Thresholds for families of multisets, with an application to graph pebbling

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Abstract

In this paper we prove two multiset analogs of classical results. We prove a multiset analog of Lovász's version of the Kruskal-Katona Theorem and an analog of the Bollobás-Thomason threshold result. As a corollary we obtain the existence of pebbling thresholds for arbitrary graph sequences. In addition, we improve both the lower and upper bounds for the 'random pebbling' threshold of the sequence of paths.

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1 Introduction

Throughout this paper G will denote a simple connected graph, and $n = n(G)$ will denote the number of its vertices. The vertex set of G will be the set $[n] = \{1, 2, \ldots, n\}$.

1.1 Pebbling

Suppose t pebbles are distributed onto the vertices of a graph G. A pebbling step $[v, v']$ consists of removing two pebbles from one vertex v and then placing one pebble at an adjacent vertex v'. We say a pebble can be *moved* to a vertex z, the *root* vertex, if we can repeatedly apply pebbling steps so that in the resulting distribution z has at least one pebble.

For a graph G, a distribution D of pebbles onto the vertices of G, and a 'root' vertex z, we say that D is *z-solvable* if it is possible to move a pebble to z; otherwise, D is *z-unsolvable*. Also D is *solvable* if it is z-solvable for all z, and *unsolvable* otherwise.

The *pebbling number* $pn(G)$ is the smallest integer t such that *all* distributions of t pebbles to the vertices of G are solvable. In this paper we are concerned instead with the minimum t such that *almost all* distributions of t pebbles to the vertices of G are solvable. The interested reader is encouraged to read [6] for the history of and the many results on graph pebbling.

1.2 Random Distributions

In this paper we are interested in the probabilistic pebbling model, in which the pebbling distribution is selected uniformly at random from the set of all distributions with a prescribed number t of pebbles (we emphasize that the pebbles are unlabeled and the vertices are labeled). This is certainly not the only possible random pebbling model; for instance one could consider the model obtained by placing each of the pebbles uniformly at random on a vertex of G – this model will exhibit very different behavior.

We will study thresholds for the number t of pebbles so that if t is essentially larger than the threshold, then a random distribution is almost surely solvable, and if t is essentially smaller than the threshold, then a random distribution is almost surely unsolvable. Formally, our notion of a pebbling threshold is defined as follows. Let $\mathbf{N}$ denote the set of nonnegative integers, and let $D_n : [n] \to \mathbf{N}$ denote a distribution of pebbles on n vertices. For a particular function $t = t(n)$, we consider the probability space $\mathcal{D}_{n,t}$ of all distributions D_n of *size* t, i.e. with $t = \sum_{i\in[n]} D_n(i)$ pebbles, with each such distribution being equally likely. Given a graph sequence $\mathcal{G} = (G_1, \ldots, G_n, \ldots)$, where G_n has vertex set $[n]$, denote by $P_{\mathcal{G}}(n, t)$ the probability that an element of $\mathcal{D}_{n,t}$ chosen uniformly at random is G_n-solvable. We call a function $t = t(n)$ a *threshold* for $\mathcal{G}$, and write $t \in th(\mathcal{G})$, if the following two statements hold for every sequence $\omega = \omega(n)$ tending to infinity (we write $\omega \gg 1$): (*i*) $P_{\mathcal{G}}(n, t\omega) \to 1$ as $n \to \infty$, and (*ii*) $P_{\mathcal{G}}(n, t/\omega) \to 0$ as $n \to \infty$. (Here and elsewhere, if $t\omega$ and t/ω are not integers, they should be interpreted as taking on the nearest integer value.) Of course, the definition mimics the important threshold concept in random graph theory. Unlike the situation in random graphs, however, it did not seem obvious that even "natural" families of graphs have pebbling thresholds, although the existence of a threshold for any graph sequence is conjectured in [6]. The random pebbling model is also studied in [4], where the following result is proved.

Theorem 1.1 *For any $\epsilon > 0$ and any graph sequence $\mathcal{G} = (G_1, \ldots, G_n, \ldots)$, where $V(G_n) = [n]$ for each n, $th(\mathcal{G}) \subseteq \Omega(n^{1/2}) \cap\ o(n^{1+\epsilon})$.*

In other words, *if* there is a threshold function $t(n)$ for a graph sequence, then $t(n)$ cannot be essentially smaller than $n^{1/2}$ (the threshold function for the family of complete graphs) or as large as $n^{1+\epsilon}$. Our results in this paper serve to improve the upper bound.

Better bounds were obtained in [4] for graphs satisfying various diameter and connectivity conditions, and threshold functions were found for the classes of stars and wheels. Also in [4], bounds were found for the classes of paths and cycles.

Theorem 1.2 **(a)** *For the sequence of paths $\mathcal{P} = (P_1, P_2, \ldots, P_n, \ldots)$, we have $th(\mathcal{P}) \subseteq \Omega(n) \cap o(n^{1+\epsilon})$ for every $\epsilon > 0$.*

(b) *For the sequence of cycles $\mathcal{C} = (C_1, C_2, \ldots, C_n, \ldots)$, we have $th(\mathcal{C}) \subseteq \Omega(n) \cap\ o(n^{1+\epsilon})$ for every $\epsilon > 0$.*

Our purpose in this paper is firstly to show that every graph sequence does have a threshold, and secondly to prove much tighter bounds for the threshold functions of classes of paths and cycles.

