Girth, Pebbling, and Grid Thresholds

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Abstract

The pebbling number of a graph is the smallest number t such that from any initial configuration of t pebbles one can move a pebble to any prescribed vertex by a sequence of pebbling steps. It is known that graphs whose connectivity is high compared to their diameter have pebbling number as small as possible. We will use the above result to prove two related theorems. First, answering a question of the second author, we show that there exist graphs of arbitrarily high constant girth and least possible pebbling number. In the second application, we prove that the product of two graphs of high minimum degree has pebbling number equal to the number of vertices of the product. This shows that Graham's product conjecture is true in the case of high minimum degree graphs. In addition, we consider a probabilistic variant of the pebbling problem and establish a pebbling threshold result for products of paths. The last result shows that the sequence of paths satisfies the probabilistic analog of Graham's product conjecture.

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1 Introduction

1.1 Pebbling

A *pebbling configuration* $\mathbf{C}$ on a graph G is a distribution of pebbles on the vertices of G. Given a particular configuration, one is allowed to move the pebbles about the graph according to this simple rule: if two or more vertices sit at vertex v, then one of them can be moved to a neighbor provided another is removed from v. Given a specific *root* vertex r, we say that $\mathbf{C}$ is r-*solvable* if one can move a pebble to r after a finite number of pebbling steps, and that $\mathbf{C}$ is *solvable* if it is r-solvable for every r. The pebbling number is the least number $\pi = \pi(G)$ so that every configuration of π pebbles on G is solvable.

The two most obvious pebbling facts are for complete graphs and paths. The pigeonhole principle implies that $\pi(K_n) = n$, and $\pi(P_n) = 2^{n-1}$ follows by induction or a simple weight function method. In fact, $\pi(G) \geq \min\{n(G), 2^{\text{diam}(G)}\}$ for every G. Results for trees (a formula based on the maximum path partition of a tree in [12], see also [3]), d-dimensional cubes Q^d (see [3]), and many other graphs with interesting properties are known (see the survey [11]).

A probabilistic version of pebbling was introduced in [6]. Let $\mathcal{G} = (G_i)_{i=1}^{\infty}$ be a sequence of graphs with strictly increasing numbers of vertices $N = n(G_i)$. For a function $t = t(N)$ let $\mathbf{C}_t$ denote a configuration on G_i that is chosen uniformly at random from all configurations of t pebbles. The sequence $\mathcal{G}$ has *pebbling threshold* $\tau = \tau(\mathcal{G})$ if, for every $\omega \gg 1$, (1) $\Pr[\mathbf{C}_t \text{ is solvable}] \rightarrow 0$ for $t = \tau/\omega$ and (2) $\Pr[\mathbf{C}_t \text{ is solvable}] \rightarrow 1$ for $t = \omega\tau$.

It was proven in [4] that the sequence of cliques has threshold $\tau(\mathcal{K}) = \Theta(N^{1/2})$. Bekmetjev, et al. [1], showed recently that every graph sequence has a pebbling threshold. Bounds on the sequence of paths have undergone several improvements, the results of which are summarized as follows.

Result 1 *The pebbling threshold for the sequence of paths $\mathcal{P} = (P_n)_{n=1}^{\infty}$ satisfies*

$$\tau(\mathcal{P}) \in \Omega\Big(N2^{c\sqrt{\lg N}}\Big) \cap O\Big(N2^{c'\sqrt{\lg N}}\Big)$$

for every $c < 1/\sqrt{2}$ and $c' > 1$.

The lower bound is found in [1] and the upper bound is found in [9].

It is important to draw a distinction between this random pebbling model and the one in which each of t pebbles independently chooses uniformly at random a vertex on which to be placed. In the world of random graphs, the analogs of these two models are asymptotically equivalent. However, in the pebbling world, they are vastly different. For example, in the independent model the pebbling threshold for paths is at most $N \lg N$ since, with more than that many pebbles, almost always every vertex already has a pebble on it.

1.2 Results

Pachter et al. [13] proved that every graph of diameter two on N vertices has pebbling number either N or $N+1$. Graphs G with $\pi(G) = n(G)$ are called *Class 0*, and in [5] a characterization of diameter two Class 0 graphs was found and used to prove that diameter two graphs with connectivity at least 3 are Class 0. The authors also conjectured that every graph of fixed diameter and high enough connectivity was Class 0. This conjecture was proved by Czygrinow, Hurlbert, Kierstead and Trotter [7] in the following result.

