THE PEBBLING THRESHOLD OF THE SQUARE OF CLIQUES

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Abstract. Given an initial configuration of pebbles on a graph, one can move pebbles in pairs along edges, at the cost of one of the pebbles moved, with the objective of reaching a specified target vertex. The pebbling number of a graph is the minimum number of pebbles so that every configuration of that many pebbles can reach any chosen target. The pebbling threshold of a sequence of graphs is roughly the number of pebbles so that almost every (resp. almost no) configuration of asymptotically more (resp. fewer) pebbles can reach any chosen target. In this paper we find the pebbling threshold of the sequence of squares of cliques, improving upon an earlier result of Boyle and verifying an important instance of a probabilistic version of Graham's product conjecture.

1. Pebbling Number

Consider a connected graph G on n vertices. Suppose that a *configuration* C of t pebbles is placed onto the vertices of graph G. A *pebbling step* from u to v consists of removing two pebbles from vertex u and then placing one pebble on an adjacent vertex v. We say that a pebble can be *moved* to a vertex r (called *root* vertex) if after

Date: December 30, 2003.

1991 *Mathematics Subject Classification.* Primary 05C35, 00A43; Secondary 05C80, 60C05.

Key words and phrases. pebbling, threshold, product of graphs, random configuration.

* Partially supported by National Security Agency grant #MDA9040210095.

finitely many steps r has at least one pebble. If it is possible to move a pebble to the root vertex r then we say that C is *r-solvable*; otherwise, C is *r-unsolvable*. Finally, we call C *solvable* if it is r-solvable for all r, and *unsolvable* otherwise. Define the *pebbling number* $\pi(G)$ to be the smallest integer t such that every configuration of t pebbles on the vertices of G is solvable. A fair amount is known about the pebbling numbers of typical graphs like complete graphs, paths, cycles, cubes, etc. (see [18] for a survey), relations to known parameters such as connectivity [13], diameter [7], girth [12], and domination number [8], and interesting variations such as optimal pebbling [22], and cover pebbling [10] are being investigated.

2. Random Configurations

In this paper we consider a random pebbling model in which a particular configuration of pebbles is selected uniformly at random from the set of all configurations with a fixed number of pebbles. One can think of the configuration of pebbles as a placement of unlabeled balls in labeled distinct urns. This is analogous to the so-called *static* model of random graphs, whose sample space consists of all graphs with a fixed number of edges. Since vertices may have more than one pebble, a particular configuration is a multiset of t elements with the ground set $[n]$. We construct the probability space $\mathbf{C}_{n,t}$ by choosing configurations randomly and assuming that they all are equally likely to occur.

The size of the set $\mathbf{C}_{n,t}$ is the number of possible arrangements of t identical balls placed in n distinct urns, so $|\mathbf{C}_{n,t}| = \binom{n+t-1}{t}$, which we denote by $\left\langle {n \atop t} \right\rangle$ (the reader may find it useful to use the terminology "n *pebble* t"). We will be interested in the probability spaces associated with sequences of graphs $\mathcal{G} = (G_1, G_2, \ldots, G_n, \ldots)$. In this notation the index n represents the position of the graph G_n. In some of the graph sequences, such as cubes, for example, the size of the vertex set of G_n is not the same as the position. Therefore, we define $N = N_n = N(G_n) = |V(G)|$ to be the number of vertices of G_n. Graphs in $\mathcal{G}$ are in ascending order with respect to this number, i.e. $N_n > N_m$ for $n > m$.

We will study the pebbling threshold phenomenon that occurs in this model, as it does for many random graph properties. For two functions $f(n)$ and $g(n)$ we write $f \ll g$, (equivalently $g \gg f$) if $\lim_{n\to\infty} f(n)/g(n) = 0$. We set $o(g) = \{f \mid f \ll g\}$ and $\omega(f) = \{g \mid f \ll g\}$. Also, we write $f \in O(g)$, or equivalently $g \in \Omega(f)$, when there are positive constants c and k such that $f(n)/g(n) < c$, for all $n > k$. In particular, if $f(n)/g(n) \to 1$ as $n \to \infty$, we write $f \sim g$. Furthermore, we define $\Theta(g)$ $= O(g) \cap \Omega(g)$. Finally, for two sets of functions F and G we write $F \lesssim G$ if $f \in O(g)$ for all $f \in F, g \in G$.

A function $f = f(n)$ is called a *threshold* for the graph sequence $\mathcal{G}$, and we write $f \in \tau_{\mathcal{G}}$, if $P_{\mathcal{G}}(n,t) \to 1$ whenever $t \gg f$, and $P_{\mathcal{G}}(n,t) \to 0$, whenever $t \ll f$. In other words, if $f = f(n) \in \tau_{\mathcal{G}}$, then for any function $\varpi = \varpi(n)$ tending to infinity with n, $P_{\mathcal{G}}(n, \varpi f) \to 1$ and $P_{\mathcal{G}}(n, f/\varpi) \to 0$ as $n \to \infty$.