Theorem 1.3 *Let $\mathcal{G} = (G_1, \ldots, G_n, \ldots)$ be any graph sequence, and define $t = t(n) = \min\{r \mid P_{\mathcal{G}}(n, r) \geq 1/2\}$. Then $t \in th(\mathcal{G})$.*

Theorem 1.4 *For the sequence of paths $\mathcal{P} = (P_1, \ldots, P_n, \ldots)$ we have*

(a) $th(\mathcal{P}) \subseteq \Omega(n2^{c\sqrt{\lg n}})$, *where c is any constant less than $1/\sqrt{2}$;*

(b) $th(\mathcal{P}) \subseteq O(n2^{2\sqrt{\lg n}})$.

Here and throughout, lg denotes the logarithm to base 2. Our proofs apply equally well to the sequence of cycles $\mathcal{C} = (C_1, \ldots, C_n, \ldots)$.

In fact, Theorem 1.3 follows from a very general result about thresholds in random multiset models. For a natural number n, let $\mathcal{M}_n$ denote the partially ordered set (*poset*) of all multisets of $[n]$, ordered by inclusion. For $\mathcal{F}_n$ a subfamily of $\mathcal{M}_n$, and t a natural number, let $\mathcal{F}_n(t)$ denote the family of t-element multisets in $\mathcal{F}_n$. If $\mathcal{F}_n \subseteq \mathcal{M}_n$, the family $\mathcal{F}_n$ is said to be *increasing* if $E \supseteq F \in \mathcal{F}_n$ implies $E \in \mathcal{F}_n$, and *decreasing* if $E \subseteq F \in \mathcal{F}_n$ implies $E \in \mathcal{F}_n$.

The size of $\mathcal{M}_n(t)$ is $\left\langle {n \atop t} \right\rangle = \binom{n+t-1}{t}$, and therefore $P_t(\mathcal{F}_n(t)) \equiv |\mathcal{F}_n(t)|/\left\langle {n \atop t} \right\rangle$ is the probability that a uniformly randomly chosen t-multiset of $[n]$ is in the family $\mathcal{F}_n$. We say that $t = t(n)$ is a *threshold* for a sequence $(\mathcal{F}_1, \ldots, \mathcal{F}_n, \ldots)$ of increasing families of multisets if, for any function $\omega = \omega(n) \gg 1$, we have $P_{t\omega}(\mathcal{F}_n(t\omega)) \to 1$ and $P_{t/\omega}(\mathcal{F}_n(t/\omega)) \to 0$.

Theorem 1.5 *Let $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_n, \ldots)$ be a sequence of increasing families, with $\mathcal{F}_n \subseteq \mathcal{M}_n$ for each n. Define $t = t(n) = \min\{r \mid P_r(\mathcal{F}_n(r)) \geq 1/2\}$. Then $t \in th(\mathcal{F})$.*

Theorem 1.5 is an analog of a result of Bollobás and Thomason [1], stating that any sequence of increasing families of the *subset lattice* has a threshold function.

One can view a particular distribution of t pebbles to n vertices as a particular multiset of t elements from the ground set $[n]$. Thus the set of all solvable distributions for a particular graph on n vertices corresponds to a family of multisets on $[n]$. Since the set of all solvable distributions for a particular graph has the increasing property, Theorem 1.3 is an immediate consequence of Theorem 1.5.

In our proof of Theorem 1.5, the crucial tool will be the following 1969 result of Clements and Lindström, extending an earlier result of Macauley [10], which is the multiset analog of the celebrated Kruskal-Katona Theorem [5, 7, 8] for the subset lattice.

Given any subfamily $\mathcal{A} \subseteq \mathcal{M}_n(t)$, we define its *shadow* $\partial\mathcal{A} = \{C \in \mathcal{M}_n(t-1) \mid C \subset A \text{ for some } A \in \mathcal{A}\}$, and set $\partial^{i+1}\mathcal{A} = \partial\partial^i\mathcal{A}$.

For a multiset $A \in \mathcal{M}_n(t)$, and $i \in [n]$, let $A(i)$ denote the multiplicity of i in A. The *colexicographic order* on $\mathcal{M}_n(t)$ is defined by setting $A < B$ if $A \neq B$ and, for some $i \in [n]$, $A(i) < B(i)$ while $A(j) = B(j)$ for $j > i$.

Theorem 1.6 *(Clements-Lindström) Suppose that $\mathcal{F}$ is a subset of $\mathcal{M}_n(t)$, and that $\mathcal{G}$ consists of the first $|\mathcal{F}|$ elements of $\mathcal{M}_n(t)$ in colexicographic order. Then, for any $k \geq 1$,*

$$|\partial^k\mathcal{F}| \geq |\partial^k\mathcal{G}|.$$

In other words, the size of the shadow (at any level) of a subset of $\mathcal{M}_n(t)$ is minimized by taking an initial segment of the colexicographic order on $\mathcal{M}_n(t)$.