Result 2 *Let d be a positive integer and set $k = 2^{2d+3}$. If G is a graph of diameter at most d and connectivity at least k, then G is of Class 0.*

In this note, we present two applications of this result. Our first application concerns the following girth problem posed in [11].

Question 3 *Does there exist a constant C such that if G is a connected graph on n vertices with $girth(G) > C$ then $\pi(G) > n$?*

We answer the above question in the negative. Let $g_0(n)$ denote the maximum number g such that there exists a graph G on at most n vertices with finite $girth(G) \geq g$ and $\pi(G) = n(G)$. That is, $g_0(n)$ is the highest girth, as a function of n, among all Class 0 graphs. It is easy to see that

$$g_0(n) \leq 1 + 2 \lg n$$

(because the cycle on k vertices has pebbling number at least $2^{\lfloor k/2 \rfloor}$ — see [13]) and we prove the following lower bound.

Theorem 4 *For all $n \geq 3$ we have*

$$g_0(n) \geq \lfloor \sqrt{(\lg n)/2 + 1/4} - 1/2 \rfloor \ .$$

We prove this theorem in Section 2.1 using Result 2.

Our second application concerns the following conjecture of Graham [3].

Conjecture 5 *Every pair of graphs G and H satisfy $\pi(G \square H) \leq \pi(G)\pi(H)$.*

Here, the Cartesian product has vertices $V(G \square H) = V(G) \times V(H)$ and edges $E(G \square H) = \{u \times E(H)\}_{u \in V(G)} \cup \{E(G) \times v\}_{v \in V(H)}$. A number of theorems have been published in support of this conjecture, including the recent work of Herscovici [10] which verifies the case for all pairs of cycles. We show the following.

Theorem 6 *Let G and H be connected graphs on n vertices with minimum degrees $\delta(G)$, $\delta(H)$ and let $\delta = \min\{\delta(G), \delta(H)\}$. If $\delta \geq 2^{12n/\delta+15}$ then $G \square H$ is of Class 0.*

In particular, there is a constant c such that if $\delta > c\frac{n}{\lg n}$, then $G \square H$ is of Class 0. We prove this in section 2.2, again using Result 2. As a corollary we obtain that Graham's Conjecture is satisfied for graphs with minimum degree $\delta > c\frac{n}{\lg n}$.

Corollary 7 *Let G and H be as in Theorem 6, with $\delta \geq 2^{12n/\delta+15}$. Then $\pi(G \square H) \leq \pi(G)\pi(H)$.*

Proof. We have $\pi(G \square H) = n(G \square H) = n(G)n(H) \leq \pi(G)\pi(H)$. $\square$

Finally, in this paper we also consider the following probabilistic analog of Graham's Conjecture 5, which we consider a correction of one from [11].

Problem 8 *Let $\mathcal{G} = (G_n)_{n=1}^{\infty}$ and $\mathcal{H} = (H_n)_{n=1}^{\infty}$ be two graph sequences. Define the product sequence $\mathcal{G} \square \mathcal{H} = (G_n \square H_n)_{n=1}^{\infty}$. Find $\tau(\mathcal{G} \square \mathcal{H})$.*

Let $N_1 = N(G_n)$, $N_2 = N(H_n)$ denote the number of vertices of graphs G_n and H_n from Problem 8. It would be interesting to determine for which sequences $\mathcal{G} = (G_n)_{n=1}^{\infty}$ and $\mathcal{H} = (H_n)_{n=1}^{\infty}$, we have

$$f(N_1N_2) \in O\Big(g(N_1)h(N_2)\Big), \tag{1}$$

where $f \in \tau(\mathcal{G} \square \mathcal{H})$, $g \in \tau(\mathcal{G})$ and $h \in \tau(\mathcal{H})$. We call pairs of sequences which satisfy (1) *well-behaved*. One might conjecture that all pairs of sequences are well-behaved, but we believe counterexamples might exist.