Roughly speaking, the pebbling number describes the "worst-case" scenario, as it is one more than the size of the largest unsolvable configuration. The threshold function, on the other hand, deals with "typical" configurations and estimates the average chance of being solvable. For example, the threshold of family of cliques $\mathcal{K}$ is $\tau_{\mathcal{K}} = \Theta(\sqrt{N})$. This problem is similar to the well-known "birthday" problem — how many people must be in a room so that with high probability two people share the same birth date? — but here the pebbles are unlabelled. The general existence of the pebbling threshold is established in [4], and in [11] it is shown that every graph sequence $\mathcal{G}$ satisfies $\tau_{\mathcal{G}} \subset \Omega(f) \cap O(g)$, where $f \in \tau_{\mathcal{K}}$ and $g \in \tau_{\mathcal{P}}$, for the sequence of paths $\mathcal{P}$. We are going to compute the pebbling threshold of the sequence of squares of cliques, thereby verifying an instance of the threshold analogue of Graham's product conjecture.

3. Cartesian Products and Graham's Conjecture

Chung's paper [9] raised a natural question about the relationship between the pebbling numbers of individual graphs and the pebbling number of their cartesian product.

Definition 1. The *Cartesian product* of two graphs G_1 and G_2, denoted $G_1 \square G_2$ is the graph with vertex set

$$V(G_1 \square G_2) = \{(v_1, v_2) \mid v_1 \in V(G_1), v_2 \in V(G_2)\}$$

and edge set

$$\begin{aligned} E(G_1 \square G_2) \;\; = \;\; \{ \; ((v_1, v_2), (w_1, w_2)) \mid v_1 = w_1 \text{ and } (v_2, w_2) \in E(G_2) \text{ or} \\ v_2 = w_2 \text{ and } (v_1, w_1) \in E(G_1) \; \} \; . \end{aligned}$$

The general conjecture about the pebbling number of the cartesian product of graphs was originally stated by Graham ([9]).

Conjecture 2. *For all graphs G_1 and G_2 we have that*

$$\pi(G_1 \square G_2) \leqslant \pi(G_1)\pi(G_2) \ .$$

There are several results supporting this conjecture. It is known [9] that the m-dimensional cube and that the product of cliques satisfy this conjecture. Also, Moews [21] proved it holds for the product of trees. Pachter et al. [24] proved the conjecture for the product of cycles with some exceptions: it holds for $C_m \square C_n$ where m and n are not both from the set $\{5, 7, 9, 11, 13\}$. Herscovici and Higgins in [17] proved it for $C_5 \square C_5$. Recently, Herscovici [16] found a proof for all these exceptions confirming Graham's conjecture for the product of cycles. Finally, the conjecture holds for dense graphs [12].

4. Threshold Version and Main Theorem

For the graph sequences $\mathcal{G} = (G_1, \ldots G_n, \ldots)$ and $\mathcal{H} = (H_1, \ldots, H_n, \ldots)$ let us define the sequence $\mathcal{G} \square \mathcal{H} = (G_1 \square H_1, \ldots, G_n \square H_n, \ldots)$. The sequence $\mathcal{G} \square \mathcal{H}$ is called the *cartesian product* of $\mathcal{G}$ and $\mathcal{H}$. The number of vertices of the n^{th} element of $\mathcal{G} \square \mathcal{H}$ is $N(G_n \square H_n) = N(G_n)N(H_n)$. Here we are interested in the following probabilistic version of Conjecture 2.

Conjecture 3. *Let $\mathcal{F}$ and $\mathcal{G}$ be two graph sequences with numbers of vertices $R = N(F_n)$ and $S = N(G_n)$, respectively, and with pebbling thresholds $\tau_{\mathcal{F}}$ and $\tau_{\mathcal{G}}$, respectively. Let $f \in \tau_{\mathcal{F}}$, $g \in \tau_{\mathcal{G}}$, and $h \in \tau_{\mathcal{H}}$, where $\mathcal{H} = \mathcal{F} \square \mathcal{G}$, having $T = N(H_n) = RS$ vertices. Then*

$$h(T) \in O(f(R)g(S)) \ .$$

This conjecture is shown to hold for d-dimensional grids (products of paths) in [12]. We are going to verify Conjecture 3 for the cartesian product of cliques $\mathcal{K}^2 = \mathcal{K}\Box\mathcal{K} = (K_1\Box K_1, \ldots, K_n\Box K_n, \ldots)$. If true, the pebbling threshold for the product of cliques should be

$$\tau_{\mathcal{K}^2} \subseteq \Theta(\sqrt{N^{1/2}}\sqrt{N^{1/2}}) = \Theta(\sqrt{N}) \ ,$$

where N is the number of vertices of $\mathcal{K}^2$, namely $N = n^2$. This would improve Boyle's [6] result that $\tau_{\mathcal{K}^2} \subseteq O(N^{3/4})$ and give the exact result (recall the lower bound for all sequences mentioned above). Our main result is the following theorem.