In 1979, Lovász proved a version of the Kruskal-Katona theorem which was used by Bollobás and Thomason [1] to prove the existence of threshold functions. An analogous version of the Clements-Lindström theorem was conjectured in [6]. We prove this conjecture in the next section.

For x a non-negative real number, let $\left\langle {x \atop t} \right\rangle = (x)(x+1)\cdots(x+t-1)/t!$. (Note that this coincides with our earlier definition if x is a natural number.)

Theorem 1.7 *Suppose that $\mathcal{A} \subseteq \mathcal{M}_n(t)$ and define x by $|\mathcal{A}| = \left\langle {x \atop t} \right\rangle$. Then $|\partial\mathcal{A}| \geq \left\langle {x \atop t-1} \right\rangle$.*

For the case in which $\mathcal{A} = \left\langle {m \atop t} \right\rangle$ for m a natural number, the first $|\mathcal{A}|$ elements of $\mathcal{M}_n(t)$ in colexicographic order are the t-multisets of $\{1,\ldots,m\}$. The shadow of the family consisting of these multisets is the family of $(t-1)$-multisets of $\{1,\ldots,m\}$, of size $\left\langle {x \atop t-1} \right\rangle$, so Theorem 1.7 is equivalent to Theorem 1.6 in this case. For families of intermediate sizes, Theorem 1.7 is a "smoothed" version of Theorem 1.6.

The rest of this paper is organized as follows. In Section 2 we prove Theorem 1.7 (Section 2.1) and Theorem 1.5 (Section 2.2). Section 3 is devoted to a proof of Theorem 1.4.

2 Set Theory

In this section, we prove the two results concerning the multiset lattice. First, verifying a conjecture from [6], we prove our multiset analog of Lovász's version of the Kruskal-Katona theorem. Second, we establish our multiset analog of the Bollobás-Thomason threshold theorem [1]. As an immediate corollary we obtain the existence of the pebbling threshold for any graph sequence.

2.1 Multiset analog of Lovász's Theorem

Let $\left\langle{[n] \atop t}\right\rangle$ denote the family of t-element multisets of $[n]$, and as before let $\left\langle{n \atop t}\right\rangle = \binom{n+t-1}{t}$ denote its cardinality. (For convenience in reading, the reader may enjoy using the terminology "*n pebble t*".) Also as before, let $\left\langle{x \atop t}\right\rangle$ denote the polynomial evaluation of $\left\langle{n \atop t}\right\rangle$ for any real number x; that is,

$$\left\langle{x \atop t}\right\rangle = (x)(x+1)\cdots(x+t-1)/t!.$$

Let $\mathcal{A} \subseteq \left\langle{[n] \atop t}\right\rangle$, and for any $A \in \mathcal{A}$ and $i \in [n]$, let $A(i)$ denote the multiplicity of i in A. We adopt the convention of writing an element of a multiset just once, with its multiplicity written as an exponent; for example $\{1^j\}$ denotes the multiset of j ones. Also, for any multisubset I of $[n]$ we set $\mathcal{A} - I = \{A - I \mid A \in \mathcal{A}\}$.

In the proof of Theorem 1.7, we make use of various different partitions of a family $\mathcal{A} \subseteq \left\langle{[n] \atop t}\right\rangle$. For $0 \leq j \leq t$, define the sets $\mathcal{A}_j = \{A \in \mathcal{A} \mid A(1) = j\}$, forming a partition of $\mathcal{A}$. If $\mathcal{A} = \left\langle{[n] \atop t}\right\rangle$, this partition gives rise to the relation

$$\left\langle{n \atop t}\right\rangle = \sum_{j=0}^{t} \left\langle{n-1 \atop j}\right\rangle,$$

which has the polynomial equivalent

$$\left\langle{x \atop t}\right\rangle = \sum_{j=0}^{t} \left\langle{x-1 \atop j}\right\rangle. \tag{1}$$

For each $i \in [n]$, define the sets $\mathcal{A}^i = \{A \in \mathcal{A} \mid A(i) > 0 \text{ and } A(j) = 0 \text{ for all } j < i\}$, forming a second partition of $\mathcal{A}$. More important for us is that the sets $\mathcal{A}_1, \ldots, \mathcal{A}_t$ partition $\mathcal{A}^1$.

A third partition of $\mathcal{A}$ is given by $\mathcal{A} = \mathcal{A}^1 \cup \mathcal{A}_0$, which in the case that $\mathcal{A} = \left\langle{[n] \atop t}\right\rangle$ gives rise to the relation

$$\left\langle{n \atop t}\right\rangle = \left\langle{n \atop t-1}\right\rangle + \left\langle{n-1 \atop t}\right\rangle,$$

having the polynomial equivalent

$$\left\langle {x \atop t} \right\rangle = \left\langle {x \atop t-1} \right\rangle + \left\langle {x-1 \atop t} \right\rangle . \tag{2}$$

Given a family $\mathcal{A} \subseteq \mathcal{M}_n(t)$, and indices i, j with $1 \leq i < j \leq n$, a *compression* of $\mathcal{A}$ is obtained by taking each member A of $\mathcal{A}$ such that $A(j) \geq 1$ and $A - \{j\} + \{i\} \notin \mathcal{A}$, and replacing it by $A - \{j\} + \{i\}$.