We define the two-dimensional grid $P_n^2 = P_n \Box P_n$, and in general the d-dimensional grid $P_n^d = P_n \Box P_n^{d-1}$. It is easy to show that $P_n^d = P_n^\alpha \Box P_n^\beta$ for all α and β for which $\alpha + \beta = d$. If we denote $\mathcal{P}^d = (P_n^d)_{n=1}^\infty$ then we have $\mathcal{P}^d = \mathcal{P}^\alpha \Box \mathcal{P}^\beta$. Thus, for example, in light of Result 1, the truth of (1) would imply that

$$\tau(\mathcal{P}^2) \in O\left(\left(\sqrt{N} 2^{c'\sqrt{\lg \sqrt{N}}}\right)^2\right) = O\left(N 2^{c'\sqrt{2 \lg N}}\right) .$$

Here we prove the following stronger theorem.

Theorem 9 *Let $\mathcal{P}^d = (P_n^d)_{n=1}^\infty$ be the sequence of d-dimensional grids, where $P_n^d = (P_n)^d$ is the cartesian product of d paths on n vertices each, and let $N = n^d$ be the number of vertices of $\mathcal{P}_n^d$. Then*

$$\tau(\mathcal{P}^d) \subseteq \Omega\left(N 2^{c_d (\lg N)^{1/(d+1)}}\right) \cap O\left(N 2^{c'_d (\lg N)^{1/(d+1)}}\right)$$

for all $c_d < 2^{-d/(d+1)}$ and $c'_d > d + 1$.

This verifies (1) in the case of grids.

Corollary 10 *Let α, β be any pair of positive integers; then for $\mathcal{G} = \mathcal{P}^\alpha$ and $\mathcal{H} = \mathcal{P}^\beta$, (1) holds.*

Proof. Indeed, if $g \in \tau(\mathcal{G})$ and $h \in \tau(\mathcal{H})$ then Theorem 9 says that

$$\begin{aligned} g(N^{\overline{\alpha}})h(N^{\overline{\beta}}) &\in \Omega\left(N^{\overline{\alpha}} 2^{c_\alpha (\lg N^{\overline{\alpha}})^{1/(\alpha+1)}} N^{\overline{\beta}} 2^{c_\beta (\lg N^{\overline{\beta}})^{1/(\beta+1)}}\right) \\ &\subseteq \Omega\left(N 2^{c (\lg N)^{1/(\gamma+1)}}\right) \\ &\subseteq \Omega\left(N 2^{c (\lg N)^{1/(d/2+1)}}\right) , \end{aligned}$$

for some c, where $\gamma = \min\{\alpha, \beta\}$, $d = \alpha + \beta$, $\overline{\alpha} = \alpha/d$ and $\overline{\beta} = \beta/d$. On the other hand, Theorem 9 also says that

$$\tau(\mathcal{P}^{\alpha+\beta}) = \tau(\mathcal{P}^d) \in O\left(N 2^{c'_d (\lg N)^{1/(d+1)}}\right) ,$$

which is asymptotically smaller. $\Box$

We prove Theorem 9 in Section 2.3.

2 Proofs

2.1 Proof of Theorem 4

We will make use of Mader's theorem (see [8]), below.

Result 11 *Every graph having average degree at least $\bar{d}$ has a subgraph of connectivity at least $\lfloor \bar{d}/4 \rfloor$.*

We will also make use of the following result from [2] (Chapter III, Theorem 1.1).

Result 12 *For any $g \geq 3$ and $\delta \geq 3$ there exists some graph H with girth at least g, minimal degree at least δ, and no more than $(2\delta)^g$ vertices.*

Proof of Theorem 4. Set $\delta = 2^{2g+1}$ and $n = 2^{2g(g+1)}$; then $g = \lfloor \sqrt{(\lg n)/2 + 1/4} - 1/2 \rfloor$. Let H be a graph guaranteed to exist by Result 12. By Result 11, H has some subgraph, F say, which is 2^{2g-1}-connected; clearly, F also has girth at least g. Now let $\hat{F}$ be an edge-maximal graph on the same vertices as F such that F is a subgraph of $\hat{F}$ and $\hat{F}$ has girth at least g. $\hat{F}$ can have diameter no more than $g-2$, for if there existed vertices x and y in $\hat{F}$ such that the shortest path between x and y had length $g-1$ or more, adding the edge xy to $\hat{F}$ would give a graph of girth g or more, contradicting maximality. Therefore $\hat{F}$ has diameter at most $g-2$ and is 2^{2g-1}-connected, so by Result 2, it is of Class 0, and it has no more than $(2\delta)^g = 2^{2g(g+1)}$ vertices. □

2.2 Proof of Theorem 6

Theorem 6 follows from the following two lemmas and Result 2.