Theorem 4. *Let $\mathcal{K}^2$ be the sequence of the cartesian products of cliques, with $N = N(K_n^2)$. Then the pebbling threshold of $\mathcal{K}^2$ is*

$$\tau_{\mathcal{K}^2} = \Theta(\sqrt{N}) \ .$$

This theorem is perhaps surprising, considering that the graph K_n^2 is fairly sparse. It seems that the structure of the graph is what keeps its threshold small.

5. Cops and Robbers

Let us consider a particular configuration of pebbles on the cartesian product of cliques $K_n\Box K_n$. Note that this graph can be thought of as a rectangular grid with each row and column a complete graph. Therefore, to pebble to a specific root r one needs to collect two or more pebbles on any vertex that belongs to the *row* $r\Box K_n$ or to the *column* $K_n\Box r$ (see Figure 1). This suggests the following interpretation of the pebbling problem. We partition the vertices of K_n^2 into three distinct sets: *police*, or

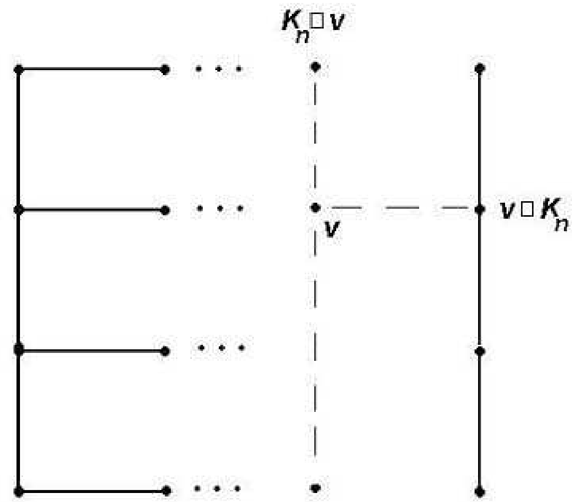

FIGURE 1. Schematic Presentation of $K_n\square K_n$.

cops (**P**), *citizens* (**T**) and *robbers* (**R**). Vertices in the set **P** are those with two or more pebbles on them, **T** is the set of vertices with one pebble, and **R** is the remaining set of empty vertices. (This approach is motivated by a variety "Cops and Robbers" games, one of the more prevalent types of games on graphs. More information on these types of games can be found, for example, in [1, 2, 23, 26].)

In our case, the robber is immobile and cops can move only in certain directions and their number may change during the game. If root r is chosen in **R** then for a pebbling configuration to be solvable it is sufficient that there is at least one cop on $r\square K_n$ or $K_n\square r$. Any citizen can become a cop if it is possible to move at least one pebble to it from some other cop. We say that vertex u *sees* v if u and v are in the same row or column of K_n^2. Furthermore, we say that a robber $r = v_0$ can be *caught* if there is a sequence of citizens $v_1, \ldots, v_{k-1}$ and a cop $c = v_k$ so that v_i sees v_{i+1} for $0 \leqslant i < k$.

Then the pebbling configuration is r-solvable if a vertex r can be caught. For example, on Figure 2 it is possible to pebble from the vertex c (cop) to the vertex r (robber).

Any pebbling configuration C determines the *citizen subgraph* G_C of K_n^2 induced by the vertex set $\mathbf{P} \cup \mathbf{T}$. The edge set of G_C is determined by the vertices that see each other. Any component of G_C containing two or more cops we call a *police component.*

Claim 5. *Any configuration whose citizen subgraph has a police component is solvable.*

Proof. Let us consider a police component with vertices $v_1, \ldots, v_k$ such that $v_1, v_k \in \mathbf{P}$, $v_1, \ldots, v_{k-1} \in \mathbf{T}$ and v_{i-1} sees v_i for $1 \leqslant i < k$. We now use the following strategy. Without loss of generality, we assume that v_1 and v_2 are in the same row $v_1 \Box K_n$. Then we find vertex v_1' which is the intersection of $K_n \Box r$ and $v_1 \Box K_n$ and make v_1' a citizen by moving a pebble from v_1. Now $r = v_0$ can be caught by $v_1', v_2, \ldots, v_k$. ∎

Another sufficient condition for a pebbling configuration to be solvable is the existence of a "robocop", a vertex with 4 or more pebbles on it. In that case any robber r can be caught by sending two pebbles to either $K_n \Box r$ or $r \Box K_n$, making a cop there and moving a pebble to r from this new cop. In Section 8 we prove that the probability that such a "robocop" exists tends to zero. Hence, our goal is to prove that almost every configuration of asymptotically more than n pebbles on K_n^2 has a police component.