A family $\mathcal{A}$ is said to be *compressed* if, for all $1 \leq i < j \leq n$, we have $A - \{j\} + \{i\} \in \mathcal{A}$ whenever $A \in \mathcal{A}$, i.e., $\mathcal{A}$ is unchanged by any compression. Note that any family can be transformed into a compressed one by a sequence of compressions. Note also that initial segments of the colexicographic order are compressed families, but that these are not the only ones.

If $\mathcal{A}$ is compressed and $C \in \partial\mathcal{A}$, then $C = A - \{i\}$ for some $A \in \mathcal{A}$, $i \in [n]$. Because $\mathcal{A}$ is compressed, the set $E = A - \{i\} + \{1\}$ is in $\mathcal{A}$. Since $C = E - \{1\}$, we see that $\partial\mathcal{A} \subseteq \mathcal{A}^1 - \{1\}$, which implies that

$$|\partial\mathcal{A}| \leq |\mathcal{A}^1| .$$

On the other hand, we know that $\mathcal{A}^1 - \{1\} \subseteq \partial\mathcal{A}^1 \subseteq \partial\mathcal{A}$, and so

$$|\mathcal{A}^1| \leq |\partial\mathcal{A}| .$$

These two facts together imply the following lemma.

Lemma 2.1 *Let $\mathcal{A} \subseteq \mathcal{M}_n(t)$ and suppose that $\mathcal{A}$ is a compressed family. Then $|\partial\mathcal{A}| = |\mathcal{A}^1|$.* ◇

The following lemma is proved by Clements in [2].

Lemma 2.2 *Suppose that $\mathcal{A} \subseteq \mathcal{M}_n(t)$ and let $q(\mathcal{A})$ be a compression of $\mathcal{A}$. Then $|\partial\mathcal{A}| \geq |\partial q(\mathcal{A})|$.* ◇

With these tools we now can prove Theorem 1.7.

Proof of Theorem 1.7. We use induction on n. Because of Lemma 2.2 we may assume that $\mathcal{A}$ is compressed. If $|\mathcal{A}^1| \geq \left\langle {x \atop t-1} \right\rangle$ then we are done because of Lemma 2.1. So we will assume that $|\mathcal{A}^1| < \left\langle {x \atop t-1} \right\rangle$ and argue to a contradiction.

Claim For each $0 \leq j \leq t$ we have $|\mathcal{A}_j| \geq \left\langle {x-1 \atop t-j} \right\rangle$.

The truth of this claim yields the following contradiction. Because the families $\mathcal{A}_1, \ldots, \mathcal{A}_t$ partition the family $\mathcal{A}^1$, we have

$$|\mathcal{A}^1| \;=\; \sum_{j=1}^{t} |\mathcal{A}_j| \;\geq\; \sum_{j=1}^{t} \left\langle {x-1 \atop t-j} \right\rangle \;=\; \sum_{k=0}^{t-1} \left\langle {x-1 \atop k} \right\rangle \;=\; \left\langle {x \atop t-1} \right\rangle ,$$

using equation (1). Thus we need only to prove the claim.

Induction on j shows that $|\partial^j \mathcal{A}_0| \geq \left\langle {x-1 \atop t-j} \right\rangle$. Indeed, for $j = 0$ we know from equation (2) that $|\mathcal{A}_0| > \left\langle {x-1 \atop t} \right\rangle$. Also, $\partial^j \mathcal{A}_0$ is a family of $(t-j)$-multisets of the set $\{2, \ldots, n\}$ of size $n-1$, and so if $|\partial^j \mathcal{A}_0| \geq \left\langle {x-1 \atop t-j} \right\rangle$, then $|\partial^{j+1} \mathcal{A}_0| \geq \left\langle {x-1 \atop t-j-1} \right\rangle$ (by induction on n for the theorem).

Now, if $C \in \partial^j \mathcal{A}_0$ then $C = A - I$ for some $A \in \mathcal{A}_0$ and some submultiset I of A of size j. Because $\mathcal{A}$ is compressed, the set $E = A - I + \{1^j\}$ is in $\mathcal{A}$, and in particular is in $\mathcal{A}_j$. Since $C = E - \{1^j\}$ we see that $\partial^j \mathcal{A}_0 \subseteq \mathcal{A}_j - \{1^j\}$, which implies that

$$|\mathcal{A}_j| \;=\; |\mathcal{A}_j - \{1^j\}| \;\geq\; |\partial^j \mathcal{A}_0| \;\geq\; \left\langle {x-1 \atop t-j} \right\rangle .$$

This proves the claim, and the theorem follows. ◇

2.2 Thresholds

The main result of this section is the analog of the Bollobás-Thomason threshold theorem.