Lemma 13 *Let G be a connected graph on n vertices with minimum degree δ. Then the diameter of G is at most $3\frac{n}{\delta} + 3$.*

Proof. Fix two vertices x, y in G and consider the shortest path $x = x_1, \ldots, x_k = y$ between x and y. Let $i = \lfloor \frac{k-1}{3} \rfloor$. Then $x_1, x_4, x_7 \ldots, x_{3i+1}$ must have disjoint neighborhoods, and so $i(\delta + 1) \leq n$ which yields $\frac{k-3}{3} \leq \lfloor \frac{k-1}{3} \rfloor = i \leq \frac{n}{\delta+1}$, so that $k < \frac{3n}{\delta+1} + 3 \leq \frac{3n}{\delta} + 3$. □

The next lemma was proved by Czygrinow and Kierstead. We reproduce the proof here.

Lemma 14 *For connected graphs G and H, the product $G\Box H$ has connectivity $\kappa(G\Box H) \geq \min\{\delta(G), \delta(H)\}$.*

Proof. Set $\delta = \min\{\delta(G), \delta(H)\}$. Let $v_1 = (g, h_1), v_2 = (g, h_2), \ldots, v_\delta = (g, h_\delta)$, $w_1 = (g_1, h), w_2 = (g_2, h), \ldots, w_\delta = (g_\delta, h)$ be distinct vertices (other than perhaps $v_1 = w_1$) in $G\Box H$ that satisfy

$$dist_G(g_i, g) \leq dist_G(g_{i+1}, g) \tag{2}$$

and

$$dist_H(h_i, h) \leq dist_H(h_{i+1}, h), \tag{3}$$

for $i = 1, \ldots, \delta - 1$. We shall construct vertex-disjoint paths $P_1, \ldots, P_\delta$ such that P_i connects v_i with w_i. Construct P_1 as follows: let $g_1\bar{g}(1)\ldots\bar{g}(k)g$ be any shortest path in G connecting g_1 with g and let $h\bar{h}(1)\ldots\bar{h}(l)h_1$ be any shortest path in H connecting h with h_1. Then P_1 is the path:

$$w_1 = (g_1, h)(g_1, \bar{h}(1))\ldots(g_1, h_1)(\bar{g}(1), h_1)\ldots(g, h_1) = v_1.$$

Delete v_1 and w_1 and construct $P_2, \ldots, P_\delta$ inductively. We claim that $P_2, \ldots, P_\delta$ are vertex-disjoint with P_1. Indeed, suppose that $V(P_j) \cap V(P_1) \neq \emptyset$ for some $j = 2, \ldots, \delta$. There are two similar cases to consider. First, suppose that $(g_j, f) \in V(P_j) \cap V(P_1)$. Since $g_j \neq g_1$, $f = h_1$ and $g_j = \bar{g}(i)$ for some $i = 1, \ldots, k$. Then however

$$dist_G(g_j, g) < dist_G(g_1, g),$$

contradicting (2). Similarly, if $(f, h_j) \in V(P_j) \cap V(P_1)$ then $f = g_1$ and $h_j = \bar{h}(i)$ for some $i = 1, \ldots, l$ which implies that

$$dist_H(h_j, h) < dist_H(h_1, h),$$

contradicting (3).