The next argument transforms the original problem of the solvability of a pebbling configuration on K^2 to connectedness properties of a related bipartite multigraph $B'_{n,n}$. First, we observe that K_n^2 is isomorphic to the line graph of the complete bipartite graph $K_{n,n}$. Indeed, both vertex sets are isomorphic to $\{1, \ldots, n\}^2$, and both edge sets are

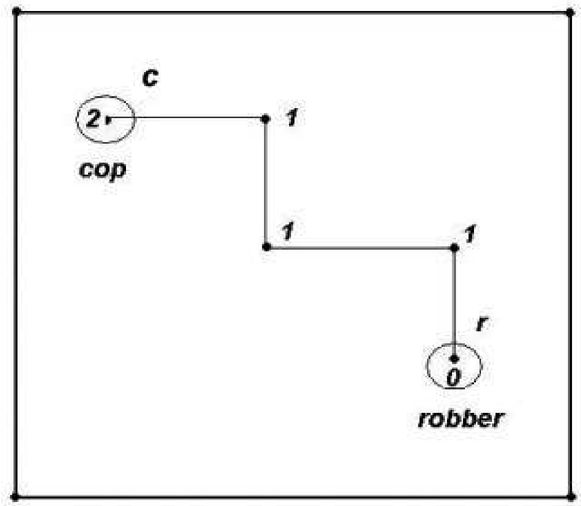

FIGURE 2. Pebbling from "Cop" to "Robber".

isomorphic to pairs from $\{1,\ldots,n\}^2$ that share a coordinate. Similarly we construct a bipartite graph $B_{n,n}$ whose line graph is isomorphic to G_C. The bipartite multigraph $B'_{n,n}$ is constructed from $B_{n,n}$ by adding multiple edges according to the multiplicity of pebbles on the vertices of G_C. In other words, for every vertex $(u,v)\in G_C$ we place the edge $uv\in B'_{n,n}$ with multiplicity $C((u,v))$.

6. Model Descriptions

In this section we describe three different models for random bipartite graphs and multigraphs . We compare them and determine asymptotic implications from one to another which we can apply then to the pebbling threshold on the product of graphs. In particular, we will be interested in the property of having a large component (which will be shown to be a police component almost surely).

The first model (Model A) is an analogue of the probabilistic model for random graphs. In model A edges between any two vertices in different parts of $B_{n,n}$ are mutually independent and have the same probability p. Computations are easiest in this model, in which all graphs are simple. The probability space corresponding to model A we denote by $\mathcal{B}_{n,p}$.

The second model (Model B) is an analogue of the static model for random graphs. First, the set of all bipartite simple graphs on n by n vertices is denoted $\mathcal{B}(n)$. The set of graphs in $\mathcal{B}(n)$ with M edges we denote by $\mathcal{B}(n, M)$. This model consists of $|\mathcal{B}(n, M)| = \binom{N}{M}$ different graphs, where $N = n^2$. Clearly, $\mathcal{B}(n) = \cup_{M=0}^{N} \mathcal{B}(n, M)$.

Finally, we need a generalization of the second model for the case of bipartite multigraphs (Model B$'$). As it was defined in the previous section, the edges of the line graph represent pebbles; therefore we need a multiple edge model to reflect this situation. We denote by $\mathcal{B}'(n, m)$ the set of all bipartite multigraphs on n by n vertices with m edges. Model B$'$ consists of precisely $|\mathcal{B}'(n, m)| = \left\langle {N \atop m} \right\rangle$ different graphs, where $N = n^2$. Finally, we define $\mathcal{B}'(n) = \cup_{m=0}^{\infty} \mathcal{B}'(n, m)$.

The multiple edge model for random graphs was considered in [3]. It was shown that the differences between simple graphs and multigraphs are negligible in most cases. Janson et al. [19] give a detailed analysis of the multigraph model using an algebraic approach. We are going to show that, for the right translation of parameters, certain properties that hold in model A will transfer to hold in B, and then to B$'$ as well.

7. Connections Between Models

Models A and B are very closely related to each other, provided that M is about pN, which is the expected number of edges of a graph in $\mathcal{B}_{n,p}$. In fact, these two models are asymptotically equivalent to each other for any convex property. Call a family of multisets $\mathcal{M}$ *increasing* if $A \subseteq B$ and $A \in \mathcal{M}$ implies that $B \in \mathcal{M}$, *decreasing* if $A \subseteq B$ and $B \in \mathcal{M}$ implies that $A \in \mathcal{M}$. A family which is either increasing or decreasing is called *monotone*. Finally, a family $\mathcal{M}$ is *convex* if $A \subseteq B \subseteq C$ and $A, C \in \mathcal{M}$ imply that $B \in \mathcal{M}$. Also, given a property S we shall say that *almost every* (*a.e.*) graph in the probability space $\mathcal{M}$ has property S if $\Pr[G \in \mathcal{M} : G \text{ has } \mathbf{S}] \to 1$, as $n \to \infty$.

