2\epsilon - 1)\lg n}{[\omega n^{1-\epsilon}]^i} \;. \tag{22}$$

Let A be the number of pebbles that can be accumulated on v_m using the pebbles assigned to vertices from S. Then

$$\mathbf{E}[A] \;=\; \mathbf{E}[|S \cap L_2|] + \mathbf{E}[|S \cap L_3|] + 2\mathbf{E}[|S \cap L_4|] + \ldots \lfloor \frac{t}{2} \rfloor \mathbf{E}[|S \cap L_t|] \;. \tag{23}$$

Using (22) we can bound $\mathbf{E}[A]$ from above by

$$\begin{aligned} \mathbf{E}[A] \;&<\; \frac{n - (2\epsilon - 1)\lg n}{[\omega n^{1-\epsilon}]^2} \sum_{k \geq 0} \frac{(k+1)}{[\omega n^{1-\epsilon}]^k} && (24) \\ &<\; \frac{2(n - (2\epsilon - 1)\lg n)}{[\omega n^{1-\epsilon}]^2} && (25) \\ &<\; \frac{2n^{2\epsilon - 1}}{\omega^2} \;. && (26) \end{aligned}$$

Define the following random variable

$$Y \;=\; \sum_{k=0}^{m-1} \frac{D(v_{k+1})}{2^k} + \frac{A}{2^{m-1}}$$

and note that $Y \geq 1$ if and only if D is v_1-solvable. Then by (19)

$$\mathbf{E}[Y] \;\leq\; \frac{2}{n^{1-\epsilon}\omega} + \frac{\mathbf{E}[A]}{2^{m-1}} \;,$$

and by (24-26)

$$\mathbf{E}[Y] \;<\; \frac{2}{n^{1-\epsilon}\omega} + \frac{n^{2\epsilon - 1}}{\omega^2 2^{m-2}} \;=\; \frac{2}{n^{1-\epsilon}\omega} + \frac{4}{\omega^2} \;\to\; 0 \;.$$

Therefore, by Markov's inequality,

$$\mathbf{Pr}[Y \geq 1] \;\leq\; \mathbf{E}[Y] \;\to\; 0 \;.$$

□

Proof of Theorem 4. By Proposition 14 and Proposition 15, for $m = (2\epsilon - 1)\lg n$,

$$\tau_{\mathbf{M}}(\mathcal{B}_{m,n}) \;=\; \Theta(n^{\epsilon}) \;.$$

□

5 Appendix

In this section we will give a proof of Lemma 8.

Proof of Lemma 8. Let $l = 2(a+w)$, $C = \sum_{i=1}^{l} C_i$, $\bar{C} = \sum_{i=0}^{l} \bar{C}_i$ and recall that $t = n2^w$, $w = (1-\delta)\sqrt{\lg n}$, $a = \lg k + \lg\ln n$. It is enough to prove that, for distinct vertices $v_1, \ldots v_l, w_1, \ldots, w_l$ and for $C \le 2^{w+2}$ and $\bar{C} \le 2^{w+2}$, we have

$$\mathbf{Pr_M}[\wedge_{i=1}^{l}(D(v_i) = C_i) \wedge_{i=1}^{l} (D(w_i) = \bar{C}_i)] \le$$

$$\mathbf{Pr_M}[\wedge_{i=1}^{l}(D(v_1) = C_1)]\mathbf{Pr_M}[\wedge_{i=1}^{l}(D(w_i) = \bar{C}_i)] .$$

Also,

$$\mathbf{Pr_M}[\wedge_{i=1}^{l}(D(v_i) = C_i) \wedge_{i=1}^{l} (D(w_i) = \bar{C}_i)] = \frac{\binom{t+n-1-2l-C-\bar{C}}{t-C-\bar{C}}}{\binom{t+n-1}{t}} ,$$

$$\mathbf{Pr_M}[\wedge_{i=1}^{l}(D(v_1) = C_1)] = \frac{\binom{t+n-1-C-l}{t-C}}{\binom{t+n-1}{t}}$$

and

$$\mathbf{Pr_M}[\wedge_{i=1}^{l}(D(w_i) = \bar{C}_i)] = \frac{\binom{t+n-1-\bar{C}-l}{t-\bar{C}}}{\binom{t+n-1}{t}} .$$

Therefore it is enough to prove that

$$\binom{t+n-1-2l-C-\bar{C}}{t-C-\bar{C}}\binom{t+n-1}{t} \le$$

$$\binom{t+n-1-C-l}{t-C}\binom{t+n-1-\bar{C}-l}{t-\bar{C}} ,$$

which can be re-written as

$$\frac{\binom{t+n-1}{t}}{\binom{t+n-1-C-l}{t-C}} \le \frac{\binom{t+n-1-\bar{C}-l}{t-\bar{C}}}{\binom{t+n-1-2l-C-\bar{C}}{t-C-\bar{C}}} . \qquad (27)$$

The left-hand side of (27) is equal to

$$\left(\prod_{i=0}^{l-1} \frac{t+n-1-i}{n-1-i}\right)\left(\prod_{j=0}^{C-1} \frac{t+n-1-l-j}{t-j}\right)$$

and the right-hand side equals

$$\left(\prod_{j=0}^{C-1} \frac{t+n-1-\bar{C}-l-j}{t-\bar{C}-j}\right)\left(\prod_{i=0}^{l-1} \frac{t+n-1-C-\bar{C}-l-i}{n-1-l-i}\right).$$

Clearly, for any j as above, we have

$$\frac{t+n-1-l-j}{t-j} \leq \frac{t+n-1-\bar{C}-l-j}{t-\bar{C}-j}.$$

In addition, since $n(C+\bar{C}) \ll lt$, for any i as above we have

$$\frac{t+n-1-i}{n-1-i} \leq \frac{t+n-1-C-\bar{C}-l-i}{n-1-l-i}.$$

The result follows. □

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