Proof of Theorem 1.5. We consider the two "reference" families:

$$\begin{aligned} \mathcal{M}_n(r;b) &= \{A \in \mathcal{M}_n(r) \mid A(n) < b\} \quad (1 \leq b \leq r), \\ \mathcal{N}_n(r;b) &= \{A \in \mathcal{M}_n(r) \mid A(n-b+1) = \cdots = A(n) = 0\} \\ & \quad\; (1 \leq b \leq n-1). \end{aligned}$$

Note that each family $\mathcal{M}_n(r;b)$ and $\mathcal{N}_n(r;b)$ is an initial segment of the colexicographic order on $\mathcal{M}_n(r)$, and that, for any $k \geq 1$,

$$\partial^k \mathcal{M}_n(r;b) = \mathcal{M}_n(r-k;b) \;; \qquad \partial^k \mathcal{N}_n(r;b) = \mathcal{N}_n(r-k;b) \;.$$

We use these families, rather than general initial segments of the colexicographic order, since their sizes are a little easier to estimate. Our strategy is, for each r, to compare $\overline{\mathcal{F}_n(r)} = \mathcal{M}_n(r) - \mathcal{F}_n(r)$ with an appropriate member of one of these reference families, at levels r near t, somewhat above t, and somewhat below t.

First, we need some estimates on the probabilities of the reference families, which we shall use repeatedly. For any positive integers n, r and b with $b \leq r$,

$$|\mathcal{M}_n(r;b)| = |\mathcal{M}_n(r)| - |\mathcal{M}_n(r-b)| = \left\langle {n \atop r} \right\rangle - \left\langle {n \atop r-b} \right\rangle ,$$

and so

$$\begin{aligned} P_r(\mathcal{M}_n(r;b)) &= 1 - \tfrac{(n+r-b-1)!r!}{(n+r-1)!(r-b)!} \\ &= 1 - \left(\tfrac{r}{n+r-1}\right)\left(\tfrac{r-1}{n+r-2}\right)\cdots\left(\tfrac{r-b+1}{n+r-b}\right) . \end{aligned}$$

We derive the lower bounds

$$\begin{aligned} P_r(\mathcal{M}_n(r;b)) &\geq 1 - \left(\tfrac{r}{n+r-1}\right)^b \\ &= 1 - \left(1 - \tfrac{n-1}{n+r-1}\right)^b \\ &\geq 1 - \exp(-b(n-1)/(n+r-1)) , \end{aligned}$$

and the upper bounds

$$\begin{aligned} P_r(\mathcal{M}_n(r;b)) &\leq 1 - \left(\tfrac{r-b+1}{n+r-b}\right)^b \\ &= 1 - \left(1 + \tfrac{n-1}{r-b+1}\right)^{-b} \\ &\leq 1 - \exp(-b(n-1)/(r-b+1)) . \end{aligned}$$

Similarly we see that, for positive integers n, r and b with $b \leq n-1$, $|\mathcal{N}_n(r;b)| = |\mathcal{M}_{n-b}(r)| = \left\langle {n-b \atop r} \right\rangle$, and so

$$\begin{aligned} P_r(\mathcal{N}_n(r;b)) &= \tfrac{(n-b+r-1)!(n-1)!}{(n-b-1)!(n+r-1)!} \\ &= \left(\tfrac{n-1}{n+r-1}\right)\cdots\left(\tfrac{n-b}{n-b+r}\right) . \end{aligned}$$

This gives the bounds

$$\begin{aligned} P_r(\mathcal{N}_n(r;b)) &\geq \left(\tfrac{n-b}{n-b+r}\right)^b \\ &= \left(1 + \tfrac{r}{n-b}\right)^{-b} \geq \exp(-rb/(n-b)) ; \end{aligned}$$

$$\begin{aligned} P_r(\mathcal{N}_n(r;b)) &\leq \left(\tfrac{n-1}{n+r-1}\right)^b \\ &= \left(1 - \tfrac{r}{n+r-1}\right)^b \leq \exp(-rb/(n+r-1)) . \end{aligned}$$

Recall that $t = t(n)$ is defined as the least integer such that $P_t(\mathcal{F}_n(t)) \geq 1/2$. Let $\omega = \omega(n)$ be any function tending to infinity with n such that $t(n)/\omega(n)$ takes integer values. We shall show that $P_{t/\omega}(\mathcal{F}_n(t/\omega)) \to 0$ as $n \to \infty$, or equivalently that $P_{t/\omega}(\overline{\mathcal{F}_n(t/\omega)}) \to 1$. We may assume without loss of generality that $n \geq 3$ and $\omega \geq 30$, and so $t \geq 30$.

We fix n for the moment and consider two cases.

(1) Suppose that $t = t(n) \geq 2n - 1$.