The equivalence of models A and B follows from the general equivalence of the probabilistic and static models in random graphs, which was proven by Bollobás (see [5, 25]). Here we state the result for random bipartite graphs.

Result 6. *Let $N = n^2$ and let $0 < p = p(n) < 1$ be such that $pN \to \infty$ and $(1-p)N \to \infty$ as $n \to \infty$, and let $\mathbf{S}$ be a property of graphs.*

(1) *Suppose that $\varepsilon > 0$ is fixed and that a.e. graph in $\mathcal{B}(n, M)$ has $\mathbf{S}$ whenever*

$$(1-\varepsilon)pN < m < (1+\varepsilon)pN \ .$$

Then a.e. graph in $\mathcal{B}_{n,p}$ has $\mathbf{S}$.

(2) *If $\mathbf{S}$ is a convex property and a.e. graph in $\mathcal{B}_{n,p}$ has $\mathbf{S}$, then a.e. graph in $\mathcal{B}(n, M)$ has $\mathbf{S}$ for $M = \lfloor pN \rfloor$.*

Bollobás' technique is on the boolean lattice applied to graphs, so we can apply it to bipartite graphs equally well since we are still considering the boolean algebra in models A and B. Next we establish a relationship between models B and B′.

The *support* of multigraph $G \in \mathcal{B}'(n)$ is the simple graph obtained by identifying the parallel edges of G. We denote the support by Λ_G. Obviously, $\Lambda_G \in \mathcal{B}(n)$. We call the number of edges in the support the *size* of the support of G, written $Z = Z_G = ||\Lambda_G||$. (Here we use the notation $||\cdot||$ because we are counting edges rather than vertices.) The set of all graphs $G \in \mathcal{B}'(n,m)$ with the same support size $Z_G = s$ we denote $\Lambda(n,m,k)$.

An equivalent setting for the last definition is to consider m unlabeled balls placed in N distinct urns. Then for $N = n^2$ the set Λ_G represent the set of non-empty urns and $\Lambda(n,m,s)$ is the set of distributions into exactly s of N urns. We need to find the average size of the support in this model. The probability that G has support of size s, for $0 \leqslant s \leqslant m$, is

$$\begin{aligned}
\Pr[Z_G = s] &= \frac{\binom{N}{s}\left\langle {s \atop m-s} \right\rangle}{\left\langle {N \atop m} \right\rangle} \\
&= \frac{\binom{N}{s}\binom{m-1}{m-s}}{\binom{N+m-1}{m}} \\
&= \frac{\binom{N}{s}\binom{(N+m-1)-N}{m-s}}{\binom{N+m-1}{m}} .
\end{aligned}$$

The last expression means that the random variable $Z = Z_G$ follows the hypergeometric distribution $\mathbf{H}$ with parameters $\mathbf{H}(N+m-1, N, m)$. The hypergeometric distribution

$\mathbf{H}(L, k, l)$ describes the number of white balls in the sample of size l chosen randomly (without replacement) from an urn containing L balls, of which k are white and $L-k$ are black. Direct computations give us the expected value and the variance of Z_G. Indeed, the general formula for $\mathbf{E}[Z^k]$ is

$$
\begin{aligned}
\mathbf{E}[Z^k] &= \sum_{s=0}^{N} s^k \Pr[Z=s] \\
&= \sum_{s=0}^{N} s^k \frac{\binom{N}{s}\binom{m-1}{m-s}}{\binom{N+m-1}{m}} \\
&= \frac{Nm}{N+m-1} \sum_{s=0}^{N} s^{k-1} \frac{\binom{N-1}{s-1}\binom{m-1}{m-s}}{\binom{(N+m-1)-1}{m-1}} \qquad (1) \\
&= \frac{Nm}{N+m-1} \sum_{r=0}^{N-1} (r+1)^{k-1} \frac{\binom{N-1}{r}\binom{m-1}{(m-1)-r}}{\binom{N+m-2}{m-1}} \\
&= \frac{Nm}{N+m-1} E((Y+1)^{k-1}) \ ,
\end{aligned}
$$

where Y is a hypergeometric random variable with parameters $\mathbf{H}(N+m-2, N-1, m-1)$.