By induction, paths $P_1, \ldots, P_\delta$ are vertex-disjoint. Now, for any two distinct vertices $v = (g, \tilde{h}), w = (\tilde{g}, h) \in V(G\Box H)$, let $v_1 = (g, h_1), v_2 = (g, h_2), \ldots, v_\delta = (g, h_\delta)$ be neighbors of v in the H-dimension, and let $w_1 = (g_1, h), w_2 = (g_2, h), \ldots, w_\delta = (g_\delta, h)$ neighbors of w in the G-dimension ordered according to (2) and (3). By the previous argument we can find vertex-disjoint paths $P_1, \ldots, P_\delta$ connecting the v_is with the w_js. These paths now can be used to connect v with w by δ internally vertex-disjoint paths. Indeed, if any of the paths contains v or w then it yields a shorter path between v and w which is disjoint with other paths. Therefore the connectivity of $G\Box H$ is at least δ. $\Box$

Proof of Theorem 6. By Lemma 13, the diameter d of $G\Box H$ is at most $6\frac{n}{\delta}+6$ and by Lemma 14, the connectivity k of $G\Box H$ is at least δ. Since $\delta \geq 2^{12n/\delta+15}$ the assumptions of Result 2 are satisified and so $G\Box H$ is of Class 0. $\Box$

2.3 Proof of Theorem 9

Throughout, we let $N = n^d$. Also, we define $\left\langle {a \atop b} \right\rangle = \binom{a+b-1}{b}$. Note that $\left\langle {a \atop b} \right\rangle$ is the number of ways to place b unlabeled balls into a labeled urns. For our purposes, it equals the number of configurations of b pebbles on a graph of a vertices. We will also use the fact that $\left\langle {a \atop b} \right\rangle$ counts the number of points in $\mathbb{Z}^a$ whose coordinates are nonnegative and sum to b.

We begin by proving that a configuration with relatively few pebbles almost always has no vertices having a huge number of pebbles. For natural numbers a and b, define $a^{\underline{b}} = a!/(a-b)!$. For a configuration $\mathbf{C}$ of pebbles on a graph let $\mathbf{C}(v)$ denote the number of pebbles on vertex v.

Lemma 15 *Let $s \gg 1$ and $t = sN$. Let $\mathbf{C}$ be a random configuration of t pebbles on the vertices of P_n^d, and let $p = (1+\epsilon)s\ln N$ for some $\epsilon > 0$. Then*

$$\Pr[\mathbf{C}(v) < p \text{ for all } v] \rightarrow 1 \text{ as } n \rightarrow \infty \ .$$

Proof. Let q be the probability that the vertex v satisfies $\mathbf{C}(v) \geq p$. Then q is at most

$$\begin{aligned} \frac{\left\langle {N \atop t-p} \right\rangle}{\left\langle {N \atop t} \right\rangle} &= \frac{t^{\underline{p}}}{(N+t-1)^{\underline{p}}} \\ &< \left(\frac{t}{N+t-1}\right)^p \\ &= \left(1 - \frac{1-1/N}{s+1-1/N}\right)^p \\ &\leq e^{-p(1-1/N)/(s+1-1/N)} \ . \end{aligned}$$

Hence, the probability that some vertex v satisfies $\mathbf{C}(v) \geq p$ is at most

$$Ne^{-p(1-1/N)/(s+1-1/N)} = e^{\ln N(1-\epsilon s+[(1+\epsilon)s-1]/N)/(s+1-1/N)} \rightarrow 0$$

as $n \to \infty$. Therefore, the probability that every vertex v satisfies $\mathbf{C}(v) < p$ tends to 1 as $n \to \infty$. $\square$

Next we show that a configuration with relatively few pebbles almost always has some large hole with no pebbles in it. For any set S of vertices, denote by $\mathbf{C}(S)$ the number of pebbles on its vertices.

Lemma 16 *Let $N = n^d$, $0 < c < 2^{-d/(d+1)}$, $u = c(\lg N)^{1/(d+1)}$, $s = 2^u$ and $t = \lfloor sN \rfloor$. Write $c = ((1-\epsilon)/(2+\delta)^d)^{1/(d+1)}$ for some $\epsilon, \delta > 0$, and set $m = \lfloor (2+\delta)u \rfloor$, $M = m^d$ and $k = \lfloor n/m \rfloor^d$. Let $B_1, \ldots, B_k$ be a collection of k pairwise disjoint blocks of vertices of P_n^d, each having every side of length m. Let $\mathbf{C}$ be a random configuration of t pebbles on the vertices of P_n^d. Then*

$$\Pr[\mathbf{C}(B_h) = 0 \text{ for some } h] \to 1 \text{ as } n \to \infty \ .$$