In this case, we set $b = \left\lfloor \frac{t}{2n-1} \right\rfloor$. Note that our assumption on t ensures that $b \geq 1$, and that our choice of b ensures that $b(n-1)/(t-b) \leq 1/2$. Now we have

$$\begin{aligned} P_{t-1}(\mathcal{M}_n(t-1;b)) &\leq 1 - \exp(-b(n-1)/(t-b)) \\ &\leq 1 - e^{-1/2} \\ &< 1/2 \\ &< P_{t-1}(\overline{\mathcal{F}_n(t-1)}) \ . \end{aligned}$$

Then, since $\overline{\mathcal{F}_n}$ is decreasing and $\mathcal{M}_n(t-1;b)$ is an initial segment of the colexicographic order, the Clements-Lindström Theorem implies that

$$\begin{aligned} P_{t/\omega}(\overline{\mathcal{F}_n(t/\omega)}) &\geq P_{t/\omega}(\partial^{t-1-t/\omega}\overline{\mathcal{F}_n(t-1)}) \\ &\geq P_{t/\omega}(\partial^{t-1-t/\omega}\mathcal{M}_n(t-1;b)) \\ &= P_{t/\omega}(\mathcal{M}_n(t/\omega;b)) \\ &\geq 1 - \left(\frac{t/\omega}{n+t/\omega-1}\right)^b . \end{aligned}$$

If $n-1 \geq \frac{t}{\sqrt{\omega}}$, then this gives $P_{t/\omega}(\overline{\mathcal{F}_n(t/\omega)}) \geq 1 - \frac{1}{\sqrt{\omega}+1}$. On the other hand, if $n-1 \leq \frac{t}{\sqrt{\omega}} \leq \frac{t}{5}$, then $\frac{t}{2n-1} \geq \frac{5n-5}{2n-1} \geq 2$, so $b \geq \frac{1}{2}\frac{t}{2n-1} \geq \frac{t}{5(n-1)} \geq \sqrt{\omega}/5$, so

$$\begin{aligned} P_{t/\omega}(\overline{\mathcal{F}_n(t/\omega)}) &\geq 1 - \left(\frac{t/\omega}{t/5b+t/\omega}\right)^b \\ &= 1 - \left(1 - \frac{\omega}{\omega+5b}\right)^b \\ &\geq 1 - \exp(-b\omega/(\omega+5b)) \\ &\geq 1 - \exp(\sqrt{\omega}/10) \ . \end{aligned}$$

(2) Now suppose that $t = t(n) \leq 2n - 2$.

This time we set $b = \left\lceil \frac{n+t-2}{t-1} \right\rceil$, and use exactly the same method as in (1), but this time comparing with $\mathcal{N}_n(t-1;b)$. Indeed

$$\begin{aligned} P_{t-1}(\mathcal{N}_n(t-1;b)) &\le \exp(-(t-1)b/(n+t-2)) \\ &\le e^{-1} \\ &< 1/2 \\ &< P_{t-1}(\overline{\mathcal{F}_n(t-1)}) \ . \end{aligned}$$

As before we deduce that

$$P_{t/\omega}(\overline{\mathcal{F}_n(t/\omega)}) \ge P_{t/\omega}(\mathcal{N}_n(t/\omega;b)) \ge \exp(-tb/\omega(n-b)) \ .$$

Now observe that $b \le 5n/t$, and so $P_{t/\omega}(\overline{\mathcal{F}_n(t/\omega)}) \ge \exp(-6/\omega)$.

Summarizing, we see that

$$P_{t/\omega}(\overline{\mathcal{F}_n(t/\omega)}) \ge \min\left\{1-\frac{1}{\sqrt{\omega(n)}+1}, 1-\exp(-\sqrt{\omega(n)}/10), \exp(-6/\omega(n))\right\},$$

so $P_{t/\omega}(\overline{\mathcal{F}_n(t/\omega)}) \to 1$ as $n \to \infty$.

Now let $\omega(n) \to \infty$ be such that $t(n)\omega(n)$ takes integer values; we claim that $P_{t\omega}(\overline{\mathcal{F}_n(t\omega)}) \to 0$ as $n \to \infty$. We may assume that $n \ge 2$ and $\omega(n) \ge 36$.

As before, we fix n for the moment and consider two cases.

(1) Suppose $t = t(n) \ge n/2$. In this case, we set $b = \lceil \frac{n+t-1}{n-1} \rceil$, and note that $b = 1 + \lceil t/(n-1) \rceil \le \frac{2t}{n-1} + \frac{2t}{n-1} = 4t/(n-1)$ and $b - 1 \le t\omega/2$. Observe that

$$\begin{aligned} P_t(\mathcal{M}_n(t;b)) &\ge 1-\exp(-b(n-1)/(n+t-1)) \\ &\ge 1-e^{-1} \\ &> 1/2 \\ &\ge P_t(\overline{\mathcal{F}_n(t)}). \end{aligned}$$

This implies that $P_{t\omega}(\overline{\mathcal{F}_n(t\omega)}) \le P_{t\omega}(\mathcal{M}_n(t\omega;b))$, since otherwise we would have

$$\left|\overline{\mathcal{F}_n(t)}\right| \ge \left|\partial^{t\omega-t}\overline{\mathcal{F}_n(t\omega)}\right| > \left|\partial^{t\omega-t}\mathcal{M}_n(t\omega;b)\right| = |\mathcal{M}_n(t;b)| \, .$$

Therefore we see that

$$\begin{aligned} P_{t\omega}(\overline{\mathcal{F}_n(t\omega)}) &\le 1-\exp(-b(n-1)/(t\omega-b+1)) \\ &\le 1-\exp(-4t/(t\omega/2)) \\ &= 1-\exp(-8/\omega) \ . \end{aligned}$$