Setting $k=1$ in the last line of equation (1), we obtain

$$
\mathbf{E}[Z_G] = \frac{Nm}{N+m-1} = mq \ , \qquad (2)
$$

with $q = \dfrac{N}{N+m-1}$. The intuitive idea is that, for a large value of N, the average support size is close to m. If the number of edges $m = m(n) \in o(N)$ then the value of q is close to one. According to the second moment method, if the variance of random variable Z_G is relatively small then the value of Z_G almost always stays close to the mean. Indeed, in equation (1) if $k=2$ then

$$\mathbf{E}[Z^2] = \frac{Nm}{N+m-1}\mathbf{E}[Y+1] = \frac{Nm}{N+m-1}\left(\frac{(N-1)(m-1)}{N+m-2}+1\right).$$

Therefore, the variance is

$$\begin{aligned}
\mathrm{Var}[Z_G] &= \mathbf{E}[Z^2]-(\mathbf{E}[Z])^2 \\
&= \frac{Nm}{N+m-1}\left(\frac{(N-1)(m-1)}{N+m-2}+1-\frac{Nm}{N+m-1}\right) \\
&= mq\Big[(N-1)(m-1)(N+m-1)+(N+m-2)(N+m-1)- \\
&\qquad -Nm(N+m-2)\Big] \Big/ (N+m-2)(N+m-1) \qquad (3) \\
&= \frac{mq}{N+m-2}\left(\frac{Nm-N-m+1}{N+m-1}\right) \\
&= \frac{mq}{N+m-2}(N-1)(1-q)\ .
\end{aligned}$$

For $m(n)\in o(N)$ we have from Equations (2) and (3) that

$$\frac{\mathrm{Var}[Z]}{(\mathbf{E}[Z])^2} = \frac{N-1}{N+m-2}\left(\frac{1-q}{mq}\right)\to 0\ ,$$

as $n\to\infty$. Hence, $\mathrm{Var}[Z_G]\in o((\mathbf{E}[Z_G])^2)$.

Janson et al. [20] suggested the following notation to measure more precisely the closeness of a random variable to its mean.

Notation 7. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables and $\{a_n\}_{n=1}^{\infty}$ a sequence of positive real numbers. We write*

$$X_n = o_p(a_n)$$

if, for every $\varepsilon > 0$, almost always $|X_n| < \varepsilon a_n$ (i.e. $\Pr[|X_n| < \varepsilon a_n] \to 1$ as $n \to \infty$).

This definition is analogous to $o(\cdot)$, but with probability involved.

Lemma 8. *Let $q = \dfrac{N}{N+m-1}$ and $m \in m(n) \subseteq o(N)$. Then*

$$Z_G = mq + o_p(mq) \ ,$$

for a.e. graph G in $\mathcal{B}'(n,m)$.

Proof. We are going to use the second moment method with the random variable $Z = Z_G$. We have $\mathbf{E}[Z] = mq$ by Equation 2 and, using Equation 3 and Chebyshev's inequality, we obtain

$$\Pr[|Z - mq| > \lambda] \leqslant \frac{\sigma^2}{\lambda^2} = \frac{mq}{\lambda^2}\frac{N-1}{N+m-2}(1-q) \leqslant \frac{mq}{\lambda^2} \to 0$$

for $\lambda = \varepsilon mq$ since $mq \to \infty$. ∎

Now we are ready to establish the relationship between models B and B′. The next theorem provides a criterion for any increasing property that holds in $\mathcal{B}(n,M)$ to hold in $\mathcal{B}'(n,m)$ as well.

Theorem 9. *Let* $\mathbf{S}$ *be any increasing property of graphs and* $q = N/(N+m-1)$ *for some* $m \in o(N)$. *Also, let* $B_{\mathbf{S}}(n,M) \subseteq \mathcal{B}(n,M)$ *and* $B'_{\mathbf{S}}(n,m) \subseteq \mathcal{B}'(n,m)$ *denote those bipartite graphs and bipartite multigraphs, respectively, having property* $\mathbf{S}$. *If for every sequence* $M = M(n)$ *such that* $M = mq + o_p(mq)$ *we have* $\Pr[B_{\mathbf{S}}(n,M)] \to 1$, *as* $n \to \infty$, *then also* $\Pr[B'_{\mathbf{S}}(n,m)] \to 1$, *as* $n \to \infty$.