Proof. The second moment method applies. Let X_h be the indicator variable for the event that the block B_h contains no pebbles, and let $X = \sum_{h=1}^k X_h$. Then Chebyschev's inequality yields

$$\Pr[X = 0] \ \leq \ \frac{\mathrm{var}[X]}{\mathbf{E}[X]^2} \ ,$$

and

$$\begin{aligned} \mathrm{var}[X] &= \mathbf{E}[X^2] - \mathbf{E}[X]^2 \\ &= \sum_{h,j} \mathbf{E}[X_h X_j] - \sum_{h,j} \mathbf{E}[X_h]\mathbf{E}[X_j] \\ &\leq \sum_h \mathbf{E}[X_h^2] \ , \end{aligned}$$

since $\mathbf{E}[X_h X_j] \leq \mathbf{E}[X_h]\mathbf{E}[X_j]$ for $h \neq j$. Hence,

$$\mathrm{var}[X] \ \leq \ \sum_h \mathbf{E}[X_h^2] \ = \ \sum_h \mathbf{E}[X_h] \ = \ \mathbf{E}[X] \ .$$

Moreover, we have

$$\mathbf{E}[X] \ = \ k\left\langle {N-M \atop t} \right\rangle \Big/ \left\langle {N \atop t} \right\rangle$$

$$
\begin{aligned}
&= \left\lfloor \frac{n}{m} \right\rfloor^d \frac{(N-1)^{\underline{M}}}{(N+t-1)^{\underline{M}}} \\
&\geq \left(\frac{n}{m}-1\right)^d \left(\frac{N-M}{N+t-M}\right)^M \\
&\gtrsim \left(\frac{N}{M}\right)\left(\frac{N-M}{(s+1)N-M}\right)^M \\
&> \left(\frac{N}{M(s+1)^M}\right)\left(1-\frac{M}{N}\right)^M \\
&\sim \frac{N}{M(s+1)^M} \\
&\sim \frac{N}{m^d s^{m^d(1+o(1))}} \\
&= \frac{N}{m^d 2^{u^{d+1}(2+\delta)^d(1+o(1))}} \\
&= \frac{N}{m^d N^{(1-\epsilon)(1+o(1))}} \\
&\to \infty\ .
\end{aligned}
$$

Hence $\Pr[X=0] \leq \mathrm{var}[X]/\mathbf{E}[X]^2 \leq 1/\mathbf{E}[X] \to 0$ as $n \to \infty$. $\square$

The following lemma records the structure of the d-dimensional grid in order to keep track of the results of pebbling steps.

Lemma 17 *For any intervals $I_1, \ldots, I_d$ in $\mathbb{Z}$ such that each I_j contains r integers, let $\mathbf{B} = I_1 \times \cdots \times I_d \subseteq \mathbb{Z}^d$, and for $i > 0$, let $\mathcal{S}_i$ be the set of points in $\mathbb{Z}^d$ having distance i from $\mathbf{B}$, where distance between a pair of points in $\mathbb{Z}^d$ is defined by the sum of the absolute values of the differences of their coordinates. Then*

$$R_i := |\mathcal{S}_i| \leq \sum_{1 \leq j \leq d} \binom{d}{j} 2^j r^{d-j} \binom{i-1}{j-1}.$$

Proof. We partition $\mathbb{Z}^d$ according to the number j of coordinates in which a given point differs from its nearest neighbor in $\mathbf{B}$. Given a fixed j, there are $\binom{d}{j}$ ways to pick which j coordinates to change, each of the changed coordinates can be to either side of $\mathbf{B}$, giving 2^j possibilities, and there are r ways to pick each unchanged coordinate, giving r^{d-j} possibilities. Given this information, we can specify an element of $\mathcal{S}_i$ by specifying a j-tuple of positive integers with sum i, which can be done in $\left\langle {j \atop i-j} \right\rangle = \binom{i-1}{j-1}$ ways. □

Finally, our proof of Theorem 9 in the case of the lower bound will use this technical lemma to bound the number of pebbles that can reach the empty hole.

Lemma 18 $\sum_{i=1}^{nd} \binom{i-1}{j-1} 2^{-i} < 1$.