(2) Now suppose $t \le (n-1)/2$. Here we set $b = \lfloor \frac{n}{2t+1} \rfloor \ge 1$, and observe that $tb/(n-b) \le 1/2$. We see that

$$\begin{aligned} P_t(\mathcal{N}_n(t;b)) &\ge \exp(-tb/(n-b)) \\ &\ge e^{-1/2} \\ &> 1/2 \\ &\ge P_t(\overline{\mathcal{F}_n(t)}) \ , \end{aligned}$$

which, as before, implies that

$$P_{t\omega}(\overline{\mathcal{F}_n(t\omega)}) \le P_{t\omega}(\mathcal{N}_n(t\omega;b)) \le \left(\frac{n-1}{n+t\omega-1}\right)^b .$$

If $n-1 \le t\sqrt{\omega}$, this yields $P_{t\omega}(\overline{\mathcal{F}_n(t\omega)}) \le 1/(1+\sqrt{\omega})$. On the other hand, if $n-1 \ge t\sqrt{\omega}$, then $\frac{n}{2t+1} \ge \frac{n-1}{3t} \ge 2$, so $b \ge \frac{n-1}{6t} \ge \frac{\sqrt{\omega}}{6}$, and

$$\begin{aligned} P_{t\omega}(\overline{\mathcal{F}_n(t\omega)}) &\le \left(\tfrac{6tb}{6tb+t\omega}\right)^b \\ &= \left(1 - \tfrac{\omega}{6b+\omega}\right)^b \\ &\le \exp(-\omega b/(6b+\omega)) \\ &\le \exp(-\sqrt{\omega}/12) \ . \end{aligned}$$

Combining all the cases gives

$$P_{t\omega}(\overline{\mathcal{F}_n(t\omega)}) \le \max\left\{1-\exp(8/\omega(n)), \frac{1}{1+\sqrt{\omega(n)}}, \exp(-\sqrt{\omega(n)}/12)\right\},$$

so $P_{t\omega}(\overline{\mathcal{F}_n(t\omega)}) \to 0$ as $n \to \infty$.

This completes the proof. ◇

3 The Threshold for Paths

For convenience we will assume that all logarithms and roots take on the value of their nearest integer. Let P_n be the path on n vertices, c be any constant less than $1/\sqrt{2}$, $u = c\sqrt{\lg n}$, $t = n2^u$, and $p = (1+\epsilon)2^u \ln n$, for some $\epsilon > 0$. In addition, let D_i be the random variable which is the number of pebbles on vertex i when a random distribution of t pebbles is selected, and let S_i denote the event that $D_i \le p$ and $T_i = \neg S_i$. Finally, we will also need the following random variables

$$Y_i^+ = \sum_{l=i}^{n} \frac{D_l}{2^{l-i}} \tag{3}$$

and

$$Y_i^- = \sum_{l=1}^{i} \frac{D_l}{2^{i-l}} \ . \tag{4}$$

Lemma 3.1 *Let p, t, n, u be as above. Then* $\Pr[T_1 \cup T_2 \cup \cdots \cup T_n] \to 0$.

Proof. For $1 \le i \le n$, we have

$$\Pr[T_i] = \frac{\left\langle {n \atop t-p} \right\rangle}{\left\langle {n \atop t} \right\rangle} = \frac{(t-p+1)\cdots(t)}{(t+n-p)\cdots(t+n-1)}$$
$$\le \left(\frac{t}{t+n}\right)^p \le e^{-np/(t+n)} \ .$$

Therefore,

$$\Pr[T_1 \cup \cdots \cup T_n] \le ne^{-p/(2^u+1)} = \exp\left(\ln n \left(1 - \frac{2^u}{2^u+1}(1+\epsilon)\right)\right) \to 0.$$

◇

Note that the role of u is not important in the above lemma. However, we stated the lemma as such because of its use in the proof of Theorem 1.4.

For the next lemma, we will need $m = \sqrt{2\lg n}$, and $k = \lfloor n/m \rfloor$. Partition P_n into consecutively disjoint paths (blocks) $B_1, \ldots, B_k$ of lengths m or $m+1$. For a randomly chosen distribution of t pebbles on P_n we denote by E_i the event that block B_i contains no pebbles (is empty), and set $F_i = \neg E_i$.

Lemma 3.2 *Let m, k be as above and let $s = s(n)$ be such that $n \gg m s^m e^{m/s}$. If $t = ns$ then* $\Pr[F_1 \cap \cdots \cap F_k] \to 0$.