Proof. We are going to prove that $\Pr[B'_{\bar{\mathbf{S}}}(n,m)] \to 0$. Let us consider the set

$$\mathbf{M}(\varepsilon) = \{M \mid |M - mq| \leqslant \varepsilon mq\} \ ,$$

for some $\varepsilon > 0$. We assume in the hypothesis that for any $M \in \mathbf{M}(\varepsilon)$ we have

$$\Pr[B_{\mathbf{S}}(n,M)] \to 1 \ , \tag{4}$$

whenever $n \to \infty$. Then

$$\begin{aligned} \Pr[B'_{\bar{\mathbf{S}}}(n,m)] &= \sum_{M \notin \mathbf{M}(\varepsilon)} \Pr[B'_{\bar{\mathbf{S}}}(n,m) \mid Z_G = M] \Pr[Z_G = M] \\ &\quad + \sum_{M \in \mathbf{M}(\varepsilon)} \Pr[B'_{\bar{\mathbf{S}}}(n,m) \mid Z_G = M] \Pr[Z_G = M] \ . \end{aligned}$$

The first sum in the last expression can be bounded from above by

$$\sum_{M \notin \mathbf{M}(\varepsilon)} \Pr[Z_G = M] \ = \ \Pr[Z_G \notin M(\varepsilon)] \ ,$$

which tends to zero by Lemma 8. For every graph $B \in \mathcal{B}(n,M)$ there are $\left\langle {M \atop m-M} \right\rangle$ multigraphs $B' \in \mathcal{B}'(n,m)$ with $\Lambda_{B'} = B$. Moreover, $\mathbf{S}$ is increasing. Therefore we can give an upper bound for the second sum of

$$\begin{aligned}
\sum_{M \in \mathbf{M}(\varepsilon)} \Pr[B'_{\bar{\mathbf{S}}}(n,m) \mid Z_{B'} = M] &\leqslant \sum_{M \in \mathbf{M}(\varepsilon)} \frac{\left\langle {M \atop m-M} \right\rangle |B_{\bar{\mathbf{S}}}(n,M)|}{\left\langle {N \atop M} \right\rangle} \\
&= \sum_{M \in \mathbf{M}(\varepsilon)} \frac{\left\langle {M \atop m-M} \right\rangle \binom{N}{m}}{\left\langle {N \atop M} \right\rangle} \Pr[B_{\bar{\mathbf{S}}}(n,M)] \qquad (5) \\
&\leqslant \Pr[B_{\bar{\mathbf{S}}}(n,M^*)] \sum_{M \in \mathbf{M}(\varepsilon)} \frac{\left\langle {M \atop m-M} \right\rangle \binom{N}{m}}{\left\langle {N \atop M} \right\rangle} ,
\end{aligned}$$

where M^* is the element of $\mathbf{M}(\varepsilon)$ that maximizes $\Pr[B_{\bar{\mathbf{S}}}(n,M)]$. The sum in the last expression is a partial sum of probabilities of a hypergeometric random variable and, therefore, does not exceed 1. Hence, the last line in (5) is bounded from above by $\Pr[B_{\bar{\mathbf{S}}}(n,M^*)]$, which goes to zero, as $n \to \infty$, by assumption (4). Thus, $\Pr[B'_{\bar{\mathbf{S}}}(n,m)] \to 0$ as $n \to \infty$, and the statement of the theorem follows. ∎

The particular increasing property in which we are most interested is that of containing a large component, of size proportional to $2n$. We will show that such a connected component is almost surely a police component.

8. Large Components and Police Components

We first note that, almost surely, all cops have only two edges. Recall that $B' \in \mathcal{B}'(n,m)$ is chosen uniformly at random, where $m = \varpi n$ and $\varpi \to \infty$ arbitrarily slowly as $n \to \infty$. The probability that there exists a vertex with k pebbles on it is at most

$$\frac{n^2 \left\langle {n^2 \atop \varpi n - k} \right\rangle}{\left\langle {n^2 \atop \varpi n} \right\rangle} \sim n^2 \left(\frac{\varpi n}{n^2 + \varpi n} \right)^k \sim n^2 \left(\frac{\varpi}{n} \right)^k .$$

For $k > 2$ the last expression tends to zero as $n \to \infty$.

We use this fact to show that connected components of linear size have many cops.

Lemma 10. *Let H be a connected component of size $\alpha 2n$ in $B' \in \mathcal{B}'(n,m)$, where $m = \varpi n$. Then almost surely H is a police component.*

Proof. Let $x = x(n)$ be the excess of edges in B', namely $x(n) = ||B'|| - ||\Lambda_{B'}||$. Since almost surely all cops have exactly two edges, the number of cops $s = s(n)$ in B' is almost always equal to the excess $x(n)$. Using Lemma 8 (with $q = N/(N+m-1)$) we compute

$$x(n) \;\sim\; m - mq \;=\; \varpi n(1-q) \;\sim\; \varpi^2$$

almost surely. Given that there are ϖ^2 cops in B', an upper bound of the probability that H has at most one cop is

$$\frac{\binom{qn-\alpha 2n}{\varpi^2} + \alpha 2n\binom{qn-\alpha 2n}{\varpi^2 - 1}}{\binom{qn}{\varpi^2}} \;\lesssim\; \left(\frac{qn-\alpha 2n}{qn}\right)^{\varpi^2}\left(1 + \frac{2\alpha n\varpi^2}{qn - 2\alpha n - \varpi^2}\right)$$

$$\lesssim\; e^{-2\alpha\varpi^2/q}\left(\frac{2\alpha\varpi^2}{q - 2\alpha - \varpi^2/n}\right)\;.$$