Proof. It is straightforward to use generating functions or induction to prove $\sum_{i \geq 1} \binom{i-1}{j-1} 2^{-i} = 1$. □

Turning to the case of the upper bound, we show that almost every configuration with relatively many pebbles fills every reasonably large block with plenty of pebbles.

Lemma 19 *Let $N = n^d$, $c' = d + 1 + \epsilon$ for some $\epsilon > 0$, $u' = c'(\lg N)^{1/(d+1)}$, $s' = 2^{u'}$, $t' = \lceil s'N \rceil$, $m' = \lceil (\frac{d+1}{c'})^{1/d} (\lg N)^{1/(d+1)} \rceil$, $M' = (m')^d$, and $k' = \lceil n/m' \rceil^d$. Let $B'_1, \ldots, B'_{k'}$ be a collection of k' blocks, each having every side of length m', that cover the vertices of P_n^d. Let $\mathbf{C}$ be a random configuration of t' pebbles on the vertices of P_n^d. Then*

$$\Pr[\mathbf{C}(B'_f) \geq M' 2^{dm'} \text{ for all } f] \to 1 \text{ as } n \to \infty \ .$$

Proof. Define Z_f to be the event that block B'_f contains fewer than $M^* = M' 2^{dm'}$ pebbles and approximate the probability $\Pr[\cup_{f=1}^{k'} Z_f]$ by

$$\Pr[\cup_{f=1}^{k'} Z_f] \ \leq \ k' \sum_{f=0}^{M^*-1} \left\langle {M' \atop f} \right\rangle \left\langle {N - M' \atop t' - f} \right\rangle \bigg/ \left\langle {N \atop t'} \right\rangle \ .$$

Now use the estimate

$$\left\langle {N - M' \atop t' - f} \right\rangle \ \leq \ \left(\frac{N}{N + t'} \right)^{M'} \left\langle {N \atop t'} \right\rangle$$

to obtain

$$\Pr[\cup Z_f] \;\le\; k'\left(\frac{N}{N+t'}\right)^{M'} \sum_{f=0}^{M^*-1} \left\langle {M' \atop f} \right\rangle .$$

Then use the upper bound

$$\sum_{f=0}^{M^*-1} \left\langle {M' \atop f} \right\rangle \;=\; \sum_{f=0}^{M^*-1} \left\langle {f+1 \atop M'-1} \right\rangle \;=\; \sum_{j=1}^{M^*} \left\langle {j \atop M'-1} \right\rangle \;=\; \left\langle {M^* \atop M'} \right\rangle \;\le\; M^{*M'}$$

to obtain

$$\begin{aligned}
\Pr[\cup Z_f] &\le k'\left(\frac{N}{N+t'}\right)^{M'} M^{*M'} \\
&\lesssim \frac{N}{M'}\left(\frac{M'2^{dm'}}{s'}\right)^{M'} \\
&= \frac{1}{M'}2^{\lg N - M'(u' - \lg M' - dm')} \\
&= \frac{1}{M'}2^{\lg N - (1+d)\lg N + o(\lg N) + d(\frac{1+d}{c'})^{\frac{d+1}{d}}\lg N} \\
&= \frac{1}{M' N^{d - d(\frac{1+d}{c'})^{\frac{d+1}{d}} - o(1)}} \\
&\to 0 .
\end{aligned}$$

Thus, almost surely, every f satisfies $\mathbf{C}(B'_f) \ge M'2^{dm'}$. $\square$

Proof of Theorem 9. We begin with the lower bound. Given $N = n^d$ and $0 < c < 2^{-d/(d+1)}$, we write $c = ((1-\epsilon)/(2+\delta)^d)^{1/(d+1)}$ for some $\epsilon, \delta > 0$, and set $u = c(\lg N)^{1/(d+1)}$, $s = 2^u$, $t = \lfloor sN \rfloor$, $m = \lfloor (2+\delta)u \rfloor$, $M = m^d$ and $k = \lfloor n/m \rfloor^d$. Let $B_1, \ldots, B_k$ be a collection of k pairwise disjoint blocks of vertices of P_n^d, each having every side of length m. Let $\mathbf{C}$ be a random configuration of t pebbles on the vertices of P_n^d. By Lemma 16 we know that, almost surely, some block B_h has no pebbles on its vertices. By Lemma 15 we know that, almost surely, no vertex has more that p pebbles on it, where $p = (1+\epsilon)s \ln N$ for some $\epsilon > 0$.