Proof. We will apply the second moment method. Let X_i be the indicator variable of E_i and let $X = \sum_{i=1}^{k} X_i$. We have

$$\Pr(X=0) \le \frac{\sigma^2(X)}{(\mathbf{E}(X))^2}$$

and

$$\begin{aligned}\sigma^2(X) &= \mathbf{E}(X^2) - (\mathbf{E}(X))^2 \\ &= \sum_{i,j} \mathbf{E}(X_i X_j) - \sum_{i,j} \mathbf{E}(X_i)\mathbf{E}(X_j) \\ &\le \sum_i \mathbf{E}(X_i^2) ,\end{aligned}$$

where the last inequality follows from the fact that $\mathbf{E}(X_i X_j) \le \mathbf{E}(X_i)\mathbf{E}(X_j)$ for $i \neq j$. Therefore,

$$\sigma^2(X) \le \sum_i \mathbf{E}(X_i^2) = \sum_i \mathbf{E}(X_i) = \mathbf{E}(X) .$$

We also have

$$\begin{aligned}\mathbf{E}(X) &= \left(\frac{n}{m}\right) \frac{\binom{t+n-m-1}{t}}{\binom{t+n-1}{t}} \\ &\ge \left(\frac{n}{m}\right) \left(\frac{n-m}{t+n-m}\right)^m \\ &\sim \left(\frac{n}{ms^m}\right) e^{-m^2/n - m/s} \\ &\to \infty .\end{aligned}$$

Thus $\Pr[X=0] \to 0$. ◇

Proof of Theorem 1.4 (a). We first remark that it is easy to check that $s = 2^u$ satisfies hypothesis of Lemma 3.2. Next, for each block $B = [i,j]$, let $Y_B^- = Y_i^-$ and $Y_B^+ = Y_j^+$. Then Lemma 3.1 and Lemma 3.2 imply that with probability tending to one there is a block B of length m which is empty and such that both Y_B^- and Y_B^+ are less than or equal to $(1+\epsilon)2^u \ln n$. Since $(1+\epsilon)2^u \ln n < 2^{m/2}$ for large enough n, there is a vertex of B, namely the center vertex, to which it is not possible to pebble using Y_B^- and Y_B^+. ◇

We also shall prove that $th(P_n) \subseteq O(n 2^{2\sqrt{\lg n}})$. The argument is a modification of an idea from [4]. We will use the fact [4] that any distribution of at least 2^m pebbles on a connected graph of $m+1$ vertices is solvable.

Proof of Theorem 1.4 (b). Let $m = \sqrt{\lg n}$, $k = \lfloor n/m \rfloor$, $s = 2^{2m+2}$, $t = ns$, and partition P_n into consecutively disjoint paths (blocks) $B_1, \ldots, B_k$, each with m or $m+1$ vertices. Let Z_i denote the event that the ith block contains less than 2^m pebbles. We will show that $\Pr[Z_1 \cup \cdots \cup Z_k] \to 0$, which, by the remark preceding the proof, implies the result.

We have that

$$\Pr[Z_1 \cup \cdots \cup Z_k] \le k \sum_{i=0}^{2^m-1} \frac{\binom{i+m-1}{i}\binom{t+n-m-i-1}{t-i}}{\binom{t+n-1}{t}} .$$

Since $\binom{t+n-m-i-1}{t-i} \le \left(\frac{t}{t+n-m-1}\right)^i \binom{t+n-m-1}{t} \le \binom{t+n-m-1}{t} \le \left(\frac{n}{n+t}\right)^m \binom{t+n-1}{t}$, we have

$$\binom{t+n-m-i-1}{t-i} \le \left(\frac{n}{n+t}\right)^m \binom{t+n-1}{t} . \tag{5}$$

For $0 \le i < 2^m$, we have

$$\binom{i+m-1}{i} \le 2^m 2^{m^2} . \tag{6}$$

Using (5) and (6), we see that

$$\Pr[Z_1 \cup \cdots \cup Z_k] \le k 2^{2m+m^2} \left(\frac{n}{n+t}\right)^m \le \frac{n}{m}\left(\frac{1}{s}\right)^m 2^{2m+m^2} = \frac{1}{m} \to 0 .$$

◇

References

[1] B. Bollobás and A. Thomason, *Threshold functions*, Combinatorica **7** (1987), 35–38.

[2] G.F. Clements, *On existence of distinct representative sets for subsets of a finite set*, Can. J. Math **22** (1970), 1284–1292.

[3] G.F. Clements and B. Lindström, *A generalization of a combinatorial theorem of Macaulay*, J. Combin. Theory **7** (1969), 230–238.

[4] A. Czygrinow, N. Eaton, G. Hurlbert and P.M. Kayll, *On pebbling thresholds functions for graph sequences*, Discrete Math. **247** (2002), 93–105.

[5] P. Frankl, *A new short proof for the Kruskal-Katona theorem*, Discrete Math. **48** (1984), 327–329.

[6] G.H. Hurlbert, *A survey of graph pebbling*, Congress. Numer. **139** (1999), 41–64.

[7] G.O.H. Katona, *A theorem on finite sets*, in Theory of Graphs (P. Erdős and G.O.H. Katona, eds.), Akadémiai Kiadó, Budapest (1968), 187–207.

[8] J.B. Kruskal, *The number of simplices in a complex*, in Mathematical Optimization Techniques, Univ. California Press, Berkeley (1963), 251–278.

[9] L. Lovász, *Combinatorial Problems and Exercises*, North Holland Pub., Amsterdam, New York, Oxford (1979).

[10] F.S. Macaulay, *Some properties of enumeration in the theory of modular systems*, Proc. Lond. Math. Soc. **26** (1927), 531–555.