We may assume that H is small, so if $4\alpha < q$ the last term is at most $\varpi^2 e^{-\varpi^2/2} \to 0$ as $n \to \infty$. ∎

Finally, we prove that there is a connected component of linear size in $\mathcal{B}_{n,p}$. The following theorem was proven in [15] for the random graph $\mathcal{G}_{n,p}$. The proof involved analyzing the hitting time of a certain parameter in a random walk and used no special property of the graph structure. Here we modify the result for the random bipartite graph $\mathcal{B}_{n,p}$. The same method yields the following result, which we state without proof.

Result 11. *Let $\beta > \ln 16$, $p = \beta/n$, and $B_n \in \mathcal{B}_{n,p}$. Then almost surely there is a path in B_n of length at least $(1 - (\ln 16)/\beta)2n$.*

9. Proof of Theorem 4

Now we prove that $\tau_{\mathcal{K}^2} = \Theta(\sqrt{N})$.

Proof. We recall that the pebbling threshold of every graph sequence is in $\Omega(\sqrt{N})$. Therefore we need only show that $\tau_{\mathcal{K}^2} = O(\sqrt{N})$. Write $N = n^2$, let $m = \varpi n$, where $\varpi = \varpi(n) \to \infty$ arbitrarily slowly, and let C be a randomly chosen configuration from $\mathbf{C}_{N,m}$. Let $B'_{n,n} \in \mathcal{B}'(n,m)$ be the bipartite multigraph associated with C, and $B_{n,n} \in \mathcal{B}(n,M)$ be the simple bipartite graph determined by the support of $B'_{n,n}$. Lemma 8 implies $M = mq + o_p(mq)$, where $q = M/(M+n-1)$.

Let $p = M/N$ and consider the probability space $\mathcal{B}_{n,p}$. For a graph G let $\mathbf{S} = \mathcal{S}(G)$ be the property that G has a connected component of size at least $\alpha|G|$, where $\alpha = q/4$. Define $\beta > \ln 16$ by $\alpha = 1 - (\ln 16)/\beta$ and let $p' = \beta/n$. Then Result 11 implies that almost every graph in $\mathcal{B}_{n,p'}$ has $\mathbf{S}$. Since almost surely $p \sim mq/N \sim (\varpi/n)e^{-\varpi/n} > p'$, and $\mathbf{S}$ is an increasing property, almost every graph in $\mathcal{B}_{n,p}$ has $\mathbf{S}$. Every increasing property is also convex. Thus Theorem 6 assures that almost every graph in $\mathcal{B}(n,M)$ has $\mathbf{S}$. Then Theorem 9 implies that almost every graph in $\mathcal{B}'(n,m)$ has $\mathbf{S}$. Let H be such a connected component of $B'_{n,n}$ of size at least $\alpha 2n$. According to Lemma 10 H is almost surely a police component. Finally, let H_C be the corresponding connected component of the citizen subgraph G_C of the configuration C. Since H_C is a police component, Claim 5 implies that C is solvable. This finishes the proof. ∎

10. Future Research

Consider the graph $K_n^d = K_n^{d-1} \square K_n$, and the sequence $\mathcal{K}^d = \{K_1^d, \ldots, K_n^d, \ldots\}$. If Conjecture 3 is true then induction would show that $\tau_{\mathcal{K}^d} = \Theta(\sqrt{N})$ for all d. On the surface such a result might be surprising, considering the sparcity of the graphs (size n^d, degree $d(n-1)$). However, its low diameter and high structure make such a result believable.

Another interesting test for Conjecture 3 is the sequence of n-dimensional cubes $\mathcal{Q} = \{Q^1, \ldots, Q^n, \ldots\}$, where $Q^n = Q^{n-1} \square Q^1$, and Q^1 is the path on two vertices. Because $\mathcal{Q}^2$ is a subsequence of $\mathcal{Q}$, we must have $\tau_{\mathcal{Q}^2} = \tau_{\mathcal{Q}}$. Therefore, if $\tau_{\mathcal{Q}} = \Theta(N^\alpha f(N))$ for some function $f(N)$, one can see that $f(N)$ must submultiplicative; i.e. $f(xy) \leqslant f(x)f(y)$ must hold. The best result to date is that $\tau_{\mathcal{Q}} \in \Omega(N^{1-\epsilon}) \cap O(N/\lg N)$ for all $\epsilon > 0$ (see [14]).

Acknowledgments

The authors wish to thank Graham Brightwell for useful hallway conversations during his visit to ASU.

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