Let $\overline{B}_h$ be the boundary of B_h. Any vertex v with $\mathbf{C}(v)$ pebbles on it can contribute at most $\mathbf{C}(v)/2^i$ pebbles to $\overline{B}_h$, where i is the distance from v to $\overline{B}_h$. Also, the number of vertices of $P_n^d - B_h$ at distance i from $\overline{B}_h$ is at most R_i. Thus, according to Lemmas 17 and 18, the number of pebbles that can be amassed on $\overline{B}_h$ via pebbling steps almost surely is less than or equal to

$$\begin{aligned}
\sum_{i=1}^{nd} pR_i/2^i &\leq \sum_{i=1}^{nd} p \sum_{j=1}^{d} \binom{d}{j} 2^j m^{d-j} \binom{i-1}{j-1} 2^{-i} \\
&\leq p \sum_{j=1}^{d} \binom{d}{j} 2^j m^{d-j} \sum_{i=1}^{nd} \binom{i-1}{j-1} 2^{-i} \\
&< p \sum_{j=1}^{d} \binom{d}{j} 2^j m^{d-j} \\
&< p(m+2)^d \\
&\ll 2^{m/2} .
\end{aligned}$$

The last line holds because the dominant term in $p(m+2)^d$ is 2^u, and we have $m = \lfloor (2+\delta)u \rfloor$. Therefore, almost surely, too few vertices are amassed on $\overline{B}_h$ to be able to move a single pebble to the center of B_h. This shows that $\tau(\mathcal{P}^d) \in \Omega(sN)$, as required.

Next we prove the upper bound. Given $N = n^d$ and $c' = d + 1 + \epsilon$ for some $\epsilon > 0$, set $u' = c'(\lg N)^{1/(d+1)}$, $s' = 2^{u'}$, $t' = \lceil s'N \rceil$, $m' = \lceil (\frac{d+1}{c'})^{1/d} (\lg N)^{1/(d+1)} \rceil$, $M' = (m')^d$ and $k' = \lceil n/m' \rceil^d$. Let $B'_1, \ldots, B'_{k'}$ be a collection of k' blocks, each having every side of length m', that cover the vertices of P_n^d. Let $\mathbf{C}$ be a random configuration of t' pebbles on the vertices of P_n^d. Then Lemma 19 states that, almost surely, every block B'_f has at least $M' 2^{dm'}$ pebbles. Since (see [6]) every graph G is solvable by $n(G)2^{\mathrm{diam}(G)}$ pebbles, any given vertex v in P_n^d almost surely is solvable by the pebbles in the block B'_f which contains v. This shows that $\tau(\mathcal{P}^d) \in O(s'N)$, as required.

□

3 Remarks

Let $l = l(n)$ and $d = d(n)$ and denote by $\mathcal{P}_l^d$ the sequence of graphs $(P_{l(n)}^{d(n)})_{n=1}^{\infty}$, where $P_l^d = (P_l)^d$. For $l(n) = 2$, $\mathcal{P}_l^n = \mathcal{Q}$, which can be shown to have a threshold asymptotically less than N.

We conjecture that the same result holds for all fixed l.

Conjecture 20 *Let $\mathcal{P}_l$ denote the graph sequence $(P_l^n)_{n=1}^{\infty}$. Then for fixed l we have $\tau(\mathcal{P}_l) \in o(N)$.*

In contrast, we have proved that $\tau(\mathcal{P}^d) \in \omega(N)$ for fixed d. Thus we believe there should be some relationship between two functions $l = l(n)$ and $d = d(n)$, both of which tend to infinity, for which the sequence $\mathcal{P}_l^d$ has threshold on the order of N.

Problem 21 *Denote by $\mathcal{P}^d$ the graph sequence $(P_n^{d(n)})_{n=1}^{\infty}$. Find a function $d = d(n) \rightarrow \infty$ for which $\tau(\mathcal{P}^d) = \Theta(N)$. In particular, how does d compare to n?*

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